F2 June 2017 (MA)

Q1) $z^5 = 32(\cos 0 + i\sin 0)$

$z = 32^{\frac{1}{5}}\left(\cos(2k\pi) + i\sin(2k\pi)\right)^{\frac{1}{5}}$

$z = 2\left(\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)\right)$ //

$\underline{k=0}$: $z = 2(\cos 0 + i\sin 0)$

$\underline{k=1}$: $z = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$\underline{k=2}$: $z = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$\underline{k=3}$: $z = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$\underline{k=4}$: $z = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

Q2) $\frac{x-4}{x+3} \leq \frac{5}{x(x+3)}$ *

$\underline{\times(x+3)^2}$: $(x-4)(x+3) \leq \frac{5(x+3)}{x}$

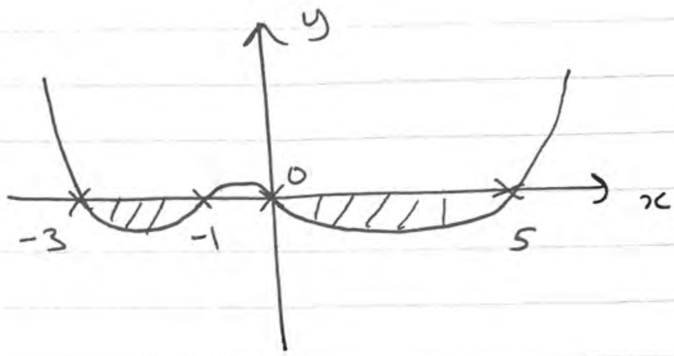
$\underline{\times(x^2)}$: $x^2(x-4)(x+3) \leq 5x(x+3)$

$x(x+3)[x(x-4) - 5] \leq 0$

$x(x+3)(x^2-4x-5) \leq 0$

$x(x+3)(x-5)(x+1) \leq 0$

Critical values : $x = 0$
$x = -3$
$x = 5$
$x = -1$



we want where $y \leq 0$

required region : $\boxed{\begin{array}{c} -3 \leq x \leq -1 \\ 0 < x \leq 5 \end{array}}$

<u>Remember</u> : $(x = 0)$ and $(x = -3)$ can <u>not</u> be part of our solutions as $(x=0)$ and $(x=3)$ are asymptotes to the two equations in the given inequality *.
Using $x=0$ and $x = -3$ would yield a 0 on the denominator in those equations and anything divided by 0 is undefined.

Q3a) $r^3 - (r-1)^3 = \text{LHS} = r^3 - (r^2 - 2r + 1)(r-1)$

$= r^3 - (r^3 - r^2 - 2r^2 + 2r + r - 1)$

$= r^3 - r^3 + 3r^2 - 3r + 1$

$= 3r^2 - 3r + 1$

$\square$

b) from (a), $3r^2 \equiv r^3 - (r-1)^3 + 3r - 1$

$\therefore \sum_1^n 3r^2 = \sum_1^n r^3 - (r-1)^3 + 3r - 1$

$\Rightarrow \sum_1^n 3r^2 = \underbrace{\sum_1^n r^3 - (r-1)^3}_{1} + \underbrace{\sum_1^n 3r - 1}_{2}$

1 : $\sum_1^n r^3 - (r-1)^3$ by method of differences,

$\underline{n=1}$: $\cancel{1} - (0)$

$\underline{n=2}$: $\cancel{8} - (\cancel{1})$

$\ldots$

$\underline{n=n-1}$: $(n-\cancel{1})^3 - (n-\cancel{2})^3$

$\underline{n=n}$: $n^3 - (n-\cancel{1})^3$

$= n^3$ //

2 : $\sum_1^n 3r - 1 = 3\sum_1^n r - \sum_1^n (1) = \frac{3n}{2}(n+1) - n$ //

b cont.) so $3\sum_{1}^{n} r^2 = n^3 + \frac{3n}{2}(n+1) - n$

$$= n\left(n^2 + \frac{3n}{2} + \frac{3}{2} - 1\right)$$

$$= n\left(n^2 + \frac{3n}{2} + \frac{1}{2}\right)$$

$$= \frac{n}{2}\left(2n^2 + 3n + 1\right)$$

$$= \frac{n}{2}(2n+1)(n+1) \quad //$$

$$\therefore \sum_{1}^{n} r^2 = \frac{\frac{n}{2}(2n+1)(n+1)}{3} = \frac{n}{6}(2n+1)(n+1) \quad \square$$

Q4a) $y = \boxed{e^{-x}(3\cos 3x + A\sin 3x)}$

$$y' = -e^{-x}(3\cos 3x + A\sin 3x) + e^{-x}(-9\sin 3x + 3A\cos 3x)$$
$$= \boxed{e^{-x}\left((-9-A)\sin 3x + (3A-3)\cos 3x\right)}$$

$$y'' = -e^{-x}\left((-9-A)\sin 3x + (3A-3)\cos 3x\right) + e^{-x}\left((-27-3A)\cos 3x + (9-9A)\sin 3x\right)$$

$$= \boxed{e^{-x}\left((-24-6A)\cos 3x + (18-8A)\sin 3x\right)}$$

substituting into diff. eqn...

