F3 IAL Model Answers Kprime2

June 2016

1. The curve C has equation

$$y = 9\cosh x + 3\sinh x + 7x$$

Use differentiation to find the exact x coordinate of the stationary point of C, giving your answer as a natural logarithm. **(6)**

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1. $\frac{\partial y}{\partial x} = 9\sinh x + 3\cosh x + 7$

$\frac{\partial y}{\partial x} = 0 \Rightarrow 9\sinh x + 3\cosh x + 7 = 0$

$\therefore \frac{9e^{x} - 9e^{-x} + 3e^{x} + 3e^{-x}}{2} = -7$

$\therefore 12e^{x} - 6e^{-x} = -14$

$\times e^{x} \Rightarrow 12e^{2x} + 14e^{x} - 6 = 0$

$6e^{2x} + 7e^{x} - 3 = 0$

$(3e^{x} - 1)(2e^{x} + 3) = 0$

$e^{x} \neq -\frac{3}{2}$

$\Rightarrow e^{x} = \frac{1}{3}$

$\therefore x = \ln\left(\frac{1}{3}\right)$

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2. An ellipse has equation

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

The point P lies on the ellipse and has coordinates $(5\cos\theta,\ 2\sin\theta)$, $0 < \theta < \frac{\pi}{2}$

The line L is a normal to the ellipse at the point P.

(a) Show that an equation for L is

$$5x\sin\theta - 2y\cos\theta = 21\sin\theta\cos\theta$$

(5)

Given that the line L crosses the y-axis at the point Q and that M is the midpoint of PQ,

(b) find the exact area of triangle OPM, where O is the origin, giving your answer as a multiple of $\sin 2\theta$

(6)

2(a) @ P_1 $\frac{\partial y}{\partial x} = \frac{\partial y/\partial\theta}{\partial x/\partial\theta} = \frac{2\cos\theta}{-5\sin\theta}$

$\therefore$ gradient of normal $= \frac{5\sin\theta}{2\cos\theta}$

$y - y_1 = m(x - x_1)$

$\Rightarrow y - 2\sin\theta = \frac{5\sin\theta}{2\cos\theta}(x - 5\cos\theta)$

$\times 2\cos\theta \Rightarrow$ $2y\cos\theta - 4\sin\theta\cos\theta = 5x\sin\theta - 25\sin\theta\cos\theta$

$\therefore$ $5x\sin\theta - 2y\cos\theta = 25\sin\theta\cos\theta - 4\sin\theta\cos\theta$

$\therefore$ $5x\sin\theta - 2y\cos\theta = 21\sin\theta\cos\theta$

as required.

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Question 2 continued

(b) @ Q

$x=0 \Rightarrow -2y\cos\theta = 21\sin\theta\cos\theta$

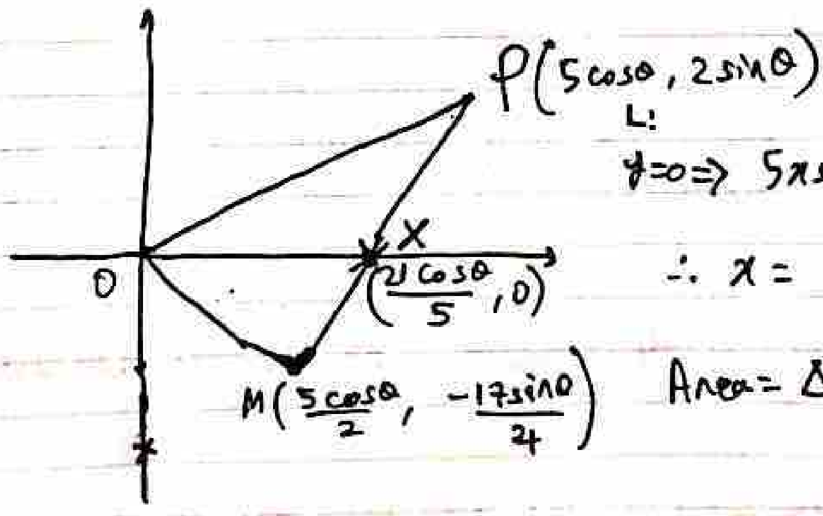
$\therefore\ y_Q = -\dfrac{21\sin\theta}{2}$

$Q:\left(0, \dfrac{-21}{2}\sin\theta\right) \qquad P:\left(5\cos\theta, 2\sin\theta\right)$

$\therefore$ M has coordinates $x_M = \dfrac{5\cos\theta}{2}$

$y_M = \dfrac{2\sin\theta - \frac{21}{2}\sin\theta}{2} = \dfrac{-17\sin\theta}{4}$

$\Rightarrow$



L:

