Pearson
Edexcel

Mark Scheme (Results)

Summer 2019

Pearson Edexcel GCE In A level Further Mathematics Paper 9FM0/4A

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Summer 2019
Publications Code 9FM0_4A_1906_MS

General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- M marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- A marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- B marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- * The answer is printed on the paper
- The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

Question	Scheme	Marks	AOs
1(a)	$(x-3)^2+y^2=16\left((x+1)^2+y^2\right)$	M1	1.1b
	$x^2-6x+9+y^2=16x^2+32x+16+16y^2$ $15x^2+15y^2+38x+7=0$*	A1*	2.1
		(2)	
(b)	$15x^2+15y^2+38x+7=15\left(x\pm\frac{19}{15}\right)^2-...+15y^2+7=0$	M1	2.1
	Centre is $\left(-\frac{"19"}{5},0\right)$ and radius is $\sqrt{\left(\frac{"19"}{15}\right)^2-\frac{7}{15}}\left(=\frac{16}{15}\right)$	M1	2.2a
	$\max\lvert z\rvert=\frac{16}{15}+\frac{19}{15}=\frac{7}{3}$	A1	3.1a
		(3)	
			(5 marks)

Notes

(a)

M1: Obtains an equation in terms of x and y using the given information. Allow if the 4 is not squared, but i^2 must have been dealt with correctly (ie positive y^2 terms). Condone invisible brackets for the M mark.

A1*: Expands and simplifies and obtains a circle equation correctly. Accept terms in different order but must include =0. No errors seen, so bracketing errors in solution are A0.

(b)

M1: Completes the square on the x term achieving $A\left(x\pm\frac{19}{15}\right)^2-B$, or uses other appropriate method in order to attempt the radius and/or centre of the circle. Award if correct x coordinate of centre or radius is found.

M1: Deduces both centre and radius for their completed square form, either seen used in work clearly as centre and radius, stated or labelled on a diagram, not just embedded within the equation. This is implied by the correct calculation being carried out for their centre and radius.

A1cso: Realises the need to add distance of centre from origin to radius to achieve the correct answer. Must come from correct work.

Note that completing the square as $\left(x-\frac{19}{15}\right)^2-...$ can score a maximum M1M1A0

Question	Scheme	Marks	AOs
2(a)	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=(6-\lambda)[...]-(-2)[...]+2[...]=...$	M1	1.1b
	$(6-\lambda)\left((3-\lambda)^2-1\right)+2\left(2(\lambda-3)+2\right)+2\left(2-2(3-\lambda)\right)(=0)$ $\left(\lambda^3-12\lambda^2+36\lambda-32=0\right)$	A1	1.1b
	$=(\lambda-2)\left(\lambda^2+...\lambda+...\right)$	M1	2.1
	$=(\lambda-2)\left(\lambda^2-10\lambda+16\right)=(\lambda-2)^2(\lambda-8)\Rightarrow\lambda=2$ is a repeated eigenvalue *	A1*	2.2a
	$\lambda=8$	B1	1.1b
		(5)	
(b)	$\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ or $\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=8\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Rightarrow\mathbf{v}=...$	M1	1.1b
	Obtains any multiple of $\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}$ for $\lambda=8$	A1	1.1b
	Obtains any (non-zero) multiple or linear combination of $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	1.1b
	Obtains a **different** linear combination or (non-zero) multiple of different vector from $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	3.1a
		(4)	
(c)	Forms a matrix with their eigenvectors as columns	M1	1.2
	E.g. $\begin{pmatrix}-1 & 1 & 2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{pmatrix}$	A1ft	1.1b
		(2)	
			(11 marks)