$$e^{-x}\left[(-24-6A)\cos 3x + \underbrace{(18-8A)\sin 3x}\right] - 2e^{-x}\left[\underbrace{(-9-A)\sin 3x} + (3A-3)\cos 3x\right]$$

$$+ 10e^{-x}\left[3\cos 3x + \underbrace{A\sin 3x}\right] = \underbrace{40e^{-x}\sin 3x}$$

compare coefficients of $\sin 3x$ terms:

$$18 - 8A + 18 + 2A + 10A = 40$$

$36 + 4A = 40$

$4A = 4$

$\boxed{A = 1}$

b) the P.I is: $y = 3e^{-x}\cos 3x + e^{-x}\sin 3x$

considering original equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 40e^{-x}\sin 3x$$

Aux: $\lambda^2 - 2\lambda + 10 = 0$

By Quadratic formula, $\lambda = 1+3i$

$\lambda = 1-3i$ //

C.F: $y = e^{x}(A\sin 3x + B\cos 3x)$

∴ General solution

↳ $y = e^{x}(A\sin 3x + B\cos 3x) + 3e^{-x}\cos 3x + e^{-x}\sin 3x$

$y = e^{x}(A\sin 3x + B\cos 3x) + e^{-x}(3\cos 3x + \sin 3x)$

c) $x=0,\ y=3$: $3 = B + 3$

$B = 0$ //

$$\frac{dy}{dx} = e^{x}(A\sin 3x) + e^{x}(3A\cos 3x) + e^{-x}(-9\sin 3x + 3\cos 3x) + (-e^{-x})(3\cos 3x + \sin 3x).$$

$\frac{dy}{dx} = 3,\ x=0$: $3 = 3A + 3 - (3)$

$A = 1$ //

Particular solution : $y = e^{x}(\sin 3x) + e^{-x}(3\cos 3x + \sin 3x)$

Q5a) $y = e^{(\cos x)^2}$

$\frac{dy}{dx} = (-2\sin x\cos x)\, e^{\cos^2 x} = -\sin 2x\, e^{\cos^2 x}$ //

$\frac{d^2y}{dx^2} = -2\cos 2x\, e^{\cos^2 x} - \sin 2x\left(-\sin 2x\, e^{\cos^2 x}\right)$

$= (\sin^2 2x)e^{\cos^2 x} + (-2\cos 2x)e^{\cos^2 x}$

$= e^{\cos^2 x}\left(\sin^2 2x - 2\cos 2x\right)$ // $\square$

b) $f(0) = e^{(\cos 0)^2} = e$

$f'(0) = 0$

$f''(0) = e\cdot 1(0-2) = -2e$ //

$e^{\cos^2 x} \simeq e + 0x - 2e\cdot\frac{x^2}{2}$

$\boxed{e^{\cos^2 x} \simeq e - ex^2}$

$e^{\cos^2 x} \simeq e(1 - x^2)$

Q6) $\cos x \frac{dy}{dx} + y\sin x = (\cos^2 x)\ln x$

$\div \cos x$: $\frac{dy}{dx} + y\tan x = \cos x \ln x$.

$I = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$ //

$\times \sec x$: $\sec x \frac{dy}{dx} + y\sec x \tan x = \ln x$

$\Rightarrow \frac{d}{dx}(y\sec x) = \ln x$

$\Rightarrow y\sec x = \int (\ln x)\, dx$

By Parts

$u = \ln x \quad v' = 1$

$u' = \frac{1}{x} \quad v = x$

$\Rightarrow y\sec x = [x\ln x] - \int [1]\, dx + c$

$\Rightarrow y\sec x = x\ln x - x + c$

$\times[\cos x]$

$\Rightarrow \boxed{y = \cos x(x\ln x - x + c)}$

Q7a) SP is a tangent to the points where $\frac{dy}{d\theta} = 0$.

$r = 4\cos 2\theta$

$r\sin\theta = 4\sin\theta\cos 2\theta$

$y = 4\sin\theta\cos 2\theta$

$\frac{dy}{d\theta} = 4\cos\theta\cos 2\theta + (-8)\sin\theta\sin 2\theta = 0$

$\Rightarrow \cos\theta\cos 2\theta = 2\sin\theta\sin 2\theta$

$\Rightarrow \cos\theta\cos 2\theta = 4\sin^2\theta\cos\theta$

$\Rightarrow \cos\theta(\cos 2\theta - 4\sin^2\theta) = 0$

$\cos\theta = 0$

$\theta = \pm\frac{\pi}{2}, \frac{3\pi}{2}$

(no solutions within required interval)