$y=0 \Rightarrow 5x\sin\theta = 21\sin\theta\cos\theta$

$\therefore\ x = \dfrac{21\cos\theta}{5}$

Area = $\Delta OPX + \Delta OMX$

$\text{Area} = \left|\frac{1}{2}\left(\frac{21\cos\theta}{5}\right)(2\sin\theta)\right| + \left|\frac{1}{2}\left(\frac{21\cos\theta}{5}\right)\left(-\frac{17\sin\theta}{4}\right)\right|$

$= \frac{21}{5}\sin\theta\cos\theta + \frac{357}{40}\sin\theta\cos\theta = \frac{105}{8}\sin\theta\cos\theta$

$\text{Area} = \frac{105}{16}\sin 2\theta$

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3. Without using a calculator, find

(a) $\displaystyle\int_{-2}^{1} \frac{1}{x^2 + 4x + 13}\,dx$, giving your answer as a multiple of π, **(5)**

(b) $\displaystyle\int_{-1}^{4} \frac{1}{\sqrt{4x^2 - 12x + 34}}\,dx$, giving your answer in the form $p\ln\left(q + r\sqrt{2}\right)$,

where p, q and r are rational numbers to be found. **(7)**

3.(a).

$$\int_{-2}^{1} \frac{1}{x^2+4x+13}\,dx = \int_{-2}^{1} \frac{1}{(x+2)^2+9}\,dx$$

$$= \left[\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\right]_{-2}^{1}$$

$$= \frac{1}{3}\arctan(1) - \frac{1}{3}\arctan(0)$$

$$= \frac{1}{3}\arctan(1) = \frac{\pi}{12}$$

(b) $4x^2 - 12x + 34 = 4\left(x^2 - 3x + \frac{17}{2}\right)$

$$= 4\left[\left(x - \frac{3}{2}\right)^2 + \frac{25}{4}\right]$$

Question 3 continued

$$\therefore \int_{-1}^{4} \frac{1}{\sqrt{4x^2-12x+34}}\,\partial x = \int_{-1}^{4} \frac{1}{\sqrt{4\left[(x-3/2)^2+\frac{25}{4}\right]}}\,\partial x$$

$$= \frac{1}{2}\int_{-1}^{4} \frac{1}{\sqrt{(x-\frac{3}{2})^2+\frac{25}{4}}}\,\partial x$$

$$= \frac{1}{2}\left[\operatorname{arsinh} \frac{x-\frac{3}{2}}{5/2}\right]_{-1}^{4} = \frac{1}{2}\left[\operatorname{arsinh} \frac{2x-3}{5}\right]_{-1}^{4}$$

$$= \frac{1}{2}\operatorname{arsinh}(1) - \frac{1}{2}\operatorname{arsinh}(-1)$$

$$= \frac{1}{2}\ln(1+\sqrt{2}) - \frac{1}{2}\ln(-1+\sqrt{2})$$

$$= \frac{1}{2}\ln(3+2\sqrt{2})$$

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4.

$$\mathbf{M} = \begin{pmatrix} 1 & k & 0 \\ -1 & 1 & 1 \\ 1 & k & 3 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

(a) Find $\mathbf{M}^{-1}$ in terms of k. **(5)**

Hence, given that $k = 0$

(b) find the matrix $\mathbf{N}$ such that

$$\mathbf{MN} = \begin{pmatrix} 3 & 5 & 6 \\ 4 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$$

(4)

4(a). $\det(M) = \begin{vmatrix} 1 & 1 \\ k & 3 \end{vmatrix} - k \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3-k - k(-4) = \underline{3+3k}$

$$\begin{pmatrix} 3-k & -4 & -k-1 \\ 3k & 3 & 0 \\ k & 1 & k+1 \end{pmatrix}$$

apply $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}^T \Rightarrow \begin{pmatrix} 3-k & 4 & -k-1 \\ -3k & 3 & 0 \\ k & -1 & k+1 \end{pmatrix}^T$

$$= \begin{pmatrix} 3-k & -3k & k \\ 4 & 3 & -1 \\ -k-1 & 0 & k+1 \end{pmatrix}$$

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$$\therefore M^{-1} = \frac{1}{3k+3}\begin{pmatrix} 3-k & -3k & k \\ 4 & 3 & -1 \\ -k-1 & 0 & k+1 \end{pmatrix}$$

(b) $k=0 \Rightarrow M^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 4 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$M^{-1}MN = N$

$$\Rightarrow N = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 4 & 3 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 & 6 \\ 4 & -1 & 1 \\ 3 & 2 & -3 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 9 & 15 & 18 \\ 21 & 15 & 30 \\ 0 & -3 & -9 \end{pmatrix}$$

$$N = \begin{pmatrix} 3 & 5 & 6 \\ 7 & 5 & 10 \\ 0 & -1 & -3 \end{pmatrix}$$

Q3

(Total 12 marks)

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5. Given that $y = \text{artanh}(\cos x)$

(a) show that

$$\frac{dy}{dx} = -\text{cosec}\, x$$

(2)

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{6}} \cos x \,\text{artanh}(\cos x)\, dx$$

giving your answer in the form $a\ln\left(b + c\sqrt{3}\right) + d\pi$, where a, b, c and d are rational numbers to be found.

