



1. Given that

$$2z^3 - 5z^2 + 7z - 6 \equiv (2z - 3)(z^2 + az + b)$$

where  $a$  and  $b$  are real constants,

(a) find the value of  $a$  and the value of  $b$ .

(2)

(b) Given that  $z$  is a complex number, find the three exact roots of the equation

$$2z^3 - 5z^2 + 7z - 6 = 0$$

(3)

$$a) -5z^2 \equiv 2az^2 - 3z^2 \quad \therefore 2a - 3 = -5 \Rightarrow 2a = -2 \therefore \underline{a = -1}$$

$$-6 \equiv -3b \quad \therefore \underline{b = 2}$$

$$b) (2z - 3)(z^2 - z + 2)$$

$$\rightarrow (z - \frac{1}{2})^2 - \frac{1}{4} + 2 = 0$$

$$(z - \frac{1}{2})^2 = -\frac{7}{4}$$

$$z - \frac{1}{2} = \pm \frac{1}{2}\sqrt{7}i$$

$$z = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}i$$

$$f(z) = 0 \Rightarrow z = \frac{3}{2}, \frac{1}{2} + \frac{1}{2}\sqrt{7}i, \frac{1}{2} - \frac{1}{2}\sqrt{7}i$$

2. Use the standard results for  $\sum_{r=1}^n r$  and for  $\sum_{r=1}^n r^2$  to show that

$$\sum_{r=1}^n (3r - 2)^2 = \frac{n}{2}(an^2 + bn + c)$$

where  $a$ ,  $b$  and  $c$  are integers to be found.

(5)

$$\sum_{r=1}^n 9r^2 - 12r + 4 = 9 \sum r^2 - 12 \sum r + 4 \sum 1$$

$$= \frac{9}{6}n(n+1)(2n+1) - \frac{12}{2}n(n+1) + 4n$$

$$= \frac{9}{6}n(n+1)(2n+1) - \frac{36}{6}n(n+1) + \frac{24}{6}n$$

$$= \frac{1}{6}n [18n^2 + 27n + 9 - 36n - 36 + 24]$$

$$= \frac{1}{6}n [18n^2 - 9n - 3] = \frac{3}{6}n(6n^2 - 3n - 1)$$

$$= \frac{1}{2}n(6n^2 - 3n - 1)$$

2

3. It is given that  $\alpha$  and  $\beta$  are roots of the equation

$$2x^2 - 7x + 4 = 0$$

(a) Find the exact value of  $\alpha^2 + \beta^2$

(3)

(b) Find a quadratic equation which has roots  $\frac{\alpha}{\beta}$  and  $\frac{\beta}{\alpha}$ , giving your answer in the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers.

(3)

$$x^2 - \frac{7}{2}x + 2 = 0 \qquad x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\alpha + \beta = \frac{7}{2} \qquad \alpha\beta = 2 \qquad (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$(\alpha + \beta)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

$$\frac{49}{4} = \alpha^2 + \beta^2 + 2(2) \qquad \therefore \alpha^2 + \beta^2 = \frac{49}{4} - \frac{16}{4} = \frac{33}{4}$$

$$b) \quad x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 0$$

$$x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0$$

$$x^2 - \frac{33}{8}x + 1 = 0$$

4.

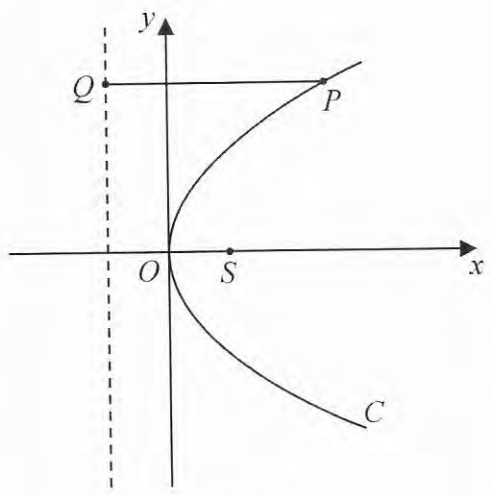


Figure 1

Figure 1 shows a sketch of the parabola  $C$  with equation  $y^2 = 4ax$ , where  $a$  is a positive constant. The point  $S$  is the focus of  $C$  and the point  $Q$  lies on the directrix of  $C$ . The point  $P$  lies on  $C$  where  $y > 0$  and the line segment  $QP$  is parallel to the  $x$ -axis.