Notes
(a) M1: Attempts to expand the determinant to find the characteristic polynomial. Note: other methods of expanding the determinant are possible. If unsure send to review. A1: Correct expansion need not be simplified. (Need not see set equal to zero) Allow recovery of missing brackets if indicated by later working. M1: Attempts to take out a factor of $(\lambda - 2)$ of their equation (may first expand to cubic or may spot the factor and take out without full expansion). E.g $(6-\lambda)\left((3-\lambda)^2-1\right)+2\left(2(\lambda-3)+2\right)+2\left(2-2(3-\lambda)\right)=(6-\lambda)(4-\lambda)(2-\lambda)+4(\lambda-2)+4(\lambda-2)$ $=(\lambda-2)\left((6-\lambda)(4-\lambda)+4+4\right)$ This is for a method that will allow λ to be shown as a repeated eigenvalue, so just stating two solutions is not sufficient, factorisation must be seen. A1*: Obtains a correct factor of $(\lambda - 2)^2$ and deduces that 2 is a repeated eigenvalue. Must see **statement** about 2 being repeated. (Just listing 2 twice is not sufficient.) B1: (Note this is A1 on ePEN) Obtains and identifies 8 as the other eigenvalue (B0 if not identified in (a) but full marks can be scored in (b) and (c) for use of 8 as eigenvalue) (b) M1: Uses a correct method to find at least one eigenvector A1: Obtains one correct eigenvector for $\lambda = 8$ A1: Obtains one correct eigenvector for $\lambda = 2$ A1: Obtains two correct linearly independent eigenvectors for $\lambda = 2$ Note some other common eigenvectors for $\lambda = 2$ are $\begin{pmatrix}1\\1\\-1\end{pmatrix},\begin{pmatrix}1\\3\\1\end{pmatrix},\begin{pmatrix}2\\5\\1\end{pmatrix},\begin{pmatrix}1\\4\\2\end{pmatrix}$ (c) M1: Forms a matrix with their three **different non-zero** eigenvectors as columns or with their normalised (or any scaled version) of their eigenvectors. A1ft: Correct matrix with the eigenvectors (normalised/scaled) as columns in any order (follow through their three **different** vectors which are **not multiples** of any other)

Question	Scheme	Marks	AOs
3(a)	$V_{n+2} = V_{n+1} + kV_n$	B1	3.3
		(1)	
(b)	$\lambda^2 - \lambda - 0.24 = 0 \Rightarrow \lambda = ...(1.2, -0.2)$	M1	1.1b
	$V_n = a(1.2)^n + b(-0.2)^n$	A1	2.2a
	$65 = a(1.2)^1 + b(-0.2)^1$ **and** $71 = a(1.2)^2 + b(-0.2)^2$	B1ft	3.4
	E.g. $\left.\begin{array}{l} 78 = 1.44a - 0.24b \\ 71 = 1.44a + 0.04b \end{array}\right\} \Rightarrow 7 = -0.28b \Rightarrow b = ...$	M1	2.1
	$a = 50, b = -25 \Rightarrow V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	
(c)	$50(1.2)^N > 10^6 \Rightarrow N = ...$	M1	3.1b
	$\Rightarrow N = 55$ i.e. month 55	A1	3.2a
		(2)	
			(8 marks)

Notes

(a)
B1: A correct expression for the model using the information given
(b)
M1: Forms and solves the auxiliary equation for their answer to (a) with $k = 0.24$
A1: The **correct** closed form deduced from their solutions. This must be consistent with their equation. Note the answer is given so check carefully. This is **not** a follow through mark.
B1ft: Applies initial conditions to their general equation – correct two equations for their general form with $V_1 = 65$ and $V_2 = 71$
M1: Attempts to solve their equations showing a correct method, reaching a value for at least one variable. It is a show that question and answers are on the paper, so method is needed. Look for one equation multiplied through to give same coefficients before attempting eliminating or substitution. If a matrix system is used the inverse must be found, not just solutions stated.
A1*: Correct expression formed following suitable working with no errors seen. With fractions instead of decimals is fine.
(c)
M1: Selects a suitable method to solve the problem. For example, realises that in the model, $(-0.2)^n$ is negligible for large n and so attempts to solve e.g. $50(1.2)^N = 10^6$, or tries at least one value either side of $N = 55$ as a process of trial and improvement, or uses a calculator/graphical approach – implied by a value of $N = 55$ or $N = 54$ stated.
A1: $N = 55$.
The correct answer will imply both marks for this part. Ignore erroneous working if correct answer is stated as a restart.

3(b)	$V_n = 50(1.2)^n - 25(-0.2)^n$, $V_1 = 65$, $V_2 = 71$		

Alt	$V_1 = 50\times1.2 - 25\times -0.2 = 60 + 5 = 65$ $V_2 = 50\times(1.2)^2 - 25\times(-0.2)^2 = 72 - 1 = 71$	M1	1.1b
	Hence true for n = 1 and n =2 Assume true for $n = k$ and $n = k + 1$ (for some $k > 0$)	A1	2.2a
	$V_{k+2} = V_{k+1} + 0.24V_k$ $= 50(1.2)^{k+1} - 25(-0.2)^{k+1} + 0.24\left(50(1.2)^k - 25(-0.2)^k\right)$	B1ft	3.4
	$= \frac{50}{1.2}(1.2)^{k+2} - \frac{25}{-0.2}(-0.2)^{k+2} + \frac{12}{1.2^2}(1.2)^{k+2} - \frac{6}{(-0.2)^2}(-0.2)^{k+2}$ $= \frac{125}{3}(1.2)^{k+2} + 125(-0.2)^{k+2} + \frac{25}{3}(1.2)^{k+2} - 150(-0.2)^{k+2} = \ldots$	M1	2.1
	So $V_{k+2} = 50(1.2)^{k+2} - 25(-0.2)^{k+2}$ Hence true for $n = k + 2$. So the result is true for n =1 and n = 2, and if true for $n = k$ and $n = k + 1$ then it is true for $n = k + 2$. Hence by mathematical induction, for all $n \in \mathbb{N}$ $V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	

