IAL F1 Jan 16 (kprime 2)

Leave blank

1. $z = 3 + 2\text{i}, \quad w = 1 - \text{i}$

Find in the form $a + b\text{i}$, where a and b are real constants,

(a) zw

(2)

(b) $\frac{z}{w^*}$, showing clearly how you obtained your answer.

(3)

Given that

$$|z + k| = \sqrt{53}, \text{ where } k \text{ is a real constant}$$

(c) find the possible values of k.

(4)

(a) $z = 3 + 2i, \quad w = 1 - i$

$\therefore zw = (3+2i)(1-i) = 3 - 3i + 2i - 2i^2$

$\therefore$

$zw = 5 - i$

(b) $w^* = 1 + i$

$\therefore \frac{z}{w^*} = \frac{3+2i}{1+i} = \frac{(3+2i)(1-i)}{(1+i)(1-i)}$

$= \frac{5-i}{1-i^2} = \frac{5-i}{2} = \frac{5}{2} - \frac{1}{2}i$

$\therefore \frac{z}{w^*} = \frac{5}{2} - \frac{1}{2}i$

(c) $|z+k| = |3+k+2i| = \sqrt{53}$

$\Rightarrow (3+k)^2 + 2^2 = 53$

$\therefore (3+k)^2 = 49$

$\therefore (k+3) = \pm 7 \Rightarrow \begin{array}{l} k = 4 \\ k = -10 \end{array}$

P 4 6 9 4 8 A 0 2 3 2

2.

$$f(x) = x^2 - \frac{3}{\sqrt{x}} - \frac{4}{3x^2}, \qquad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [1.6, 1.7]

(2)

(b) Taking 1.6 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to find a second approximation to α. Give your answer to 3 decimal places.

(5)

(a) $f(1.6) = 1.6^2 - \frac{3}{\sqrt{1.6}} - \frac{4}{3(1.6)^2} = -0.3325...$

$f(1.7) = 1.7^2 - \frac{3}{\sqrt{1.7}} - \frac{4}{3(1.7)^2} = 0.1277...$

$\Rightarrow$ $f(1.6) < 0$ & $f(1.7) > 0$

$\therefore$ There is a sign change in the interval $[1.6, 1.7]$.

$\therefore$ $\alpha \in [1.6, 1.7]$

(b) $\alpha_0 \simeq 1.6$

$\alpha_1 \approx \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$

$f(x) = x^2 - 3x^{-1/2} - \frac{4}{3}x^{-2}$

$\therefore f'(x) = 2x + \frac{3}{2}x^{-3/2} + \frac{8}{3}x^{-3}$

$f(1.6) = -0.3325...$

$f'(1.6) = 4.5922...$

$\therefore$ $\alpha_1 \simeq 1.6 - \frac{-0.3325...}{4.5922...} = 1.672$ (3dp)

$\therefore$ $\alpha_1 \simeq 1.672$ (3dp)

P 4 6 9 4 8 A 0 4 3 2

Leave blank

3. The quadratic equation

$$x^2 - 2x + 3 = 0$$

has roots α and β.

Without solving the equation,

(a) (i) write down the value of $(\alpha + \beta)$ and the value of $\alpha\beta$

(ii) show that $\alpha^2 + \beta^2 = -2$

(iii) find the value of $\alpha^3 + \beta^3$

(5)

(b) (i) show that $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

(ii) find a quadratic equation which has roots

$$(\alpha^3 - \beta) \text{ and } (\beta^3 - \alpha)$$

giving your answer in the form $px^2 + qx + r = 0$ where p, q and r are integers.

(6)

3. $x^2 - 2x + 3$

(a) (i) $x^2 - 2x + 3 \equiv (x - \alpha)(x - \beta)$

$\therefore x^2 - 2x + 3 \equiv x^2 - \beta x - \alpha x + \alpha\beta$

$\therefore x^2 - 2x + 3 \equiv x^2 - (\alpha + \beta)x + \alpha\beta$

Compare coefficients:

$\Rightarrow \alpha + \beta = 2$

$\alpha\beta = 3$

(ii) LHS $= \alpha^2 + \beta^2 \equiv (\alpha + \beta)^2 - 2\alpha\beta$

$= 2^2 - 2(3)$

$= 4 - 6 = -2 = $ RHS

Q.E.D

P 4 6 9 4 8 A 0 6 3 2

(iii) $(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$ (via Binomial expansion)

$(\alpha+\beta)^3 \equiv \alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$\therefore \alpha^3+\beta^3 = 2^3 - 3(3)(2)$