(5)

5.(a). $y = \text{artanh}(\cos x)$

$\therefore \tanh y = \cos x$

$\therefore \frac{\partial y}{\partial x} \text{sech}^2 y = -\sin x$

$\text{sech}^2 y = 1 - \tanh^2 y$

$\therefore \frac{\partial y}{\partial x} = \frac{-\sin x}{1 - \tanh^2 y}$

$\therefore \frac{\partial y}{\partial x} = \frac{-\sin x}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x}$

$= \frac{-1}{\sin x} = -\text{cosec}\, x$

as required.

(b) Let $u = \text{artanh}(\cos x)$ $\quad u' = -\operatorname{cosec} x$

Let $v' = \cos x$ $\quad v = \sin x$

$$\therefore \int_0^{\pi/6} \cos x \, \text{artanh}(\cos x) \, dx$$

$$= \left[\sin x \, \text{artanh}(\cos x) \right]_0^{\pi/6} + \int_0^{\pi/6} 1 \, dx$$

$$= \frac{1}{2} \text{artanh}\left(\frac{\sqrt{3}}{2}\right) + [x]_0^{\pi/6}$$

$$= \frac{1}{4} \ln\left(7 + 4\sqrt{3}\right) + \frac{\pi}{6}$$

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6. The coordinates of the points A, B and C relative to a fixed origin O are $(1, 2, 3)$, $(-1, 3, 4)$ and $(2, 1, 6)$ respectively. The plane Π contains the points A, B and C.

(a) Find a cartesian equation of the plane Π.

(5)

The point D has coordinates $(k, 4, 14)$ where k is a positive constant.

Given that the volume of the tetrahedron $ABCD$ is 6 cubic units,

(b) find the value of k.

(4)

6.

(a) $\overrightarrow{AB} = \begin{pmatrix}-1\\3\\4\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\1\\1\end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix}2\\1\\6\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$

$\therefore \begin{pmatrix}-2\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{matrix}-2 & 1 & 1 & -2 & 1 & 1 \\ 1 & -1 & 3 & 1 & -1 & 3\end{matrix}$

$\underset{\sim}{n} = \begin{pmatrix}4\\7\\1\end{pmatrix}$

$\therefore \underset{\sim}{r} \cdot \underset{\sim}{n} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\7\\1\end{pmatrix} = 21$

$\therefore \quad 4x + 7y + z = 21$

(b) $\frac{1}{6}\left|\left(\vec{AD} \cdot \left(\vec{AB} \times \vec{AC}\right)\right)\right| = 6$

$\therefore \left|\vec{AD} \cdot \left(\vec{AB} \times \vec{AC}\right)\right| = 36$

$\vec{AD} = \begin{pmatrix} k \\ 4 \\ 14 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} k-1 \\ 2 \\ 11 \end{pmatrix}$

$\begin{pmatrix} k-1 \\ 2 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix} = 36$

$\Rightarrow \quad 4k - 4 + 14 + 11 = 36$

$\therefore \quad 4k = 15$

$\Rightarrow \quad k = \frac{15}{4}$

7. The curve C has parametric equations

$$x = 3t^4, \quad y = 4t^3, \qquad 0 \leqslant t \leqslant 1$$

The curve C is rotated through 2π radians about the x-axis. The area of the curved surface generated is S.

(a) Show that

$$S = k\pi\int_0^1 t^5(t^2+1)^{\frac{1}{2}}\,dt$$

where k is a constant to be found.

(4)

(b) Use the substitution $u^2 = t^2 + 1$ to find the value of S, giving your answer in the form $p\pi\left(11\sqrt{2} - 4\right)$ where p is a rational number to be found.

(7)

7(a).
$$S = 2\pi\int_0^1 y\sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}\,\partial t$$

$$= 2\pi\int_0^1 4t^3\sqrt{(12t^3)^2 + (12t^2)^2}\,\partial t$$

$$= 2\pi\int_0^1 4t^3\sqrt{144t^6 + 144t^4}\,\partial t$$

$$= 2\pi\int_0^1 4t^3\sqrt{144t^4(t^2+1)}\,\partial t$$

$$= 2\pi\int_0^1 48t^5(t^2+1)^{1/2}\,\partial t$$

$$= 96\pi\int_0^1 t^5(t^2+1)^{1/2}\,\partial t$$

$k = 96$

as required.