Given that the length of  $PS$  is 13

(a) write down the length of  $PQ$ . (1)

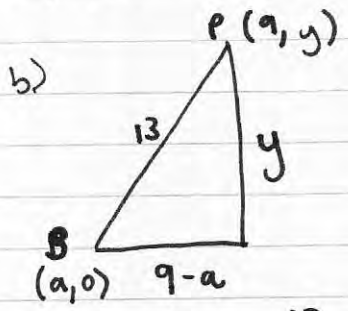
Given that the point  $P$  has  $x$  coordinate 9

find

(b) the value of  $a$ , (2)

(c) the area of triangle  $PSQ$ . (3)

a) 13

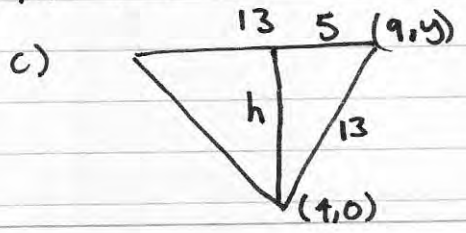


$$= 4ax = 36a$$

$$(9-a)^2 + y^2 = 13^2 \Rightarrow 81 - 18a + a^2 + 36a = 169$$

$$a^2 + 18a - 88 = 0 \quad (a+22)(a-4) = 0$$

$$\therefore \underline{a = 4}$$



$$h^2 = 13^2 - 5^2 \quad h^2 = 144 \quad \therefore h = 12$$

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(13) \times 12 = 78$$

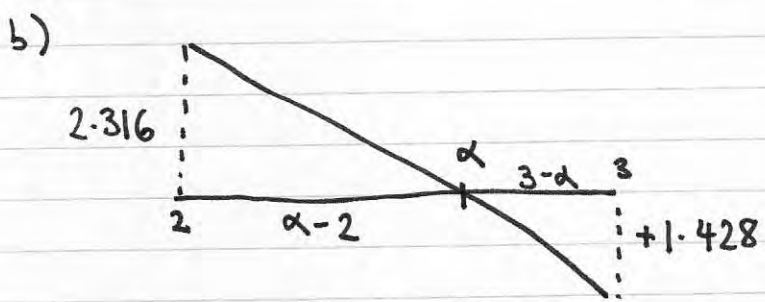
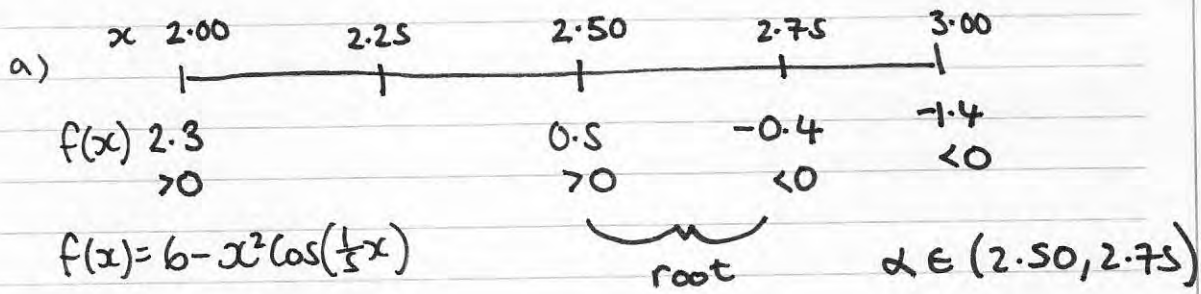
5. In the interval  $2 < x < 3$ , the equation

$$6 - x^2 \cos\left(\frac{x}{5}\right) = 0, \text{ where } x \text{ is measured in radians}$$

has exactly one root  $\alpha$ .