Notes

M1: Substitutes into equation for n = 1 and n = 2 to verify true for these cases.
A1: Deduces true for base cases and makes a **correct** assumption statement. This must include two successive cases assumed true, so e.g. as in scheme, or with $k - 2$ and $k - 1$ etc, or may assume true for all (integers) $k \le n$. But do not allow if assumed true for just k.
B1ft: Substitutes the formula for k and $k + 1$ (or their successive values) into the recurrence formula, follow through their equation from part (a).
M1: Rearranges to the form $a(1.2)^{(k+2)} + b(-0.2)^{k+2}$
A1*: Correct work leading to the correct equation for V_{k+2} and makes suitable inductive conclusion, including the ideas of "true for n = 1 and n = 2", "if true for $n = k$ and $n = k$ +1 then true for $n = k + 2$" and "hence true for all integers".

Question	Scheme	Marks	AOs
4(i)	$6^{13-1} \equiv 1(\text{mod } 13)$ or $6^{13} \equiv 6(\text{mod } 13)$	B1	1.2
	Attempts $542 = 45\times12+2$ or $542 = 41\times13+9$ (seen or implied)	M1	1.1b
	$6^{542} = \left(6^{12}\right)^{45} \times 6^2$ or $6^{542} = \left(6^{13}\right)^{41} \times 6^9$	A1	1.1b
	$\equiv 1\times6^2 \equiv \ldots(\text{mod}13)$ or $\equiv 6^{41}\times6^9 \equiv \left(6^{13}\right)^3 \times6^2\times6^9 \equiv 6^3\times6^2\times6^9 \equiv 6^{13}\times6 \equiv 6^2 \equiv \ldots(\text{mod}13)$	M1	1.1b
	$\equiv 10(\text{mod}13)$	A1	1.1b
		(5)	
(ii)(a)	$7! = 5040$	B1	1.1b
		(1)	
(b)	$4!\times4! = 576$	M1 A1	3.1b 1.1b
		(2)	
(c)	$5!\times2! = 240$	M1 A1	3.1b 1.1b
		(2)	
(d)	$7!-6!\times2! = 3600$ or $5!\times(2\times5+2\times4+2\times4+4) = 3600$	M1 A1	3.1b 1.1b
		(2)	
		(12 marks)	

Notes

(i)
B1: Recalls Fermat's Little Theorem correctly. May be implied in their work.
M1: Attempts 542 in the form 12*a* + *b*. Score if an attempt at $(6^{12})^{45}$ or similar is seen with attempt to combine with another term. Allow M1 for attempt "542 = their 12"×*a* +*b*
A1: Uses their *a* and *b* to write 6^{542} correctly in terms of 6^{12} or 6^{13} (must be one of these two)
M1: Completes the process to find the residue (more convoluted roots are possible but look for a complete process to reach a residue using their attempt at Fermat's Little Theorem at least once).
A1: Correct residue. Allow if the "45" was incorrect so long as the remainder was 2.
(ii)(a)
B1: Correct value. Accept as 7! for this part but must be evaluated in the remaining parts.
(b)
M1: Evidence that the 4 students have been considered as one unit among many and sees the problem as permutations of 4 items. Score for $4!\times k$ where $k \neq 1$
A1: Correct value
(c)
M1: Realises that the other 5 students can sit in any position – evidenced by sight of 5! (in (c))
A1: Correct value
(d)
M1: A correct strategy applied. E.g. interprets the situation as the answer to part (ii)(a) minus the ways that they can sit together So score for $7!-....$ or $5040-...$ where … is not zero.
Alternatively, considers the different positions Devindra can sit for each different position for Charles, and with 5! positions for the rest. Look for 5!×(sum of 7 terms) oe.
A1: Correct value

