

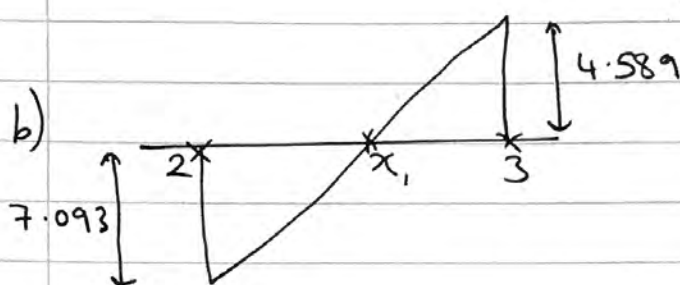
F1 Jan 2017 (MA)

Q1a) $f(2) = 2^2 - 10 \sin(2) - 2 = -7.09297 \dots$
 $f(3) = 2^3 - 10 \sin(3) - 2 = 4.5888 \dots$

change in sign over the interval $[2, 3]$

\therefore a root lies between $x=2$ and $x=3$.

remember to put your calc. in radians



$$\frac{3 - x_1}{x_1 - 2} = \frac{4.589}{7.093}$$

let $\frac{4.589}{7.093} = c,$

then $\frac{3 - x_1}{x_1 - 2} = c$

$$c x_1 - 2c = 3 - x_1$$

$$x_1 (c + 1) = 3 + 2c$$

$$x_1 = \frac{3 + 2c}{c + 1} = \boxed{2.607}$$

Q2a) $\boxed{a + \beta = \frac{1}{2}}$, $\boxed{a\beta = \frac{3}{2}}$ $\left[x^2 - \frac{x}{2} + \frac{3}{2} \right] = 0$

b) $\frac{1}{a} + \frac{1}{\beta} = \frac{\beta}{a\beta} + \frac{a}{\beta a} = \frac{a + \beta}{a\beta} = \frac{\frac{1}{2}}{\frac{3}{2}} = \boxed{\frac{1}{3}}$

c) $(x - (2a - \frac{1}{\beta})) (x - (2\beta - \frac{1}{a})) = 0 //$

$$\Rightarrow x^2 - x(2\beta - \frac{1}{\alpha}) - x(2\alpha - \frac{1}{\beta}) + (2\beta - \frac{1}{\alpha})(2\alpha - \frac{1}{\beta}) = 0$$

$$\Rightarrow x^2 - x(2\beta + 2\alpha - \frac{1}{\alpha} - \frac{1}{\beta}) + 4\alpha\beta - 2 - 2 + \frac{1}{\alpha\beta} = 0$$

$$\Rightarrow x^2 - x(2(\alpha + \beta) - (\frac{1}{\alpha} + \frac{1}{\beta})) + 4(\alpha\beta) - 4 + \frac{1}{(\alpha\beta)} = 0$$

$$\Rightarrow x^2 - x(2 \cdot \frac{1}{2} - \frac{1}{3}) + 4 \cdot \frac{3}{2} - 4 + \frac{1}{\frac{3}{2}} = 0$$

$$\Rightarrow x^2 - x(\frac{2}{3}) + \frac{8}{3} = 0$$

$$\xrightarrow{\times 3} \boxed{3x^2 - 2x + 8 = 0}$$

Q3.) $x = -1 - 3i$ will also be a root.

$$\therefore f(x) = (x - (-1 - 3i))(x - (-1 + 3i))(x^2 + ax + b) = 0$$

$$\Rightarrow [x^2 - x(-1 + 3i - 1 - 3i) + (-1 - 3i)(-1 + 3i)](x^2 + ax + b) = 0$$

$$\Rightarrow [x^2 + 2x + 10][x^2 + ax + b] = 0$$

$$\Rightarrow x^4 + ax^3 + bx^2 + 2x^3 + 2x^2a + 2bx + 10x^2 + 10ax + 10b = 0$$

$$\Rightarrow x^4 + x^3(a + 2) + x^2(b + 2a + 10) + x(2b + 10a) + 10b = 0$$

compare with given eqn : $10b = 160$

$$\therefore b = 16$$

$$a + 2 = 2$$

$$\therefore a = 0$$

$$\text{so } f(x) = (x^2 + 16)(x^2 + 2x + 10) = 0$$

$$\left. \begin{aligned} x^2 + 16 &= 0 \\ x^2 &= -16 \\ x &= \pm 4i \end{aligned} \right\} \text{all solutions : } \begin{aligned} x &= -1 + 3i \\ x &= -1 - 3i \\ x &= 4i \\ x &= -4i \end{aligned}$$