$\cos 2\theta - 4\sin^2\theta = 0$

$1 - 6\sin^2\theta = 0 \quad (\cos 2\theta = 1 - 2\sin^2\theta)$

$\sin^2\theta = \frac{1}{6}$

$\sin\theta = \pm\sqrt{\frac{1}{6}}$

$\theta = \pm\sin^{-1}\left(\sqrt{\frac{1}{6}}\right) = \pm 0.421^c$ //

$r = 4\cos(2\times 0.421^c) = \frac{8}{3}$ //

$\therefore$ points are : $\boxed{\left(\frac{8}{3}, 0.421^c\right)}$ & $\boxed{\left(\frac{8}{3}, \pi - 0.421^c\right)}$

b) Area required $= 4 \times \frac{1}{2} \int_0^{\frac{\pi}{4}} [16\cos^2 2\theta]\, d\theta$

$= 16 \int_0^{\pi/4} [2\cos^2 2\theta]\, d\theta$ $\quad \left[\begin{array}{l} \cos 4\theta = 2\cos^2 2\theta - 1 \\ \cos 4\theta + 1 = 2\cos^2 2\theta \end{array}\right]$

$= 16 \int_0^{\frac{\pi}{4}} [\cos 4\theta + 1]\, d\theta = 16\left[\frac{1}{4}\sin 4\theta + \theta\right]_0^{\pi/4}$

$= 16\left[0 + \frac{\pi}{4}\right] = \boxed{4\pi}$

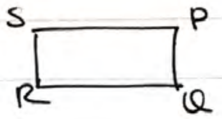
c) at $\underline{\theta = 0}$: $r = 4\cos(0) = 4$ //

$\therefore$ length $SP = 2 \times 4 = 8$

from (a), SP touches the curve at $\left(\frac{8}{3}, \pm 0.421^c\right)$

so length $SR = 2 \times \left[\frac{8}{3}\sin 0.421^c\right] = 2 \times \left[\frac{8}{3} \times \sqrt{\frac{1}{6}}\right]$

$= \frac{16}{3} \times \frac{\sqrt{6}}{6} = \frac{8\sqrt{6}}{9}$ //

$\therefore$ Area 

$= \frac{8\sqrt{6}}{9} \times 8 = \frac{64\sqrt{6}}{9}$ //

so Area shaded $= \boxed{\frac{64\sqrt{6}}{9} - 4\pi}$

Q8ai) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + \binom{5}{1}\cos^4\theta(i\sin\theta) + \binom{5}{2}\cos^3\theta(i\sin\theta)^2$

$+ \binom{5}{3}\cos^2\theta(i\sin\theta)^3 + \binom{5}{4}\cos\theta(i\sin\theta)^4$

$+ \binom{5}{5}(i\sin\theta)^5$

$\cos 5\theta + i\sin 5\theta = \cos^5\theta + i(5\cos^4\theta\sin\theta) - 10\sin^2\theta\cos^3\theta$
$+ i(-10\cos^2\theta\sin^3\theta) + 5\cos\theta\sin^4\theta + i(\sin^5\theta)$

Compare real terms: $\boxed{\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\cos\theta\sin^4\theta}$ //

ii) Compare imaginary terms: $\boxed{\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}$ //

b) $\frac{\sin 5\theta}{\cos 5\theta} = \tan 5\theta = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\cos\theta\sin^4\theta}$

[÷ top & bottom by $\cos^5\theta$] $\tan 5\theta = \frac{\frac{5\sin\theta}{\cos\theta} - \frac{10\sin^3\theta}{\cos^3\theta} + \frac{\sin^5\theta}{\cos^5\theta}}{\frac{\cos^5\theta}{\cos^5\theta} - \frac{10\sin^2\theta}{\cos^2\theta} + \frac{5\sin^4\theta}{\cos^4\theta}}$

$\Rightarrow \tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

$\Rightarrow \tan 5\theta = \frac{5t - 10t^3 + t^5}{5t^4 - 10t^2 + 1}$ //

c) notice: $\theta = \frac{\pi}{5}$ and $\theta = \frac{2\pi}{5}$ are roots of $[\tan 5\theta = 0]$ //

$\tan 5\theta = \tan^5\theta - 10\tan^3\theta + 5\tan\theta = 0$ //

$\Rightarrow \frac{(t^5 - 10t^3 + 5t)}{t} = 0$ //

$\Rightarrow t^4 - 10t^2 + 5 = 0$ // $(t = \tan\theta)$

so $\theta = \frac{\pi}{5}, \frac{2\pi}{5}$ are roots of this equation.

we want an equation where $\tan^2\frac{\pi}{5}$ and $\tan^2\frac{2\pi}{5}$ are roots.

So let $x = \tan^2\theta$,

$\Rightarrow x^2 - 10x + 5 = 0$ //

now if you put $(x = \tan^2\frac{\pi}{5})$ or $(x = \tan^2\frac{2\pi}{5})$ then the LHS will evaluate to 0.

so our answer: $\boxed{x^2 - 10x + 5 = 0}$

d) product of roots $= \left(\tan^2\frac{\pi}{5}\right)\left(\tan^2\frac{2\pi}{5}\right) = 5$ //

$\therefore \sqrt{\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5}} = \sqrt{5}$

$\Rightarrow \tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$

$\square$