MODEL ANSWERS

Write your name here

Surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number

Candidate Number

Further Mathematics

Advanced

Further Mathematics Option 2

Paper 4: Further Pure Mathematics 2

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference

9FM0/4A

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B)
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Pearson

Answer ALL questions. Write your answers in the spaces provided.

1. (i) Use the Euclidean algorithm to find the highest common factor of 602 and 161.

Show each step of the algorithm.

(3)

(ii) The digits which can be used in a security code are the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Originally the code used consisted of two distinct odd digits, followed by three distinct even digits.

To enable more codes to be generated, a new system is devised. This uses two distinct even digits, followed by any three other distinct digits. No digits are repeated.

Find the increase in the number of possible codes which results from using the new system.

(4)

1i) $602 = 161(3) + 119$

$161 = 119(1) + 42$

$119 = 42(2) + 35$

$42 = 35(1) + 7$

$35 = 7(5) + 0$

$\therefore \gcd(602, 161) = \boxed{7}$

ii) originally : 2 distinct digits from 1, 3, 5, 7, 9

& 3 distinct digits from 2, 4, 6, 8

total possible codes = ${}^5P_2 \times {}^4P_3 = 480$

no. of ways to arrange 2 odd digits

no. of ways to arrange 3 even digits

new system : 2 distinct digits from 2, 4, 6, 8

3 other digits from a list of 7

total possible codes = ${}^4P_2 \times {}^7P_3 = 2520$

hence increase in # of codes = $2520 - 480$

$= \boxed{2040}$

Question 1 continued

(Total for Question 1 is 7 marks)

2. A transformation from the z-plane to the w-plane is given by

$$w = z^2$$

(a) Show that the line with equation $\text{Im}(z) = 1$ in the z-plane is mapped to a parabola in the w-plane, giving an equation for this parabola.

(4)

(b) Sketch the parabola on an Argand diagram.

(2)

a) $w = z^2$

$u + iv = (x + iy)^2$

but $\text{Im}(z) = 1$ implies $y = 1$

$\therefore u + iv = (x + i)^2$

$u + iv = x^2 + 2xi + i^2$

$= x^2 + 2xi - 1$

$u + iv = (x^2 - 1) + 2xi$

equating real components : $u = x^2 - 1$ — ①

equating imaginary components : $v = 2x$ — ②

now eliminating x from ① and ② :

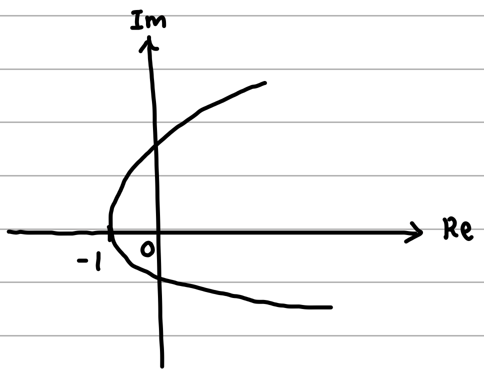
$\frac{v}{2} = x$ $\therefore u = \left(\frac{v}{2}\right)^2 - 1$

$u = \frac{v^2}{4} - 1$

$4u = v^2 - 4$

$\boxed{v^2 = 4u + 4}$

b)



$u = 0$ $v = \pm 2$

$v = 0$ $u = -1$

Question 2 continued

(Total for Question 2 is 6 marks)

3. The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$$

(a) Show that 4 is an eigenvalue of **M**, and find the other two eigenvalues. **(4)**

(b) For each of the eigenvalues find a corresponding eigenvector. **(4)**

(c) Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{MP}$ is a diagonal matrix. **(2)**

a) $M = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}$

$M - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ -1 & 0 & 4-\lambda \end{pmatrix}$

$\det(M - \lambda I) = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ -1 & 4-\lambda \end{vmatrix} + 0 = 0$