Question	Scheme	Marks	AOs
5(a)	$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx$ $u = \operatorname{cosec}^{n-2} x, \frac{dv}{dx} = \operatorname{cosec}^2 x, \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$	M1	2.1
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-3} x(-\operatorname{cosec} x \cot x)(-\cot x)\, dx$	A1	1.1b
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$		
	$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$ $I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)I_n + (n-2)I_{n-2}$	dM1	1.1b
	$(n-1)I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2)I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1} I_{n-2} - \frac{\operatorname{cosec}^{n-2} x \cot x}{n-1}$ *	A1*	2.1
		(4)	
(b)	$I_6 = \frac{4}{5} I_4 - \frac{\operatorname{cosec}^4 x \cot x}{5}$ or $[I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5}[I_4]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\operatorname{cosec}^4 x \cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	1.1b
	$= \frac{4}{5}\left(\frac{2}{3} I_2 - \frac{\operatorname{cosec}^2 x \cot x}{3}\right) - \frac{\operatorname{cosec}^4 x \cot x}{5}$ or with limits etc	M1	1.1b
	$[I_6]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{8}{15}[-\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{4\operatorname{cosec}^2 x \cot x}{15}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \left[\frac{\operatorname{cosec}^4 x \cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$	M1	2.1
	$= \frac{8}{15}\left(\frac{\sqrt{3}}{3}\right) + \frac{16\sqrt{3}}{135} + \frac{16\sqrt{3}}{135} = \frac{56}{135}\sqrt{3}$ *	A1*	2.2a
		(4)	
			(8 marks)

Notes

(a) **For Alt 1 and any other similar approaches marking follows the same pattern**

M1: Splits the integrand into the product as shown and begins the process of integration by parts
For Alt 2 this requires applying the expression for $\cot^2 x$ in terms of $\operatorname{cosec}^2 x$, splitting the integral and setting up the process for integration by parts on the composite term.

A1: Correct expression (for Alt 2 it is for a correct application of parts on their second term)

dM1: Depends on previous M. Applies $\cot^2 x = \pm 1 \pm \operatorname{cosec}^2 x$ and introduces I_n and I_{n-2} (For Alt 2 this is for complete substitution for I_n and I_{n-2})

A1*: Completes the proof by making I_n the subject with no errors seen (but condone minor notational slips). For Alt 1 a clear statement of replacing n by $n-2$ oe should be made.

(b)

M1: Begins process of application of reduction to find I_6 in terms of I_4 (need not evaluate terms) or deduces the value of I_2 (Alt 1)

M1: Uses the reduction formula correctly to find I_4 in terms of I_2 (need not be evaluated yet).

M1: A fully correct method using the reduction formula correctly to reach a value for I_6.
Substitutions must be shown for the non-zero terms but accept decimals/trig functions for the M.

A1*: Reaches the printed answer with no errors, relevant working shown and trig terms evaluated			
5(a) **ALT 1**	$I_n = \int \text{cosec}^{n+1}x \sin x \,dx$ (Allow with $n \pm 1$ in power for M's) $u = \text{cosec}^{n+1}x, \frac{dv}{dx} = \sin x, \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$	M1	2.1
	$I_n = -\text{cosec}^{n+1}x\cos x - (n+1)\int \text{cosec}^n x(-\text{cosec}x\cot x)(-\cos x)dx$	A1	1.1b
	$I_n = -\text{cosec}^n x\cot x - (n+1)\int \text{cosec}^n x\cot^2 x\,dx$		
	$I_n = -\text{cosec}^n x\cot x - (n+1)\int \text{cosec}^n x(\text{cosec}^2 x - 1)dx$ $I_n = -\text{cosec}^n x\cot x - (n+1)I_{n+2} + (n+1)I_n$	dM1	1.1b
	$(n+1)I_{n+2} = -\text{cosec}^n x\cot x + nI_n$		
	replacing n by $n-2$ gives $I_n = \frac{(n-2)}{n-1}I_{n-2} - \frac{\text{cosec}^{n-2}x\cot x}{n-1}$ *	A1*	2.1
		(4)	
5(a) **ALT 2**	$I_n = \int \text{cosec}^{n-2}x\,\text{cosec}^2 x\,dx = \int \text{cosec}^{n-2}x(1+\cot^2 x)dx$ $= \int \text{cosec}^{n-2}x\,dx + \left(\int \text{cosec}^{n-2}x\cot x\right)(\cot x)dx$ $u = \cot x, \frac{dv}{dx} = \text{cosec}^{n-2}x\cot x, \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$	M1	2.1
	$\underline{I_n = I_{n-2} + \cot x\left(-\frac{\text{cosec}^{n-2}x}{n-2}\right) - \int\left(-\frac{\text{cosec}^{n-2}x}{n-2}\right)(-\text{cosec}^2 x)dx}$	A1	1.1b
	$(n-2)I_n = (n-2)I_{n-2} - \text{cosec}^{n-2}x\cot x - \int \text{cosec}^n x\,dx$		
	$(n-2)I_n = (n-2)I_{n-2} - \text{cosec}^{n-2}x\cot x - I_n$	dM1	1.1b
	$(n-1)I_n = -\text{cosec}^{n-2}x\cot x + (n-2)I_{n-2}$		
	$I_n = \frac{(n-2)}{n-1}I_{n-2} - \frac{\text{cosec}^{n-2}x\cot x}{n-1}$ *	A1*	2.1
		(4)	
(b) **ALT**	$I_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \text{cosec}^2 x\,dx = [-\cot x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\sqrt{3}}{3}$	M1	2.2a
	$I_4 = \frac{2}{3}I_2 - \left[\frac{\text{cosec}^2 x\cot x}{3}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{2}{9}\sqrt{3} + \frac{4}{27}\sqrt{3}$	M1	1.1b
	$I_6 = \frac{4}{5}I_4 - \left[\frac{\text{cosec}^4 x\cot x}{5}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{4}{5}\left(\frac{4}{27}\sqrt{3} + \frac{2}{9}\sqrt{3}\right) + \frac{16}{135}\sqrt{3}$	M1	2.1
	$= \frac{56}{135}\sqrt{3}$ *	A1*	1.1b
		(4)	