(a) Starting with the interval  $[2, 3]$ , use interval bisection twice to find an interval of width 0.25 which contains  $\alpha$ . (4)

(b) Use linear interpolation once on the interval  $[2, 3]$  to find an approximation to  $\alpha$ . Give your answer to 2 decimal places. (3)



$$\frac{3-\alpha}{1.428} = \frac{\alpha-2}{2.316}$$

$$6.947 - 2.316\alpha = 1.428\alpha - 2.856$$

$$9.803 = 3.744\alpha$$

$$\alpha = 2.62$$

6. The rectangular hyperbola,  $H$ , has cartesian equation

$$xy = 36$$

The three points  $P \left( 6p, \frac{6}{p} \right)$ ,  $Q \left( 6q, \frac{6}{q} \right)$  and  $R \left( 6r, \frac{6}{r} \right)$ , where  $p, q$  and  $r$  are distinct, non-zero values, lie on the hyperbola  $H$ .

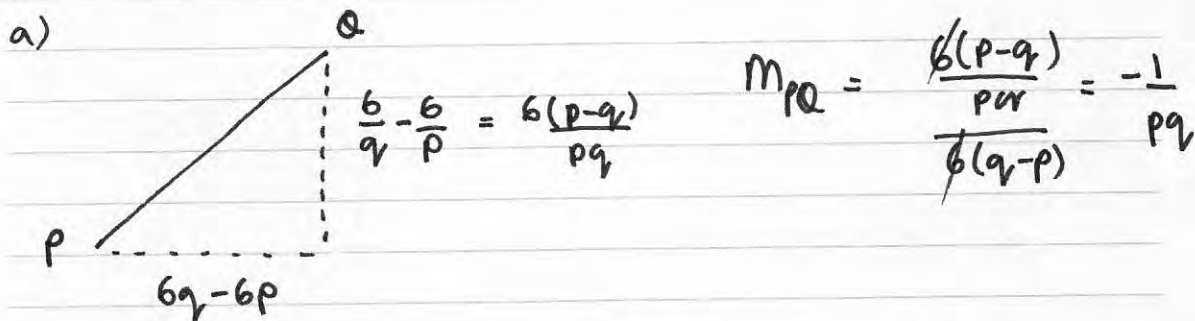
(a) Show that an equation of the line  $PQ$  is

$$pqy + x = 6(p + q) \quad (4)$$

Given that  $PR$  is perpendicular to  $QR$ ,

(b) show that the normal to the curve  $H$  at the point  $R$  is parallel to the line  $PQ$ .

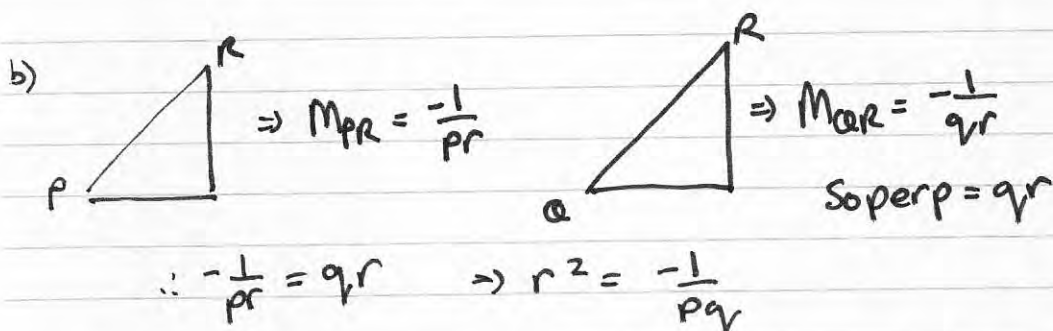
(6)



$$y - \frac{6}{p} = -\frac{1}{pq}(x - 6p) \Rightarrow pqy - 6q = 6p - x$$

$$pqy + x = 6p + 6q$$

$$\therefore pqy + x = 6(p + q) \quad \#$$



$$y = \frac{36}{x} = 36x^{-1} \Rightarrow \frac{dy}{dx} = -36x^{-2} = -\frac{36}{x^2} \quad \text{at } R \quad x = 6r$$

$$m_t \text{ at } H = -\frac{36}{36r^2} = -\frac{1}{r^2} \quad m_n \text{ at } H = r^2 \quad r^2 = -\frac{1}{pq} = \text{gradient } PQ$$

7.

