F1 January 2018 (MA)

Q1(a) $f(x) = 3x^2 - \frac{5}{3}x^{-\frac{1}{2}} - 6$

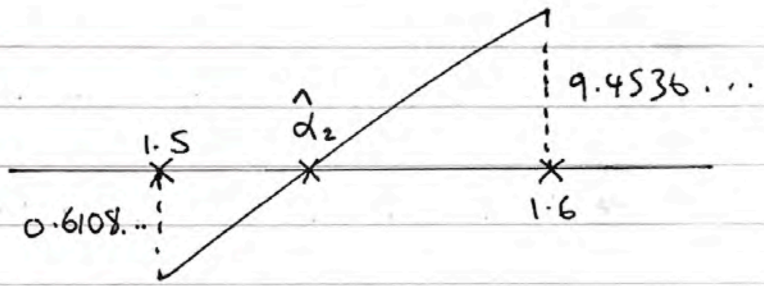
$f'(x) = 6x + \frac{5}{6}x^{-\frac{3}{2}}$

$f(1.5) = -0.6108\ldots$

$f'(1.5) = 9.4536\ldots$

$x_1 = 1.5 - \frac{-0.6108\ldots}{9.4536\ldots} = \boxed{1.565} = \alpha_1$ to 3 dp.

b)



$\frac{1.6 - \alpha}{\alpha - 1.5} = \frac{9.4536\ldots}{0.6108\ldots}$

let $\frac{9.4536\ldots}{0.6108\ldots} = c$,

then $\frac{1.6 - \hat{\alpha}}{\hat{\alpha} - 1.5} = c$

$\alpha c - 1.5c = 1.6 - \hat{\alpha}$

$\hat{\alpha}(c+1) = 1.6 + 1.5c \quad \therefore \hat{\alpha} = \frac{1.6 + 1.5c}{c+1}$ //

$\therefore \lambda = \boxed{1.563}$ to 3 d p.

Q2a) if $2+3i$ is a root then $\boxed{2-3i}$ is also a root. (complex conjugate pair).

$\therefore f(z) = (z-(2+3i))(z-(2-3i))(z^2+az+b) = 0$

$= (z^2 - z(2+2) + (2-3i)(2+3i))(z^2+az+b)$

$= (z^2 - 4z + 13)(z^2+az+b) = 0$ //

$\therefore f(z) = z^4 + az^3 + bz^2 - 4z^3 - 4az^2 - 4bz + 13z^2 + 13az + 13b = 0$

$= z^4 + (a-4)z^3 + (b-4a)z^2 + (13)z^2 + (13a-4b)z + 13b = 0$ //

compare coefficients with given eqn,

$\underline{z^3}$: $a - 4 = -6$

$\therefore a = -2$ //

$\underline{z}$: $-94 = 13a - 4b$

$\therefore b = \dfrac{13(-2)+94}{4} = 17$ //

$\therefore f(z) = (z^2-4z+13)(z^2-2z+17) = 0$

$z^2 - 2z + 17 = 0$

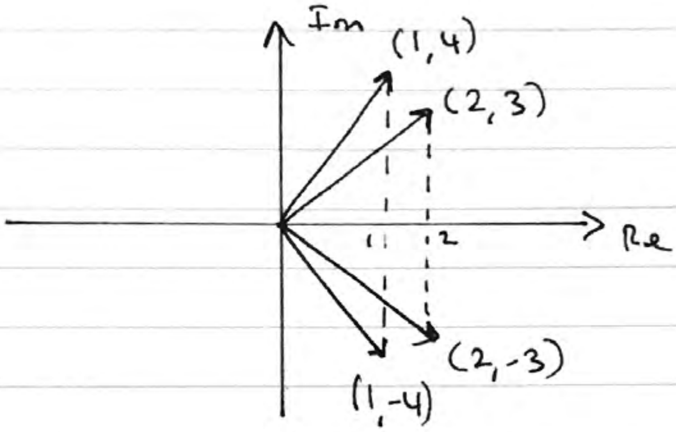
By Quadratic formula: $z = \dfrac{2 \pm \sqrt{4-4(17)}}{2}$

$a=1,\ b=-2,\ c=17$

$$z = \frac{2 \pm 8i}{2} = 1 \pm 4i$$ //

so all roots : $\boxed{\begin{array}{l} z = 2+3i \\ z = 2-3i \\ z = 1+4i \\ z = 1-4i \end{array}}$

b)