Question	Scheme	Marks	AOs
6(i)	$(a*b)*c=(a+b+ab)*c=a+b+ab+c+(a+b+ab)c$	M1	2.1
	$a*(b*c)=a*(b+c+bc)=a+b+c+bc+a(b+c+bc)$	M1	2.1
	$\underline{a+b+ab+c+(a+b+ab)c}=a+b+c+bc+ab+ac+abc$ $=\underline{a+b+c+bc+a(b+c+bc)}$	A1	2.2a
	so $(a*b)*c=a*(b*c)$ which means * is associative	A1	2.4
		(4)	
(ii)(a)	$3^2=2\quad 3^3=6\quad 3^4=4\quad 3^5=5\quad 3^6=1$ or $5^2=4\quad 5^3=6\quad 5^4=2\quad 5^5=3\quad 5^6=1$	M1	2.1
	Or special case for M1A0 if powers not shown: 3 has order 6 so generates the group		
	3 (or 5) has order 6 and so generates the group so G is cyclic	A1	2.4
		(2)	
(b)	$\{1\},\ H$	B1	1.1b
	{1, 17} **or** {1, 7, 13}	M1	1.1b
	{1, 17} **and** {1, 7, 13} (and no others)	A1	1.1b
		(3)	
(c)	G: 1, 2, 3, 4, 5, 6 → H: 1, 7, 5, 13, 11, 17 **or** G: 1, 2, 3, 4, 5, 6 → H: 1, 13, 11, 7, 5, 17	M1 A1 A1	3.1a 1.1b 1.1b
		(3)	
			(12 marks)

Notes

(i)

M1: Begins proof by correctly expanding ($a*b$)*c **or** $a*(b*c)$ to an expression in a, b and c. Note they may expand as $(a*b)*c=(a*b)+c+(a*b)c=a+b+ab+c+(a+b+ab)c$ which is equally fine.

M1: Makes progress towards the required result by attempting to expand both $(a*b)*c$ **and** $a*(b*c)$, but be generous with the attempts for this method. May achieve this by working from left to right, so look for arriving at the other expression through a chain of equalities.

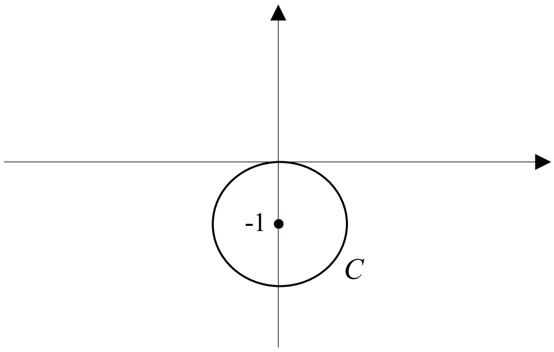
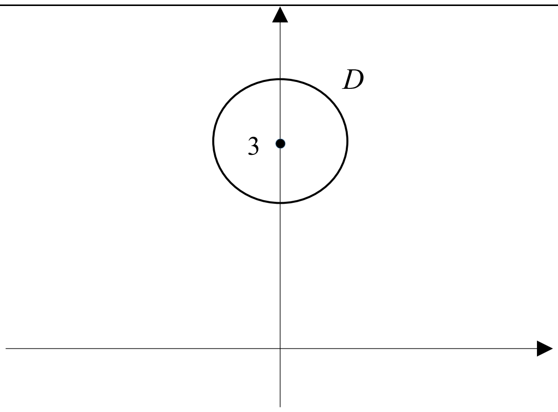
A1: For both underlined expressions (but accept eg. $c(a + b + ab)$ for $(a + b + ab)c$) and a correct expansion seen for each independently or part of a chain as shown. The expansion may have terms in different orders.