Q4a)

$$\sum_{r=1}^n r(2r+1)(3r+1) = \sum_{r=1}^n r(6r^2 + 5r + 1)$$

$$= \sum_{r=1}^n 6r^3 + 5r^2 + r = 6 \sum_{r=1}^n r^3 + 5 \sum_{r=1}^n r^2 + \sum_{r=1}^n r$$

$$= 6 \left[\frac{n^2(n+1)^2}{4} \right] + 5 \left[\frac{n}{6}(n+1)(2n+1) \right] + \frac{n}{2}(n+1)$$

$$= \frac{3}{2}n^2(n+1)^2 + \frac{5n}{6}(n+1)(2n+1) + \frac{n}{2}(n+1)$$

$$= \frac{n}{6}(n+1) \left[9n(n+1) + 5(2n+1) + 3 \right]$$

$$= \frac{n}{6}(n+1) \left[9n^2 + 9n + 10n + 5 + 3 \right]$$

$$= \frac{n}{6}(n+1)(9n^2 + 19n + 8)$$

$a = 9$ $b = 19$ $c = 8$

b)

$$\sum_{r=1}^{20} r(2r+1)(3r+1) - \sum_{r=1}^9 r(2r+1)(3r+1)$$

$$= \frac{20}{6}(21)(9(20)^2 + 19(20) + 8) - \frac{9}{6}(10)(9(9)^2 + 19(9) + 8)$$

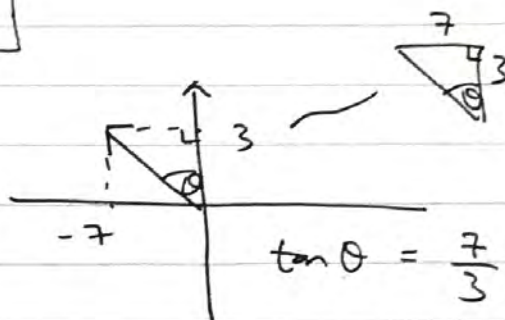
$$= 279160 - 13620 = \boxed{265540}$$

Q5a) $z = -7 + 3i$

$$|z| = \sqrt{7^2 + 3^2} = \boxed{\sqrt{58}}$$

b) $\arg z = \frac{\pi}{2} + \tan^{-1}\left(\frac{7}{3}\right)$

$$= \boxed{2.74^\circ}$$



c) $w = 3 - 6i - \frac{z}{1+i}$

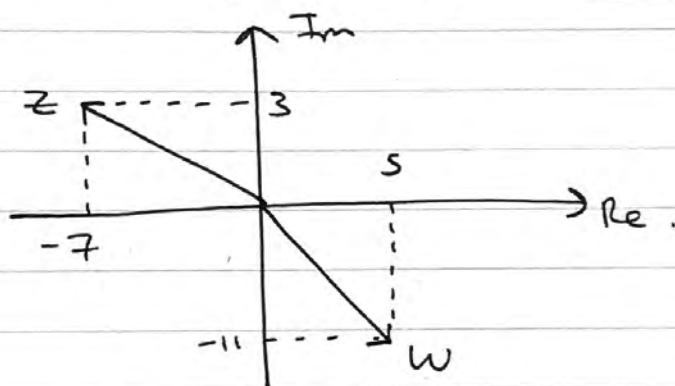
$$w = 3 - 6i - \frac{(-7 + 3i)}{1+i}$$

$$w = 3 - 6i - \left[\frac{(-7 + 3i)(1-i)}{(1+i)(1-i)} \right]$$

$$w = 3 - 6i - \left[\frac{-7 + 7i + 3i + 3}{2} \right] = 3 - 6i - \left[\frac{-4 + 10i}{2} \right]$$

$$w = 3 - 6i - [-2 + 5i] = \boxed{5 - 11i}$$

d)