Q3a) $\sum_1^n r^2(r+1) = \sum r^3 + r^2 = \sum r^3 + \sum r^2$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{n}{12}(n+1)\left[3n(n+1) + 2(2n+1)\right]$$

$$= \frac{n}{12}(n+1)\left[3n^2 + 3n + 4n + 2\right]$$

$$= \frac{n}{12}(n+1)\left[3n^2 + 7n + 2\right]$$

$$= \frac{n}{12}(n+1)(3n^2+7n+2) = \boxed{\frac{n}{12}(n+1)(n+2)(3n+1)}$$ //

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b) $\sum_{1}^{n} 3^r$ is a geometric series, where

$a = 3$

$r = 3$

$\therefore S_n = \frac{a(1-r^n)}{1-r} = \frac{3(1-3^n)}{-2}$ //

$$\sum_{5}^{25} r^2(r+1) = \sum_{1}^{25} r^2(r+1) - \sum_{1}^{4} r^2(r+1)$$

$$\sum_{5}^{25} r^2(r+1) = \frac{25}{12}(26)(27)(3(25)+1) - \frac{4}{12}(5)(6)(3(4)+1)$$

$$= 111150 - 130$$

$$= 111020 \text{ //}$$

$$\therefore 111020 + \frac{3(1-3^n)}{-2} = 140543$$

$$-\frac{3}{2}(1-3^n) = 29523$$

$$1-3^n = -19682$$

$$3^n = 19683$$

$$\log(3^n) = \log(19683)$$

$$n\log 3 = \log(19683)$$

$$\therefore n = \frac{\log 19683}{\log 3} = \boxed{9}$$

Q4a) $3x^2 + 2x + 5 = 0$

$\underline{\div 3}$: $x^2 + \frac{2}{3}x + \frac{5}{3} = 0$

$\therefore \alpha + \beta = -\frac{2}{3}$

& $\alpha\beta = \frac{5}{3}$

$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta.$

$= \left(-\frac{2}{3}\right)^2 - 2\left(\frac{5}{3}\right)$

$= \boxed{-\frac{26}{9}}$

b) $\alpha^3 + \beta^3 = ?$

$(\alpha+\beta)^3 = (\alpha^2+\beta^2+2\alpha\beta)(\alpha+\beta) = \alpha^3 + \alpha^2\beta + \beta^2\alpha + \beta^3 + 2\alpha^2\beta + 2\alpha\beta^2$

$\therefore (\alpha+\beta)^3 = \alpha^3 + \beta^3 + \alpha\beta(\alpha+\beta) + 2\alpha\beta(\alpha+\beta)$

so $(\alpha+\beta)^3 - \alpha\beta(\alpha+\beta) - 2\alpha\beta(\alpha+\beta) = \alpha^3 + \beta^3$ //

$\Rightarrow \alpha^3 + \beta^3 = \left(-\frac{2}{3}\right)^3 - \frac{5}{3}\left(-\frac{2}{3}\right) - 2\left(\frac{5}{3}\right)\left(-\frac{2}{3}\right)$

$\alpha^3 + \beta^3 = \boxed{\frac{82}{27}}$

c) $\left(x-\left(\alpha+\frac{\alpha}{\beta^2}\right)\right)\left(x-\left(\beta+\frac{\beta}{\alpha^2}\right)\right)=0$

$x^2 - x\left(\beta+\alpha+\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}\right)+\left(\alpha+\frac{\alpha}{\beta^2}\right)\left(\beta+\frac{\beta}{\alpha^2}\right)=0$

$x^2 - x\left(\alpha+\beta+\frac{\alpha^3+\beta^3}{\alpha^2\beta^2}\right)+\alpha\beta+\frac{\beta}{\alpha}+\frac{\alpha}{\beta}+\frac{\alpha\beta}{(\alpha\beta)^2}$

$x^2 - x\left(\frac{-2}{3}+\frac{\frac{82}{27}}{\left(\frac{5}{3}\right)^2}\right)+\frac{5}{3}+\frac{\beta^2+\alpha^2}{\alpha\beta}+\frac{\alpha\beta}{(\alpha\beta)^2}$

$x^2 - x\left(\frac{32}{75}\right)+\frac{5}{3}+\frac{\frac{-26}{9}}{\frac{5}{3}}+\frac{\frac{5}{3}}{\left(\frac{5}{3}\right)^2}$