$(2-\lambda)(2-\lambda)(4-\lambda) - [4-\lambda] = 0$

$(4-\lambda)[(2-\lambda)^2 - 1] = 0$

$(4-\lambda)(\lambda^2 - 4\lambda + 3) = 0$

$(4-\lambda)(\lambda - 3)(\lambda - 1) = 0$

so $\lambda = 4$, $\lambda = 3$, $\lambda = 1$ are our solutions

hence $\boxed{4}$ is an eigenvalue as are $\boxed{3}$ and $\boxed{1}$

b) $Ax = \lambda x$

$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$

$\left.\begin{array}{ll} 2x + y = 4x & \rightarrow 2x = y \\ x + 2y = 4y & \rightarrow 2y = x \\ 4z - x = 4z & \rightarrow x = 0 \end{array}\right\}$ $x, y = 0$ z coordinate can be anything here

so an eigenvector is $\boxed{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$ (corresponding to $\lambda = 4$)

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Question 3 continued

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\left.\begin{array}{ll} 2x + y = x & \rightarrow x = -y \\ x + 2y = y & \rightarrow x = -y \\ 4z - x = z & \rightarrow 3z = x \end{array}\right\} \begin{array}{l} \text{let } x = 1 \\ \text{then } y = -1 \\ \text{and } z = \frac{1}{3} \end{array}$$

so an eigenvector is $\boxed{\begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}}$ or anything parallel to this (corresponding to $\lambda = 1$)

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\left.\begin{array}{ll} 2x + y = 3x & \rightarrow x = y \\ x + 2y = 3y & \rightarrow x = y \\ 4z - x = 3z & \rightarrow x = z \end{array}\right\} \text{so } x = y = z$$

hence eigenvector corresponding to $\lambda = 3$ is $\boxed{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

c) $\boxed{P = \begin{pmatrix} 0 & 3 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{pmatrix}}$ columns can be in any order

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Question 3 continued

Question 3 continued

(Total for Question 3 is 10 marks)

4. (i) A group G contains distinct elements a, b and e where e is the identity element and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$

(4)

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15

(a) Find the order of each element of H.

(3)

(b) Find three subgroups of H each of order 4, and describe each of these subgroups.

(4)

The elements of another group J are the matrices $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$

where $k = 1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(c) Determine whether H and J are isomorphic, giving a reason for your answer.

(2)

i) proof by contradiction:

assume $ab = ba$,

then $aab = aba$

$a^2b = aba$

but we are told $a^2b = ba$, this must mean that $a = e$ from our assumption, but a is not the identity element so this is a contradiction.

hence $ab \neq ba$

iia) $\boxed{1 \text{ has order } 1}$ (identity)

$2^2 = 4 \equiv 4$

$2^3 = 8 \equiv 4$

$2^4 = 16 \equiv 1$ (mod 15) so $\boxed{\text{order of } 2 = 4}$

$4^2 = 16 \equiv 1$ (mod 15) $\boxed{\text{so order of } 4 = 2}$

$7^2 = 49 \equiv 4$

$7^3 = 7^2 \times 7 \equiv 4 \times 7 = 28 \equiv 13$

$7^4 = 7^3 \times 7 \equiv 13 \times 7 = 91 \equiv 1$ so $\boxed{\text{order of } 7 = 4}$

Question 4 continued

$8^2 = 64 \equiv 4$

$8^3 = 8^2 \times 8 \equiv 4 \times 8 = 32 \equiv 2$

$8^4 = 8^3 \times 8 \equiv 2 \times 8 = 16 \equiv 1$ $\boxed{\text{so order of } 8 = 4}$

$11^2 = 121 \equiv 1 \pmod{15}$ so $\boxed{\text{order of } 11 = 2}$

$13^2 = 169 \equiv 4$

$13^3 = 13^2 \times 13 \equiv 4 \times 13 \equiv 52 \equiv 7$

$13^4 = 13^3 \times 13 \equiv 7 \times 13 = 91 \equiv 1$ so $\boxed{\text{order of } 13 = 4}$

$14^2 = 196 \equiv 1 \pmod{15}$ so $\boxed{\text{order of } 14 = 2}$

b) 1 must be in each subgroup

A possible subgroup is $\{1, 2, 4, 8\}$

another possibility is $\{1, 4, 7, 13\}$

note that 2 is a generator of the first and 7 is a generator of the second. They are both cyclic!

another subgroup is $\{1, 4, 11, 14\}$

↳ Klein Group!

each element has order 2

c) notice that the matrix $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$ corresponds to clockwise rotation through $\left(\frac{k\pi}{4}\right)^{\circ}$ about O.

so $k = 8$ will give us the identity matrix ($360°$ rotation) which means that the element corresponding to $k=1$ has order 8, since it takes 8 consecutive $\frac{\pi}{4}$ / $45°$ rotations to reach the identity element.