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Question 7 continued

(b) $u^2 = t^2 + 1$

$\therefore 2u \frac{\partial u}{\partial t} = 2t \Rightarrow \frac{\partial u}{\partial t} = \frac{t}{u}$

$\therefore dt = \frac{u}{t} \partial u = \frac{u}{\sqrt{u^2-1}} \partial u$

$t = 1 \Rightarrow u^2 = 2 \Rightarrow u = \sqrt{2}$

$t = 0 \Rightarrow u^2 = 1 \Rightarrow u = 1$

$\Big\} \int_0^1$ becomes $\int_1^{\sqrt{2}}$

$t^5 = (u^2-1)^{5/2}$

$\therefore S = 96\pi \int_0^1 t^5 (t^2+1)^{1/2} \partial t$

$= 96\pi \int_1^{\sqrt{2}} (u^2-1)^{5/2} u \cdot \frac{u}{\sqrt{u^2-1}} \partial u$

$= 96\pi \int_1^{\sqrt{2}} u^2 (u^2-1)^2 \partial u$

$= 96\pi \int_1^{\sqrt{2}} u^2 (u^4 - 2u^2 + 1) \partial u$

$= 96\pi \int_1^{\sqrt{2}} u^6 - 2u^4 + u^2 \partial u$

Question 7 continued

$$= 96\pi\left[\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= 96\pi\left[u^3\left(\frac{1}{7}u^4 - \frac{2}{5}u^2 + \frac{1}{3}\right)\right]_1^{\sqrt{2}}$$

$$= 96\pi\left[2\sqrt{2}\left(\frac{4}{7} - \frac{4}{5} + \frac{1}{3}\right) - \frac{8}{105}\right]$$

$$= 96\pi\left(\frac{22}{105}\sqrt{2} - 8/105\right)$$

$$= 96\pi \times 2 \times \left(\frac{11\sqrt{2} - 4}{105}\right)$$

$$= \frac{64}{35}\pi\left(11\sqrt{2} - 4\right) \qquad p = \frac{64}{35}$$

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8. $$I_n = \int_0^{\ln 2} \tanh^{2n} x \, dx, \quad n \geqslant 0$$

(a) Show that, for $n \geqslant 1$

$$I_n = I_{n-1} - \frac{1}{2n-1}\left(\frac{3}{5}\right)^{2n-1}$$

(5)

(b) Hence show that

$$\int_0^{\ln 2} \tanh^4 x \, dx = p + \ln 2$$

where p is a rational number to be found.

(5)

8(a).

$$I_n = \int_0^{\ln 2} \tanh^{2n} x \, \partial x = \int_0^{\ln 2} \tanh^{2n-2} x \, \tanh^2 x \, \partial x$$

$$= \int_0^{\ln 2} \tanh^{2n-2} x \, (1 - \operatorname{sech}^2 x) \, \partial x$$

$$= \int_0^{\ln 2} \tanh^{2n-2} x - \operatorname{sech}^2 x \tanh^{2n-2} x \, \partial x$$

$$= \int_0^{\ln 2} \tanh^{2n-2} x \, \partial x - \int_0^{\ln 2} \operatorname{sech}^2 x \, (\tanh x)^{2n-2} \, \partial x$$

$$= \int_0^{\ln 2} \tanh x^{2(n-1)} \, \partial x - \left[\frac{(\tanh x)^{2n-1}}{2n-1}\right]_0^{\ln 2}$$

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Question 8 continued

$$= I_{n-1} - \left[\frac{(\tanh(\ln 2))^{2n-1}}{2n-1} - \frac{\tanh(0)^{2n-1}}{2n-1} \right]$$

$$= I_{n-1} - \left[\frac{\left(\frac{3}{5}\right)^{2n-1}}{2n-1} - 0 \right]$$

$\Rightarrow$

$$I_n = I_{n-1} - \frac{\left(\frac{3}{5}\right)^{2n-1}}{2n-1}$$ as required.

(b) $I_2 = I_1 - \frac{1}{3}\left(\frac{3}{5}\right)^3$

$I_1 = I_0 - \frac{1}{1}\left(\frac{3}{5}\right)$

$\therefore I_1 = I_0 - \frac{3}{5}$

$I_0 = \int_0^{\ln 2} \tanh^0 x \, dx = \int_0^{\ln 2} 1 \, dx = \ln 2$

Turn over

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Question 8 continued

$\therefore \quad I_1 = \ln 2 - \frac{3}{5}$

$I_2 = \ln 2 - \frac{3}{5} - \frac{1}{3}\left(\frac{3}{5}\right)^3$

$= \frac{-84}{125} + \ln 2$