A1: Explains that $(a*b)*c=a*(b*c)$ means that * is associative. Depends on both M marks and a correct expression having been found.

(ii)(a)
M1: Demonstrates understanding of the term cyclic by either attempting all the powers of 3 or 5. Accept for this a statement $\langle 3 \rangle = \{3,2,6,4,5,1\}$ which shows the elements list in order of powers.
A1: Must have evaluated all powers of 3 or 5 correctly and explains why the group is cyclic. Accept as 3 generates the group, or as 3 has the same order of *G* as reason. Must refer to cyclic in conclusion.
Special case: Allow M1A0 for a correct explanation of why *G* is cyclic if the order of 3 (or 5) is stated as 6 without justification – but must include reference to either being a generator or having the same order as *G*.
(b) (You may ignore references to the operation for this part)
B1: Identifies {1} and *H* as subgroups
M1: Identifies {1, 17} **or** {1, 7, 13} as a subgroup
A1: Identifies {1, 17} **and** {1, 7, 13} as subgroups and no others
(c)
M1: Attempts to identify an isomorphism between the groups – may be implied by

- identifying at least 2 correct non-identity pairings or
- by attempting to rearrange group tables to have the same structure, or
- by attempting to map powers of a generator to powers of a generator e.g $(\text{their } 3)^k \to (\text{their } 5)^k$ or
- by matching of non-trivial proper subgroups to each other.

A1: Identifies 4 correct pairings, or sets up a mapping with one correct generator
A1: All pairings correct, or sets up a mapping with generators of each group correct, eg. $3^k \to 5^k$

Question	Scheme	Marks	AOs
7(a)	$w = \dfrac{3\mathrm{i}z-2}{z+\mathrm{i}} \Rightarrow z = \dfrac{2+w\mathrm{i}}{3\mathrm{i}-w}$	M1	2.1
	$\lvert z+\mathrm{i}\rvert = 1 \Rightarrow \left\lvert \dfrac{2+w\mathrm{i}}{3\mathrm{i}-w}+\mathrm{i}\right\rvert = 1$	M1	2.1
	$\left\lvert \dfrac{2+w\mathrm{i}-3-\mathrm{i}w}{3\mathrm{i}-w}\right\rvert = 1 \Rightarrow \left\lvert \dfrac{1}{w-3\mathrm{i}}\right\rvert = 1$	A1	1.1b
	$\lvert w-3\mathrm{i}\rvert = 1 \Rightarrow u^2+(v-3)^2 = 1$	A1	2.2a
		(4)	
(b)		B1	1.1b
		B1ft	1.1b
		(2)	
			(6 marks)

Notes

(a)
M1: Attempts to solve the problem by attempting to make z the subject of the formula
M1: Uses the given locus to obtain an equation in w
A1: Obtains a correct simplified equation in terms of w
A1: Deduces the correct form of the equation (allow x, y or u, v etc.)
(b)
B1: The circle C correctly positioned, passing through the origin coordinates of centre labelled. Accept as coordinates or marked on axes.
B1ft: Their D correctly positioned with the centre correctly labelled (accept as coordinates or marked on axes).
Accept both drawn on the same diagram.
Allow S.C. B1B0 for two circles in correct respective positions but with no labelling.