But H doesn't have an element of order 8 so they aren't isomorphic.

Question 4 continued

Question 4 continued

(Total for Question 4 is 13 marks)

5.

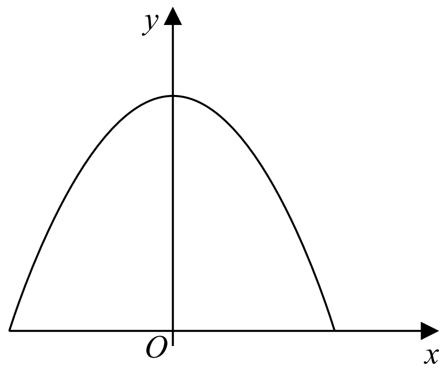

Figure 1

An engineering student makes a miniature arch as part of the design for a piece of coursework.

The cross-section of this arch is modelled by the curve with equation

$$y = A - \frac{1}{2}\cosh 2x, \qquad -\ln a \leqslant x \leqslant \ln a$$

where $a > 1$ and A is a positive constant. The curve begins and ends on the x-axis, as shown in Figure 1.

(a) Show that the length of this curve is $k\left(a^2 - \frac{1}{a^2}\right)$, stating the value of the constant k.
(5)

The length of the curved cross-section of the miniature arch is required to be 2 m long.

(b) Find the height of the arch, according to this model, giving your answer to 2 significant figures.
(4)

(c) Find also the width of the base of the arch giving your answer to 2 significant figures.
(1)

(d) Give the equation of another curve that could be used as a suitable model for the cross-section of an arch, with approximately the same height and width as you found using the first model.
(You do not need to consider the arc length of your curve)
(2)

a) arc length $= \int_{-\ln a}^{\ln a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$y = A - \frac{1}{2}\cosh 2x$

$\frac{dy}{dx} = -\frac{2}{2}\sinh 2x = -\sinh 2x$

$\frac{dy^2}{dx^2} = \sinh^2 2x$

$1+\left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 2x$

$= \cosh^2 2x$

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Question 5 continued

$\therefore$ length $= \displaystyle\int_{-\ln a}^{\ln a} \sqrt{\cosh^2 2x}\, dx$

$= \displaystyle\int_{-\ln a}^{\ln a} \cosh 2x \, dx$

$= \left[\frac{1}{2}\sinh 2x\right]_{-\ln a}^{\ln a}$

$= \left[\frac{1}{2}\sinh(2\ln a)\right] - \left[\frac{1}{2}\sinh(-2\ln a)\right]$

$\Rightarrow \sinh(2\ln a) = \dfrac{e^{2\ln a} - e^{-2\ln a}}{2} = \dfrac{a^2 - \frac{1}{a^2}}{2}$

$\Rightarrow \sinh(-2\ln a) = \dfrac{e^{-2\ln a} - e^{2\ln a}}{2} = \dfrac{\frac{1}{a^2} - a^2}{2}$

so length $= \frac{1}{2}\left(\dfrac{a^2 - \frac{1}{a^2}}{2}\right) - \frac{1}{2}\left(\dfrac{\frac{1}{a^2} - a^2}{2}\right)$

$= \frac{1}{4}\left[a^2 - \frac{1}{a^2} - \frac{1}{a^2} + a^2\right]$

$= \frac{1}{4}\left[2a^2 - \frac{2}{a^2}\right]$

$= \frac{1}{2}\left[a^2 - \frac{1}{a^2}\right]$ $\boxed{k = \frac{1}{2}}$

b) we want : $\frac{1}{2}\left(a^2 - \frac{1}{a^2}\right) = 2$

$\times 2$ $\quad a^2 - \frac{1}{a^2} = 4$

$\times a^2$ $\quad a^4 - 1 = 4a^2$

$a^4 - 4a^2 - 1 = 0$

let $a^2 = w$, $w^2 - 4w - 1 = 0$

quadratic formula: $w = 2 + \sqrt{5} = a^2$

$a = 1$ $\quad w = 2 - \sqrt{5} = a^2$

$b = -4$

$c = -1$ $\quad$ a is positive so $a^2 = 2 + \sqrt{5}$

(and $a \in \mathbb{R}$)