$\alpha^3+\beta^3 = -10$

(b) (i)

RHS $= (\alpha^2+\beta^2)^2 - 2(\alpha\beta)^2 = \alpha^4+2\alpha^2\beta^2+\beta^4 - 2\alpha^2\beta^2$

$= \alpha^4+\beta^4 =$ LHS $\square$ Q.E.D

(ii) $[x-(\alpha^3-\beta)][x-(\beta^3-\alpha)] = x^2 - x(\beta^3-\alpha) - x(\alpha^3-\beta) + (\alpha^3-\beta)(\beta^3-\alpha)$

$= x^2 - x(\beta^3-\alpha+\alpha^3-\beta) + (\alpha^3\beta^3 - \alpha^4 - \beta^4 + \alpha\beta)$

$= x^2 - x[\alpha^3+\beta^3-(\alpha+\beta)] + [(\alpha\beta)^3 - (\alpha^4+\beta^4) + \alpha\beta]$

$= x^2 + 12x + 44$

$p=1$
$q=12$
$r=44$

Leave blank

4.

$$\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation represented by the matrix **A**.

(3)

(b) Hence find the smallest positive integer value of n for which

$$\mathbf{A}^n = \mathbf{I}$$

where **I** is the 2×2 identity matrix.

(1)

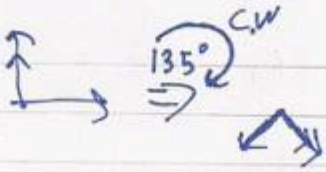
The transformation represented by the matrix **A** followed by the transformation represented by the matrix **B** is equivalent to the transformation represented by the matrix **C**.

Given that $\mathbf{C} = \begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix}$,

(c) find the matrix **B**.

(4)

(a).



A represents a 135° clockwise rotation about the origin.

(b) $A^n = I$, $n > 0$, $n \in \mathbb{Z}^+$

$n \neq 0$ $\quad n = 8$

(c) $C = BA \Rightarrow CA^{-1} = BAA^{-1} \Rightarrow B = CA^{-1}$

$\therefore$ ~~B~~ $\quad A^{-1} = \frac{1}{1}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\therefore B = \begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -3\sqrt{2} \\ -\sqrt{2} & 4\sqrt{2} \end{pmatrix}$

Leave blank

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$ to show that, for all positive integers n,

$$\sum_{r=1}^{n}(8r^3 - 3r) = \frac{1}{2}n(n+1)(2n+3)(an+b)$$

where a and b are integers to be found.

(4)

Given that

$$\sum_{r=5}^{10}(8r^3 - 3r + kr^2) = 22768$$

(b) find the exact value of the constant k.

(4)

5(a)

$$\text{LHS} = \sum_{r=1}^{n}(8r^3-3r) = 8\sum_{r=1}^{n} r^3 - 3\sum_{r=1}^{n} r$$

$$= \frac{8}{4}n^2(n+1)^2 - \frac{3n}{2}(n+1)$$

$$= 2n^2(n+1)^2 - \frac{3}{2}n(n+1)$$

$$= \frac{1}{2}n(n+1)\left[4n(n+1) - 3\right]$$

$$= \frac{1}{2}n(n+1)(4n^2+4n-3)$$

$$= \frac{1}{2}n(n+1)(2n+3)(2n-1) \quad \therefore \; a=2, \; b=-1$$

(b) $$\sum_{r=5}^{10} = \sum_{r=1}^{10} - \sum_{r=1}^{4}$$

$$= \sum_{1}^{10} 8r^3 - 3r + k\sum_{1}^{10} r^2 - \sum_{1}^{4} 8r^3 - 3r - k\sum_{1}^{4} r^2$$

$$= 24035 + k(385) - 770 - k(30)$$

$$= 23265 + 355k = 22768$$

$$\therefore 355k = -497$$

$$\Rightarrow k = -\frac{7}{5}$$

Leave blank

6. The rectangular hyperbola H has equation $xy = c^2$, where c is a non-zero constant.

The point $P\left(cp, \frac{c}{p}\right)$, where $p \neq 0$, lies on H.

(a) Show that the normal to H at P has equation

$$yp - p^3x = c(1 - p^4)$$

(5)

The normal to H at P meets H again at the point Q.

(b) Find, in terms of c and p, the coordinates of Q.