Q6a) $f(x) = x^3 - \frac{1}{2}x^{-1} + x^{\frac{3}{2}}$

$$f'(x) = 3x^2 + \frac{1}{2}x^{-2} + \frac{3}{2}x^{\frac{1}{2}}$$

$$f(0.6) = -0.152575 \dots$$

$$f'(0.6) = 3.630784 \dots$$

$$\therefore d \approx 0.6 - \frac{(-0.152575)}{3.630784} \approx \boxed{0.642}$$

$$b) f(0.6415) = (0.6415)^3 - \frac{1}{2(0.6415)} + (0.6415)^{\frac{3}{2}} = \left(\frac{-1.63}{\times 10^{-3}} \right)$$

$$f(0.6425) = (0.6425)^3 - \frac{1}{2(0.6425)} + (0.6425)^{\frac{3}{2}} = \left(\frac{2.02}{\times 10^{-3}} \right)$$

change in sign between $x = 0.6415$ and $x = 0.6425$ \therefore answer to (a) is correct to 3dp.

$$(Q7ai) A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



Reflection in the y-axis

$$b) B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$$\therefore B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ii) a) M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} u \cos \theta & -u \sin \theta \\ u \sin \theta & u \cos \theta \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -3 & -4 \end{pmatrix}$$

$$u \cos \theta = -4 \sim \textcircled{1}$$

$$u \sin \theta = -3 \sim \textcircled{2}$$

$$\textcircled{1}^2 + \textcircled{2}^2 : u^2 \cos^2 \theta + u^2 \sin^2 \theta = 3^2 + 4^2$$

$$k^2 (\sin^2 \theta + \cos^2 \theta) = 25 //$$

$$\therefore k^2 = 25$$

$$\therefore k = \sqrt{25} = 5 // \quad (k > 0)$$

b) from (a),

$$5 \cos \theta = -4 \quad 5 \sin \theta = -3$$

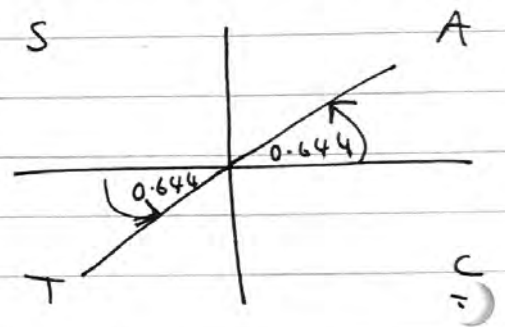
$$\textcircled{1} \quad \cos \theta = -\frac{4}{5} \quad \sin \theta = -\frac{3}{5} \quad \textcircled{2}$$

We want the value of θ where both of these are true.

$$\frac{\textcircled{2}}{\textcircled{1}} : \tan \theta = \frac{-\frac{3}{5}}{-\frac{4}{5}} = \frac{3}{4} //$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right) = 0.644^\circ, \pi + 0.644^\circ$$

$$\therefore \theta = 0.644^\circ, \pi + 0.644^\circ$$



Reject $\theta = 0.644$ because we want the value of θ that is in the 3rd quadrant

as this is the only quadrant where both $[\sin \theta = -\frac{3}{5}]$ and

$[\cos \theta = -\frac{4}{5}]$ can be true.

$$\therefore \theta = \pi + 0.644 = \boxed{3.79^\circ}$$

$$c) \det M = (-4)^2 - 3(-3) = 25$$

$$\therefore M^{-1} = \frac{1}{25} \begin{pmatrix} -4 & -3 \\ 3 & -4 \end{pmatrix}$$

Q8a)

$$y^2 = 4ax$$

$$2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{2at} = \frac{1}{t} //$$

$$\therefore \text{at normal, } m = -t // \quad (-t \times \frac{1}{t} = -1)$$

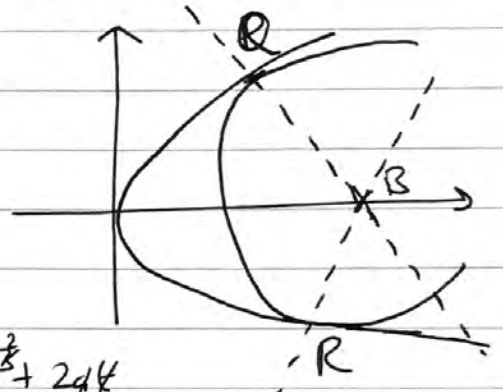
$$y = 2at = -t(x - at^2)$$

$$y = -xt + at^3 + 2at \quad \therefore \boxed{y + tx = at^3 + 2at}$$

$$b) OS = a \quad \therefore OB = 5a \quad \therefore B [5a, 0]$$

c) The normals at Q/R will pass through B.