$z = -3k - 2ki$ , where  $k$  is a real, positive constant.

(a) Find the modulus and the argument of  $z$ , giving the argument in radians to 2 decimal places and giving the modulus as an exact answer in terms of  $k$ . (3)

(b) Express in the form  $a + ib$ , where  $a$  and  $b$  are real and are given in terms of  $k$  where necessary,

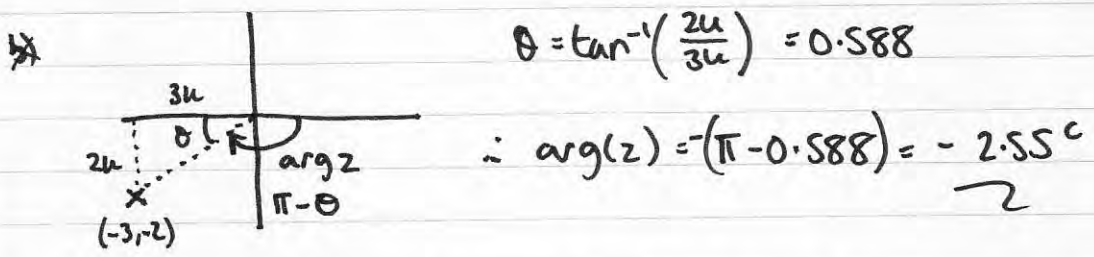
(i)  $\frac{4}{z + 3k}$

(ii)  $z^2$

(5)

(c) Given that  $k = 1$ , plot the points  $A, B, C$  and  $D$  representing  $z, z^*, \frac{4}{z + 3k}$  and  $z^2$  respectively on a single Argand diagram. (3)

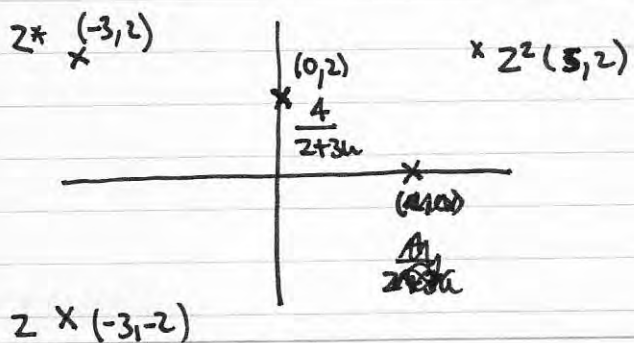
a)  $|z| = \sqrt{(3k)^2 + (2k)^2} = \sqrt{13k^2} = k\sqrt{13}$



b) i)  $\frac{4}{z + 3k} = \frac{4}{-2ki} \times \frac{2ki}{2ki} = \frac{8ki}{4k^2} = \frac{2}{k}i$

ii)  $z^2 = (-3k - 2ki)(-3k - 2ki) = 9k^2 - 4k^2 + 12ki = 5k^2 + 12ki$

c)  $k = 1 \Rightarrow z = -3 - 2i \quad z^* = -3 + 2i \quad \frac{4}{z + 3k} = 2i \quad z^2 = 5 + 12i$



$$8. \quad \mathbf{P} = \begin{pmatrix} 3a & -4a \\ 4a & 3a \end{pmatrix}, \text{ where } a \text{ is a constant and } a > 0$$

(a) Find the matrix  $\mathbf{P}^{-1}$  in terms of  $a$ .