(4)

6(a) $xy = c^2 \Rightarrow y = c^2x^{-1}$

$\therefore \frac{\partial y}{\partial x} = -\frac{c^2}{x^2}$

$\therefore$ at P, $\left(\frac{\partial y}{\partial x}\right)_{x=cp} = \frac{-c^2}{c^2p^2} = -\frac{1}{p^2}$

$\therefore$ gradient of Normal at $P = -\frac{1}{-1/p^2} = p^2$

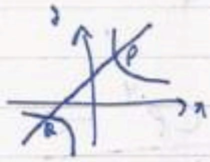
$\therefore$ $y - y_1 = m(x - x_1)$

$\therefore$ $y - \frac{c}{p} = p^2(x - cp)$

$(\times p) \Rightarrow yp - c = p^3x - cp^4$

$\Rightarrow yp - p^3x = c - cp^4$

$\Rightarrow yp - p^3x = c(1 - p^4)$ Q.E.D

(b)  $y = \frac{c^2}{x} \Rightarrow \frac{c^2}{x}p - p^3x = c(1 - p^4)$

$\times x$ $\therefore c^2p - p^3x^2 = cx(1 - p^4)$

P 4 6 9 4 8 A 0 1 8 3 2

$$\therefore c^2 p - p^3 x^2 = cx - cp^4 x$$

$$\therefore p^3 x^2 + \underbrace{(c - cp^4)}_{c(1-p^4)} x - c^2 p = 0$$

$$\therefore x = \frac{cp^4 - c \pm \sqrt{c^2(1-p^4)^2 + 4p^4 c^2}}{2p^3}$$

$$\therefore x = \frac{cp^4 - c \pm \sqrt{c^2 - 2c^2p^4 + c^2p^8 + 4p^4c^2}}{2p^3}$$

$$\therefore x = \frac{cp^4 - c \pm \sqrt{c^2p^8 + 2c^2p^4 + c^2}}{2p^3}$$

$$\therefore x = \frac{cp^4 - c \pm \sqrt{c^2(p^4+1)^2}}{2p^3}$$

$$\therefore x = \frac{cp^4 - c \pm c(p^4+1)}{2p^3}$$

$$x_p = \frac{cp^4 - c + cp^4 + c}{2p^3} = \frac{2cp^4}{2p^3} = cp$$

$$x_Q = \frac{cp^4 - c - cp^4 - c}{2p^3} = \frac{-2c}{2p^3}$$

$$\therefore x_Q = -\frac{c}{p^3}$$

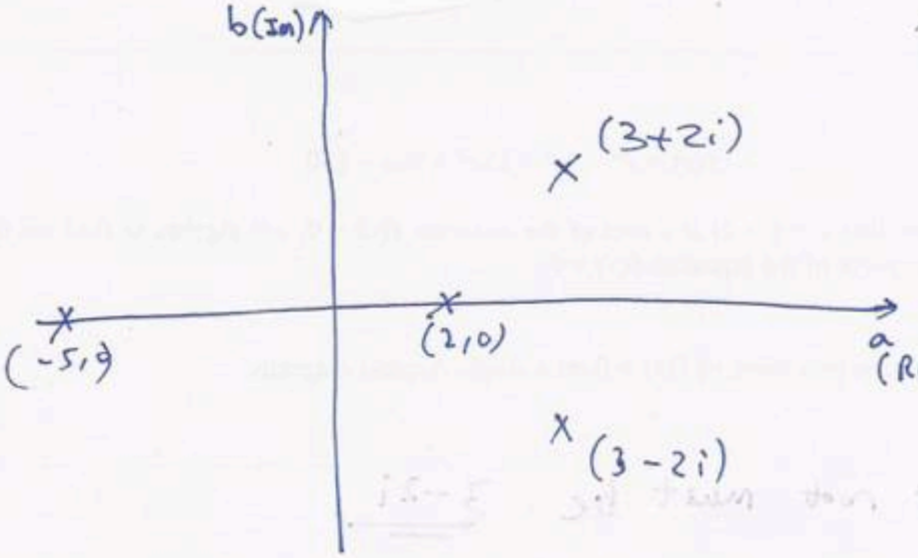
$$\therefore y = \frac{c^2}{-\frac{c}{p^3}} = -\frac{c^2p^3}{c} = -cp^3$$

$$\therefore Q: \left(-\frac{c}{p^3}, cp^3\right)$$

(b)

$x \equiv a + bi$

$a, b \in \mathbb{R}$



Leave blank

7.

$$f(x) = x^4 - 3x^3 - 15x^2 + 99x - 130$$

(a) Given that $x = 3 + 2i$ is a root of the equation $f(x) = 0$, use algebra to find the three other roots of the equation $f(x) = 0$

(7)

(b) Show the four roots of $f(x) = 0$ on a single Argand diagram.