$$\Rightarrow y + tx = at^3 + 2at$$

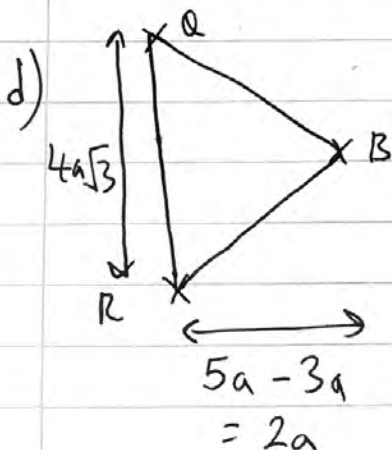


Substitute $x = 5a, y = 0$: $t(5a) = at^3 + 2at$

$$5 = t^2 + 2$$

$$\therefore -t^2 = 3 \quad \therefore t = \pm\sqrt{3} //$$

$$t = \sqrt{3} \rightarrow Q(a(3), 2a\sqrt{3}) \text{ and } R(a(3), -2a\sqrt{3})$$



$$\text{Area } \Delta BQR = \frac{1}{2} \times 4a\sqrt{3} \times 2a$$

$$= \boxed{4a^2\sqrt{3}}$$

we want to prove: $(k+1)^3(k+2) = \text{sum to } (k+1)$

(Q9i) $\sum_{r=1}^n (4r^3 - 3r^2 + r) = n^3(n+1)$

n=1: LHS = $4 - 3 + 1 = 2 = \text{RHS} = 1^3(2) = 2 //$
 \therefore relationship is true for $n=1$.

assume true for $n=k$

ie $\left[\sum_{r=1}^k 4r^3 - 3r^2 + r = k^3(k+1) \right]$

now consider $n=k+1$,

$$\sum_{r=1}^{k+1} 4r^3 + (-3r^2) + r = k^3(k+1) + 4(k+1)^3 - 3(k+1)^2 + k+1$$

$$= k^3(k+1) + 4(k+1)^3 - 3(k+1)^2 + (k+1)$$

$$= (k+1)^3 \left[\frac{k^3}{(k+1)^2} + 4 - \frac{3}{(k+1)} + \frac{1}{(k+1)^2} \right]$$

$$= (k+1)^3 \left[\frac{k^3 + 4(k+1)^2 - 3(k+1) + 1}{(k+1)^2} \right]$$

$$= (k+1)^3 \left[\frac{k^3 + 4k^2 + 8k + 4 - 3k - 3 + 1}{(k+1)^2} \right]$$

$$= (k+1)^3 \left[\frac{k^3 + 4k^2 + 5k + 2}{(k+1)^2} \right]$$

$$= (k+1)^3 \left[\frac{(k+1)^2(k+2)}{(k+1)^2} \right] = \boxed{(k+1)^3(k+2)}$$

\therefore true for $n=k+1$.

\therefore true for $n=1$

\therefore true for $n=k+1$ when $(n=k)$ is assumed true.

\therefore by Mathematical Induction true for all $n \in \mathbb{Z}^+$

$$\text{ii) } f(n) = 5^{2n} + 3n - 1$$

$$\underline{n=1}: f(1) = 5^2 + 3 - 1 = 27 = (9) \times 3 \\ \therefore \text{true for } n=1.$$

assume $[f(k) = 5^{2k} + 3k - 1]$ is div. by 9.

consider $f(k+1)$,

$$f(k+1) = 5^{2(k+1)} + 3(k+1) - 1$$

$$f(k+1) = 5^{2k+2} + 3k + 3 - 1$$

$$\left. \begin{array}{l} \text{factor} \\ \text{out} \\ f(k) \end{array} \right\} \begin{array}{l} f(k+1) = [25(5^{2k}) + 3k - 1] + 3 \\ f(k+1) = 25[5^{2k} + 3k - 1] - 24(3k) + 24(1) + 3 \\ f(k+1) = 25f(k) - 72k + 27 \\ f(k+1) = 25f(k) - 9 \times 8k + 9 \times 3 \end{array}$$

\therefore true for $n=k+1$.

\therefore true for $n=1$

\therefore true for $n=k+1$ when assumed true for $n=k$.

\therefore By Mathematical Induction true for all $n \in \mathbb{Z}^+$