(3)

The matrix  $\mathbf{P}$  represents the transformation  $U$  which transforms a triangle  $T_1$  onto the triangle  $T_2$ .

The triangle  $T_2$  has vertices at the points  $(-3a, -4a)$ ,  $(6a, 8a)$ , and  $(-20a, 15a)$ .

(b) Find the coordinates of the vertices of  $T_1$

(3)

(c) Hence, or otherwise, find the area of triangle  $T_2$  in terms of  $a$ .

(3)

The transformation  $V$ , represented by the  $2 \times 2$  matrix  $\mathbf{Q}$ , is a rotation through an angle  $\alpha$  **clockwise** about the origin, where  $\tan \alpha = \frac{4}{3}$  and  $0 < \alpha < \frac{\pi}{2}$

(d) Write down the matrix  $\mathbf{Q}$ , giving each element as an exact value.

(2)

The transformation  $U$  followed by the transformation  $V$  is the transformation  $W$ . The matrix  $\mathbf{R}$  represents the transformation  $W$ .

(e) Find the matrix  $\mathbf{R}$ .

(2)

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix} \quad P^{-1} = \frac{+1}{25a^2} \begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix}$$

$$\det P = +16a^2 + 9a^2 \\ = +25a^2$$

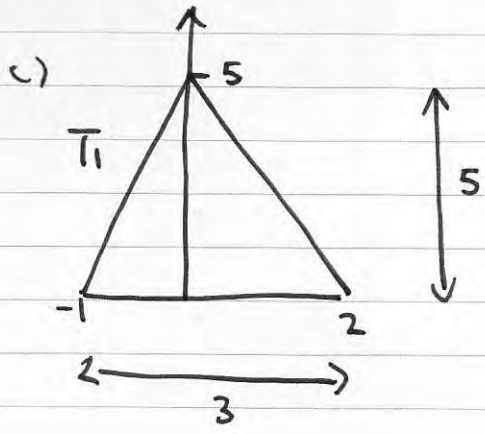
$$T_1 = \frac{+1}{25a^2} \begin{pmatrix} 3a & 4a \\ -4a & 3a \end{pmatrix} \begin{pmatrix} -3a & 6a & -20a \\ -4a & 8a & 15a \end{pmatrix}$$

$$= \frac{+1}{25a^2} \begin{pmatrix} -25a^2 & 50a^2 & 0 \\ 0 & 0 & 125a^2 \end{pmatrix} = \begin{pmatrix} -1 & +2 & 0 \\ 0 & 0 & +5 \end{pmatrix}$$

$$T_1 \quad (-1, 0) ; (+2, 0) ; (0, +5)$$

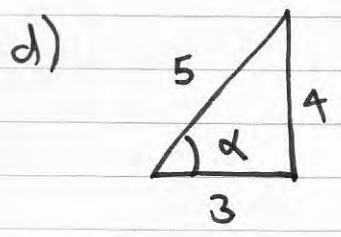


Question 8 continued

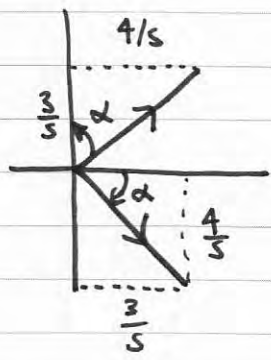
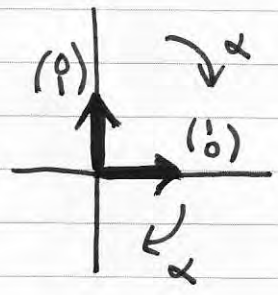
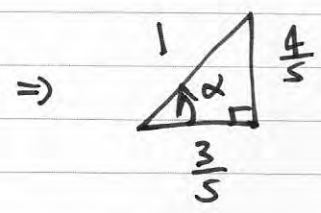


$$\text{Area } T_1 = \frac{15}{2}$$

$$\text{Area } T_2 = \frac{15}{2} \times 25a^2 = \frac{375}{2} a^2$$



$$\tan \alpha = \frac{4}{3}$$



$$Q = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$Q = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

e)

$$R = QP = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 3a & -4a \\ 4a & 3a \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 25a & 0 \\ 0 & 25a \end{pmatrix}$$

$$\therefore R = \begin{pmatrix} 5a & 0 \\ 0 & 5a \end{pmatrix}$$

9. (i) Prove by induction that, for  $n \in \mathbb{Z}^+$ ,

$$\sum_{r=1}^n r^2(2r-1) = \frac{1}{6}n(n+1)(3n^2+n-1)$$

(6)