(2)

(a) Another root must be $3-2i$

$\Rightarrow (x-(3+2i))(x-(3-2i))$ must be a factor

$(x-(3+2i))(x-(3-2i)) =$

$x^2 - x(3-2i) - x(3+2i) + (3+2i)(3-2i)$

$= x^2 - x(6) + 9 + 4 = x^2 - 6x + 13$

$\therefore (x^2-6x+13)$ is a factor

$\therefore x^4 - 3x^3 - 15x^2 + 99x - 130 \equiv (x^2-6x+13)(x^2+Bx-10)$

Compare coefficients:

$x^3: -3 = B - 6 \Rightarrow B = 3$

$\therefore (x^2-6x+13)(x^2+3x-10) = x^4 - 3x^3 - 15x^2 + 99x - 130$

$f(x) = 0$

$\Rightarrow x^2 + 3x - 10 = 0$

$\therefore \left(x + \frac{3}{2}\right)^2 - \frac{49}{4} = 0$

$\therefore x + \frac{3}{2} = \pm 7/2 \quad \therefore x = 2$

$x = -5$

$\therefore x = 3-2i \quad x = 2$

$x = -5$

Leave blank

8. The parabola P has equation $y^2 = 4ax$, where a is a positive constant. The point S is the focus of P.

The point B, which does not lie on the parabola, has coordinates (q, r) where q and r are positive constants and $q > a$. The line l passes through B and S.

(a) Show that an equation of the line l is

$$(q-a)\,y = r(x-a)$$

(3)

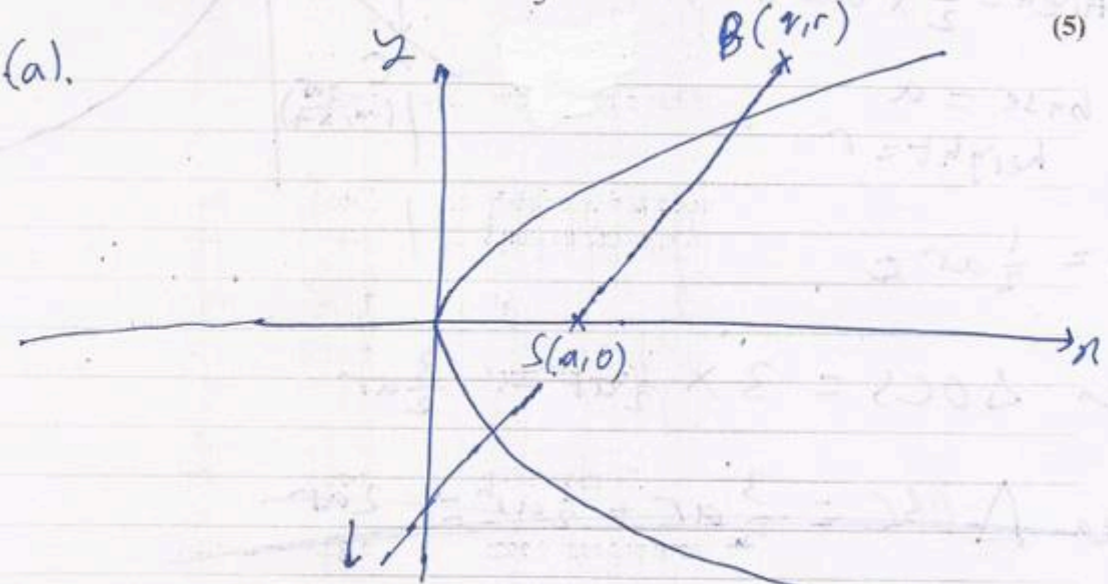
The line l intersects the directrix of P at the point C.

Given that the area of triangle OCS is three times the area of triangle OBS, where O is the origin,

(b) show that the area of triangle OBC is $\frac{6}{5}qr$

(5)

8 (a).



gradient of $l = \frac{r-0}{q-a} = \frac{r}{q-a}$

$\therefore\ y - y_1 = m(x - x_1)$

Consider S:

$\Rightarrow\ y - 0 = \frac{r}{q-a}(x-a)$

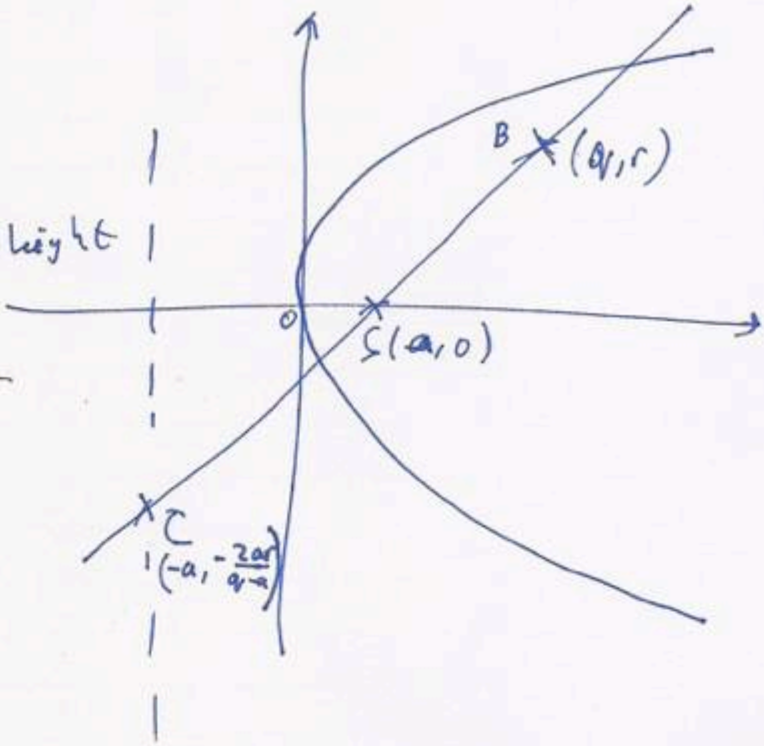
$\therefore\ y = \frac{r}{q-a}(x-a)$

$\Rightarrow\ (q-a)y = x - a$ // Q.E.D

(b) At directrix, $x = -a$

$\therefore (q-a)y = -2ar$

$\therefore y = -\dfrac{2ar}{q-a} \Rightarrow C: \left(-a, -\dfrac{2ar}{q-a}\right)$

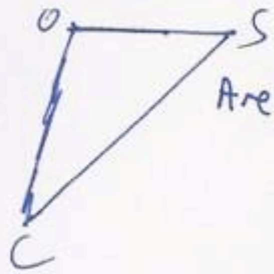


Consider ΔOBS:

Area $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times a \times r = \frac{1}{2}ar$

Consider ΔOCS:



Area $= \frac{1}{2} \times a \times \dfrac{2ar}{q-a}$

$= \dfrac{a^2 r}{q-a}$

Area $\Delta OCS = 3 \times$ Area ΔOBS

$\Rightarrow \dfrac{a^2 r}{q-a} = \dfrac{3}{2}ar$

$\div ar \Rightarrow \dfrac{a}{q-a} = \dfrac{3}{2}$

$\therefore 2a = 3q - 3a \Rightarrow a = \dfrac{3q}{5}$

Area $\Delta OBC =$ Area ΔOBS + Area $\Delta OCS = \frac{1}{2}ar + \frac{3}{2}ar$

$= 2ar = 2 \times \dfrac{3q}{5} \times r = \dfrac{6q}{5} r$ Q.E.D

Leave blank

9. Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 4^{n+1} + 5^{2n-1}$$

is divisible by 21

(6)

When $n=1$, $f(1) = 4^2 + 5^1 = 16 + 5 = 21$

$\therefore$ $f(n)$ is divisible by 21 for $n = 1$

Let us assume for $n=k$,

$f(k) = 4^{k+1} + 5^{2k-1}$ is divisible by 21.

When $n=k+1$, $f(k+1) = 4^{k+2} + 5^{2k+1}$

$$= 4(4^{k+1}) + 5^2(5^{2k-1})$$

$$= 4(4^{k+1}) + 25(5^{2k-1})$$

$$= 4(4^{k+1}) + 4(5^{2k-1}) + 21(5^{2k-1})$$

$$= 4(4^{k+1} + 5^{2k-1}) + 21(5^{2k-1})$$

$\Rightarrow$ $f(k+1) = 4f(k) + 21(5^{2k-1})$

$4f(k)$ is divisible by 21
$21(5^{2k-1})$ is divisible by 21 $\therefore$ $f(k+1)$ is divisible by 21

$\therefore$ If the result is divisible by $n=k$, it is shown to be divisible by $n=k+1$. Since it is divisible by $n=1$, by induction $f(n)$ is divisible by 21 for all $n \in \mathbb{Z}^+$

P 4 6 9 4 8 A 0 3 0 3 2