$x^2 - \frac{32}{75}x + \frac{8}{15} = 0$

$\times 75$: $\boxed{75x^2 - 32x + 40 = 0}$

Q5 i) $\dfrac{2z+3}{z+5-2i} = 1+i$

$(z+5-2i)(1+i) = 2z+3$

$z + zi + 5 + 5i - 2i + 2 = 2z + 3$

$z - zi = 2 + 2 + 3i$

$z(1-i) = 4+3i$

$z = \dfrac{4+3i}{1-i} = \dfrac{(4+3i)(1+i)}{(1-i)(1+i)}$

$z = \dfrac{4+4i+3i+3i^2}{1+1} = \dfrac{1+7i}{2}$

$\therefore\ z = \boxed{\frac{1}{2} + \frac{7}{2}i}$

ii) $w = (3+\lambda i)(2+i)$

$w = 6 + 3i + 2\lambda i - \lambda$

$w = (6-\lambda) + (3+2\lambda)i$

$|w| = \sqrt{(6-\lambda)^2 + (3+2\lambda)^2} = 15$

$\therefore\ (6-\lambda)^2 + (3+2\lambda)^2 = 15^2$

$36 - 12\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 = 225$

$\Rightarrow \quad 5\lambda^2 = 225 - 45$

$\Rightarrow \quad 5\lambda^2 = 180$

$\Rightarrow \quad \lambda^2 = 36$

$\therefore \boxed{\lambda = 6, \ \lambda = -6}$

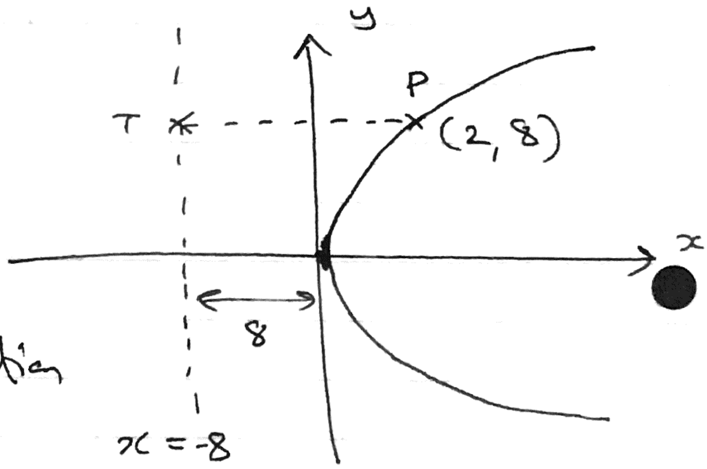
Q6a) $y^2 = \underbrace{32}_{}x$

$4a = 32$

$a = \frac{32}{4} = 8$

$\therefore \boxed{S(8, 0)}$

b) $8 + 2 = \boxed{10}$



c) $2y\frac{dy}{dx} = 32$

Implicit differentiation

$\therefore \frac{dy}{dx} = \frac{32}{2y} = \frac{16}{y}$

$= \frac{16}{8} = \underline{\underline{2}}$ at P

$\therefore \quad y - 8 = 2(x - 2)$

$y = 2x - 4 + 8$

hence $\boxed{y = 2x + 4}$

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d) $y = 2x + 4$ meets $xy = 4$

<u>substituting y</u> : $x(2x+4) = 4$

$$2x^2 + 4x = 4$$

$$2x^2 + 4x - 4 = 0$$

$$x^2 + 2x - 2 = 0$$

<u>By Quadratic Formula</u> :

$$\left.\begin{aligned} a &= 1 \\ b &= 2 \\ c &= -2 \end{aligned}\right\} \quad x = -1 \pm \sqrt{3} \;//$$

at $x = -1 + \sqrt{3}$, $y = 2(-1+\sqrt{3}) + 4$

$$= 2 + 2\sqrt{3} \;//$$

at $x = -1 - \sqrt{3}$, $y = 2(-1-\sqrt{3}) + 4$

$$= 2 - 2\sqrt{3} \;//$$

So $\quad L\left(-1+\sqrt{3},\ 2+2\sqrt{3}\right)$

$\quad M\left(-1-\sqrt{3},\ 2-2\sqrt{3}\right)$

Q7ia) $A = \begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix}$

$\det A = 6(-4) + 3u = 3u - 24$

$\boxed{\therefore A^{-1} = \frac{1}{3u-24}\begin{pmatrix} -4 & -u \\ 3 & 6 \end{pmatrix}}$ //

b) $A^2 = \begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 6 & u \\ -3 & -4 \end{pmatrix}$

$= \begin{pmatrix} 36-3u & 6u-4u \\ -18+12 & 16-3u \end{pmatrix}$

$= \boxed{\begin{pmatrix} 36-3u & 2u \\ -6 & 16-3u \end{pmatrix}}$

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c) $A^2 + 3A^{-1} = \begin{pmatrix} 5 & 9 \\ -3 & -5 \end{pmatrix}$

$= \begin{pmatrix} 36-3k & 2k \\ -6 & 16-3k \end{pmatrix} + \begin{pmatrix} \frac{-4}{k-8} & \frac{-k}{k-8} \\ \frac{3}{k-8} & \frac{6}{k-8} \end{pmatrix}$

using bottom left elements,

$-6 + \frac{3}{k-8} = -3$

$\frac{3}{k-8} = 3$

$\therefore 1 = k - 8$ so $\boxed{k = 9}$

iia) $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$

AC rotation through angle θ about O

one way stretch of s.f p parallel to y-axis.