(ii) Prove by induction that, for  $n \in \mathbb{Z}^+$ ,

$$\begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^n = \begin{pmatrix} 6n+1 & -12n \\ 3n & 1-6n \end{pmatrix}$$

(6)

$$n=1 \quad \text{LHS} = 1^2(2 \times 1 - 1) = 1 \quad \text{RHS} = \frac{1}{6}(1)(1+1)(3+1-1) \\ = \frac{1}{6} \times 2 \times 3 = 1$$

$\therefore \text{LHS} = \text{RHS} \quad \therefore \text{true for } n=1$

$$\text{assume true for } n=k \quad \therefore \sum_{r=1}^k r^2(2r-1) = \frac{1}{6}k(k+1)(3k^2+k-1)$$

$$n=k+1 \quad \sum_{r=1}^{k+1} r^2(2r-1) = \sum_{r=1}^k r^2(2r-1) + (k+1)^2(2k+2-1)$$

$$= \frac{1}{6}k(k+1)(3k^2+k-1) + \frac{1}{6}(k+1)^2(2k+1)$$

$$= \frac{1}{6}(k+1) [3k^3 + k^2 - k + 6(2k^2 + 3k + 1)]$$

$$= \frac{1}{6}(k+1) [3k^3 + 13k^2 + 17k + 6] = \text{LHS}$$

$$\text{RHS} = \frac{1}{6}(k+1)(k+1+1)(3(k+1)^2 + (k+1) - 1)$$

$$= \frac{1}{6}(k+1)(k+2)(3k^2 + 6k + 3 + k)$$

$$= \frac{1}{6}(k+1)(k+2)(3k^2 + 7k + 3)$$

$$= \frac{1}{6}(k+1)(3k^3 + 13k^2 + 17k + 6) \quad \neq$$

$\therefore \text{LHS} = \text{RHS}$

$\therefore \text{true for } n=k+1$

true for  $n=1$ , true for  $n=k+1$  if true for  $n=k$   
 $\therefore$  by mathematical induction true for all  $n \in \mathbb{Z}^+$

$$n=1 \quad \text{LHS} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^1 = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$$

$\therefore \text{LHS} = \text{RHS}$

$\therefore$  true  
for  
 $n=k$ .

$$\text{RHS} = \begin{pmatrix} 6(1)+1 & -12(1) \\ 3(1) & 1-6(1) \end{pmatrix} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$$

assume true for  $n=k$   $\therefore \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k = \begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix}$

$n=k+1$

$$\text{LHS} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^{k+1} = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}^k$$

$$= \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 6k+1 & -12k \\ 3k & 1-6k \end{pmatrix} = \begin{pmatrix} 42k+7 & -84k-12 \\ -36k & +72k \\ 18k+3 & -36k-5 \\ -15k & +30k \end{pmatrix}$$

$$\text{LHS} = \begin{pmatrix} 6k+7 & -12k-12 \\ 3k+3 & -6k-5 \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} 6(k+1)+1 & -12(k+1) \\ 3(k+1) & 1-6(k+1) \end{pmatrix} = \begin{pmatrix} 6k+7 & -12k-12 \\ 3k+3 & -6k-5 \end{pmatrix}$$

$\therefore \text{LHS} = \text{RHS} \quad \therefore$  true for  $n=k+1$

$\therefore$  true for  $n=1$ , true for  $n=k+1$  if true for  $n=k$

$\therefore$  by Mathematical Induction true for all  $n \in \mathbb{Z}^+$