7(a) **ALT 1**	$w=\frac{3\mathrm{i}z-2}{z+\mathrm{i}}\Rightarrow w(z+\mathrm{i})=3\mathrm{i}(z+\mathrm{i})+3-2$ Attempts to isolate $z+\mathrm{i}$ terms	M1	2.1
	$(z+\mathrm{i})(w-3\mathrm{i})=1\Rightarrow\lvert(z+\mathrm{i})(w-3\mathrm{i})\rvert=1\Rightarrow\lvert(w-3\mathrm{i})\rvert=1$ Gathers $z+\mathrm{i}$ terms and applies $\lvert(z+\mathrm{i})\rvert=1$	M1	2.1
	As main scheme	A1	1.1b
	As main scheme	A1	2.2a
7(a) **ALT 2**	$w=\frac{3\mathrm{i}z-2}{z+\mathrm{i}}\Rightarrow z=\frac{2+w\mathrm{i}}{3\mathrm{i}-w}$ as main scheme	M1	2.1
	$x+y\mathrm{i}=\frac{2-v+u\mathrm{i}}{-u-(v-3)\mathrm{i}}\times\frac{-u+(v-3)\mathrm{i}}{-u+(v-3)\mathrm{i}}=\ldots=\frac{u-(u^2+v^2-5v+6)\mathrm{i}}{u^2+(v-3)^2}$ $\Rightarrow\left(\frac{u}{u^2+(v-3)^2}\right)^2+\left(\frac{-(u^2+v^2-5v+6)}{u^2+(v-3)^2}+1\right)^2=1$ Applies Cartesian coordinates to both sides, extracts x and y terms and attempts to apply $x^2+(y+1)^2=1$	M1	2.1
	$\left(\frac{u}{u^2+(v-3)^2}\right)^2+\left(\frac{3-v}{u^2+(v-3)^2}\right)^2=1$ Correct expression with y + 1 term combined and simplified.	A1	1.1b
	$\Rightarrow u^2+(v-3)^2=1$	A1	2.2a
7(a) **ALT 3**	$u+\mathrm{i}v=\frac{3\mathrm{i}(x+\mathrm{i}y)-2}{x+\mathrm{i}y+\mathrm{i}}\times\frac{x-(y+1)\mathrm{i}}{x-(y+1)\mathrm{i}}=\frac{\mathrm{f}(x,y)+\mathrm{g}(x,y)\mathrm{i}}{x^2+(y+1)^2}$ Applies Cartesian coordinates to expression and use complex conjugate of denominator to reach Cartesian form.	M1	2.1
	$x^2+(y+1)^2=1\Rightarrow u+iv=x+(3x^2+3y^2+5y+2)\mathrm{i}$ $\Rightarrow u=x$ and $v=3x^2+3(y+1)^2-y-1=a+by$ Uses $x^2+(y+1)^2=1$ in their equation and extract u and v as linear terms in x and y	M1	2.1
	$u=x$ and $v=2-y$ Correct u and v	A1	1.1b
	$\Rightarrow u^2+(2-v+1)^2=1\Rightarrow u^2+(v-3)^2=1$ Uses $x^2+(y+1)^2=1$ again to find correct equation.	A1	2.2a

Note that there may be attempts via identifying images of points on a diameter. If seen, send to review.

Question	Scheme	Marks	AOs
8(a)	$SA = 2\pi\int r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\pi\int r\sin\theta\sqrt{25\cos 2\theta+...}\,d\theta$ Or $SA = 2\pi\int r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\pi\int r\cos\theta\sqrt{25\cos 2\theta+...}\,d\theta$	M1	2.1
	$r^2 = 25\cos 2\theta \Rightarrow 2r\frac{dr}{d\theta} = k\sin 2\theta$ Or $r = 5\cos^{\frac{1}{2}} 2\theta \Rightarrow \frac{dr}{d\theta} = A\cos^{-\frac{1}{2}} 2\theta \times B\sin 2\theta$ (oe)	M1	2.1
	$2r\frac{dr}{d\theta} = -50\sin 2\theta$ or $\frac{dr}{d\theta} = \frac{-50\sin 2\theta}{2r}$ Or $\frac{dr}{d\theta} = \frac{5}{2}\cos^{-\frac{1}{2}} 2\theta \times -2\sin 2\theta$ (oe)	A1	1.1b
	$SA = 2\pi\int 5\sqrt{\cos 2\theta}\sin\theta\sqrt{25\cos 2\theta + \frac{25\sin^2 2\theta}{\cos 2\theta}}\,d\theta$ $= 2\pi\int 5\sqrt{\cos 2\theta}\sin\theta\frac{5}{\sqrt{\cos 2\theta}}\,d\theta = k\pi\int\sin\theta\,d\theta$ (may use cos θ)	M1	2.1
	$= 50\pi\int\sin\theta\,d\theta$ or $50\pi\int\cos\theta\,d\theta$	A1	1.1b
	$= 50\pi\int_0^{\frac{\pi}{4}}\sin\theta\,d\theta = 50\pi\left[-\cos\theta\right]_0^{\frac{\pi}{4}}$	M1	3.4
	$= 25\pi\left(2-\sqrt{2}\right)(\text{cm}^2)$	A1	2.2a
		(7)	
(b)	Adopts the correct strategy by: Attempting $\frac{dy}{d\theta}$, finding θ when $\frac{dy}{d\theta} = 0$ and using their value of θ to find CD	M1	3.1a
	$y = r\sin\theta = 5\sqrt{\cos 2\theta}\sin\theta$ $\Rightarrow \frac{dy}{d\theta} = -\frac{5\sin 2\theta\sin\theta}{\sqrt{\cos 2\theta}} + 5\sqrt{\cos 2\theta}\cos\theta$	M1	1.1b
	$\frac{dy}{d\theta} = 0 \Rightarrow 5\cos\theta - 20\cos\theta\sin^2\theta = 0 \Rightarrow \theta = ...$	M1	2.1
	E.g. $\sin^2\theta = \frac{1}{4} \Rightarrow \theta = \frac{\pi}{6}$ or $\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{6}$	A1	1.1b
	$CD = 2r\sin\frac{\pi}{6} = 2\times 5\times\sqrt{\cos\frac{\pi}{3}}\times\frac{1}{2}$; $= \frac{5\sqrt{2}}{2}(\text{cm})$*	M1; A1*	3.4 2.1
		(6)	
			(13 marks)