$M = \begin{pmatrix} \cos\theta & -p\sin\theta \\ \sin\theta & p\cos\theta \end{pmatrix}$ //

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$\cos\theta = \frac{-1}{2}$ — ① $p\sin\theta = \sqrt{3}$ — ③

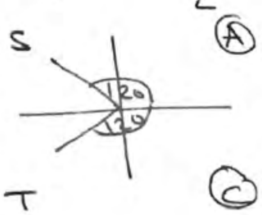
$\sin\theta = \frac{\sqrt{3}}{2}$ — ② $p\cos\theta = -1$ — ④

$\underline{③^2 + ④^2}$: $p^2\sin^2\theta + p^2\cos^2\theta = 3 + 1$

$p^2(\sin^2\theta + \cos^2\theta) = 4$

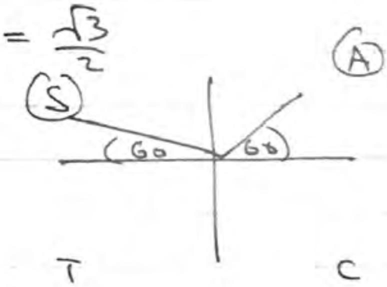
$\therefore p^2 = 4 \longrightarrow \boxed{p = 2} \quad (p > 0)$

ii b) $\cos\theta = -\frac{1}{2}$



$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = 120°, 240°$

$\sin\theta = \frac{\sqrt{3}}{2}$



$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°, 120°$

we want where both $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = -\frac{1}{2}$ . ..

So the only valid value of θ is $\boxed{120°}$ since this is a solution to both $\cos\theta = -\frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$.

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we want : $u_{k+1} = \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + 5$

Q8i) $u_1 = 3$ $\qquad u_{n+1} = u_n + 3n - 2$

prove : $u_n = \frac{3}{2}n^2 - \frac{7}{2}n + 5$

n=1 : $u_1 = \frac{3}{2}(1) - \frac{7}{2}(1) + 5 = 3$

so the statement is true for n=1.. as given.

assume true for n=k.

ie that $\left[u_k = \frac{3}{2}k^2 - \frac{7}{2}k + 5\right]$

consider n=k+1 .

$u_{k+1} = u_k + 3k - 2$ ← given identity.

$$u_{k+1} = \frac{3}{2}k^2 - \frac{7}{2}k + 5 + 3k - 2$$

$$= \frac{3}{2}k^2 - \frac{7}{2}k + 3k - 2 + 5$$

$$= \frac{3}{2}(k+1)^2 - \frac{3}{2}(2k+1) + 3k - 2 + 5 - \frac{7}{2}k$$

$$= \frac{3}{2}(k+1)^2 - \cancel{3k} - \frac{3}{2} + \cancel{3k} - \frac{7}{2}k - 2 + 5$$

$$= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + \frac{7}{2} - 2 + 5 - \frac{3}{2}$$

$$= \frac{3}{2}(k+1)^2 - \frac{7}{2}(k+1) + 5$$

∴ true for n=1.

ii) $f(n) = 3^{2n+3} + 40n - 27$

$\underline{n=1}$: $f(1) = 3^5 + 40 - 27 = 256$
$= (64) \times 4$

$\therefore$ true for $n=1$.

<u>assume true for $n=k$</u>,

(ie $f(k) = 3^{2k+3} + 40k - 27$ is div. by 64)

<u>consider $n=k+1$</u>,

$f(k+1) = 3^{2(k+1)+3} + 40(k+1) - 27$

$= 3^{2k+3+2} + 40k - 27 + 40$

$= 9(3^{2k+3}) + 40k - 27 + 40$

$= [3^{2k+3} + 40k - 27][9] + 40 - 8(40k) + 8(27)$

$= 9f(k) + 40 - 320k + 216$

$= 9f(k) - 320k + 256$

$= \underbrace{9f(k)}_{\text{divisible by 64}} - 5k\underbrace{(64)}_{\text{divisible by 64}} + \underbrace{64(4)}_{\text{divisible by 64}}$

$\therefore$ true for $n=k+1$.

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So ... true for $n=1$.
true for $n=k+1$ when
assumed true for $n=k$.

$\therefore$ By Mathematical Induction
true for all $n \in \mathbb{Z}^+$