Notes

(a)

M1: Applies the surface area formula about the x or y axis with substitution of at least the r^2 and attempt at $\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2$ as shown in scheme. May be completed in stages, so allow if correct formula quoted and the relevant "pieces" are found. The 2π may be recovered later but must be present at some stage.

M1: Attempts to find an expression in $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ via implicit differentiation or first square rooting and then using the chain rule.

A1: Correct expression for or in $\frac{\mathrm{d}r}{\mathrm{d}\theta}$ need not be simplified.

M1: Makes a complete substitution into the SA formula and applies appropriate trigonometric identities to simplify to the form $k\pi\int \sin\theta\,\mathrm{d}\theta$ or $k\pi\int \cos\theta\,\mathrm{d}\theta$ as appropriate for their method.

A1: Obtains a correct simplified integral.

M1: Uses the model with appropriate limits to determine the surface area of the top using their integral. Note that for rotation around the x axis appropriate limits will likely be 0 and $\frac{\pi}{4}$ but may be $-\frac{\pi}{4}$ and 0. For rotation about the y axis allow this mark for limits $\frac{\pi}{4}$ and $\frac{\pi}{2}$ (note that the curve is not strictly defined for these limits).

A1: Correct expression with no errors. Must come from a correct integral – so if rotation about the y axis is used they must have made clear reference to using $r^2 = -25\cos 2\theta$ as the curve.

(b)

M1: A complete method for finding CD. Need to see the maximum identified and the length of CD calculated (watch out as r is the same value as CD so check they are finding CD)

M1: Uses the product rule correctly to differentiate $r\sin\theta$ – expect the correct form

$$\alpha\sin 2\theta\sin\theta(\cos 2\theta)^{-\frac{1}{2}} + \beta\cos\theta(\cos 2\theta)^{\frac{1}{2}}$$

M1: Makes progress by setting their derivative (which may be $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta}$) equal to 0 and proceeding via correct trig work to reach a value for θ . Various routes are possible e.g.

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow 5\cos\theta - 20\cos\theta\sin^2\theta = 0 \Rightarrow \cos\theta\left(1-4\sin^2\theta\right) = 0 \Rightarrow \sin\theta = \ldots \Rightarrow \theta = \ldots$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0 \Rightarrow \cos 3\theta = 0 \Rightarrow 3\theta = \ldots \Rightarrow \theta = \ldots$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0 \Rightarrow \tan 2\theta = \frac{1}{\tan\theta} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{1}{\tan\theta} \Rightarrow \tan^2\theta = \ldots \Rightarrow \theta = \ldots$$

A1: Correct value for θ from correct working – derivative must have been correct. May be implied by correct sin and cosine values used in formulae. SC Award for $\frac{\pi}{3}$ if using $x = r\cos\theta$

M1: Uses their value of θ in the model to find CD, ie $CD = 2\times 5\sqrt{\cos\text{"}2\theta\text{"}}\times\sin\text{"}\theta\text{"}$

Allow for use of $2r\cos\theta$ for attempts stemming from $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$

A1*:cso Correct proof

NB for $x = r\cos\theta$ used, a maximum M0M0M1A1M1A0 can be gained unless $r^2 = -25\cos 2\theta$ is used, in which case full marks is possible.

ALT for (b):
M1: for correct overall strategy of finding expression for width and maximising (via any valid method, e.g completion of square, calculus).

M1: $CD = 2r\sin\theta = 10\sqrt{\cos 2\theta \sin^2\theta} = 10\sqrt{\sin^2\theta - 2\sin^4\theta}$ forms trig expression inside square root.

M1A1: $\sin^2\theta - 2\sin^4\theta = -2\left(\sin^4\theta - \tfrac{1}{2}\sin^2\theta\right) = -2\left(\left(\sin^2\theta - \tfrac{1}{4}\right)^2 - \dfrac{1}{16}\right)$ completes the square.

Alternatively may optimise via calculus – score for full method leading to a value for θ

M1: Hence max value for *CD* is $10\times\sqrt{"\frac{1}{8}"}$. Correct method to achieved *CD*.

A1: Correct proof.

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