at $y = 0$: $A - \frac{1}{2}\cosh(2\ln a) = 0$

$x = \ln a$

so $A = \frac{1}{2}\cosh(2\ln a)$

at $x = 0$, $y =$ height $= \frac{1}{2}\cosh(2\ln a) - \frac{1}{2}\cosh(0)$

$= \frac{1}{2}\cosh(2\ln a) - \frac{1}{2}$

$\simeq \boxed{0.62\text{m}}$

c) width $= 2\ln a$ look at given domain

$\simeq \boxed{1.4\text{m}}$

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Question 5 continued

d) height (y-intercept) $\simeq 0.62$

x intercepts are ± 0.7

and the arch is quite similar to a parabola, so something of the form $y = 0.62 - \alpha x^2$ would be fine (where α is a constant)

to approximate α, substitute $y=0$, $x=0.7$

$$\alpha \simeq \frac{0.62}{(0.7)^2} = 1.19 \text{ to } 3\text{ s.f.}$$

use exact values

so $\boxed{y = 0.62 - 1.19x^2}$

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Question 5 continued

(Total for Question 5 is 12 marks)

6. A curve has equation

$$|z+6| = 2\,|z-6| \qquad z \in \mathbb{C}$$

(a) Show that the curve is a circle with equation $x^2 + y^2 - 20x + 36 = 0$

(2)

(b) Sketch the curve on an Argand diagram.

(2)

The line l has equation $az^* + a^*z = 0$, where $a \in \mathbb{C}$ and $z \in \mathbb{C}$

Given that the line l is a tangent to the curve and that $\arg a = \theta$

(c) find the possible values of $\tan\theta$

(5)

a) $|z+6| = 2|z-6|$

$|x+iy+6| = 2|x+iy-6|$

$|(x+6)+iy| = 2|(x-6)+iy|$

$(x+6)^2 + y^2 = 4[(x-6)^2+y^2]$

$x^2+12x+36+y^2 = 4(x-6)^2+4y^2$

$x^2+y^2+12x+36-4y^2 = 4(x^2-12x+36)$

$x^2-3y^2+12x+36 = 4x^2-48x+144$

$3x^2+3y^2-60x+108=0$

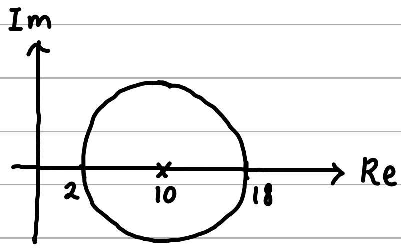
$\div 3$

$x^2+y^2-20x+36=0$

b) $(x-10)^2-100+y^2=-36$

$(x-10)^2+y^2=64$

circle centre $(10,0)$, radius $=8$



c) $a = u+iv$

$z = x+iy$

$az^* + a^*z = 0$

$(u+iv)(x-iy)+(u-iv)(x+iy) = 0$

$ux - \cancel{uiy} + \cancel{vxi} + vy + ux - \cancel{uyi} - \cancel{vix} + vy = 0$

$2ux + 2vy = 0$

$ux + vy = 0$

$y = \frac{-u}{v}x$ so l passes through O

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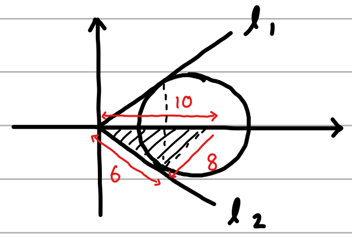
Question 6 continued

two possibilities l_1, l_2

Both have gradients $\pm\frac{8}{6} = \pm\frac{4}{3}$.

This can be seen using similar triangles if you first consider the shaded triangle.



so $\frac{-u}{v} = \pm\frac{4}{3}$

$y = \pm\frac{4}{3}x$

but we want arg a (ie $\frac{v}{u}$)

$\frac{v}{u} = \pm\frac{3}{4}$ so $\boxed{\tan\theta = \pm\frac{3}{4}}$

Question 6 continued

Question 6 continued

(Total for Question 6 is 9 marks)

7.

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx, \qquad n \geqslant 0$$

(a) Prove that, for $n \geqslant 2$,

$$nI_n = (n-1)I_{n-2}$$

(4)

(b)

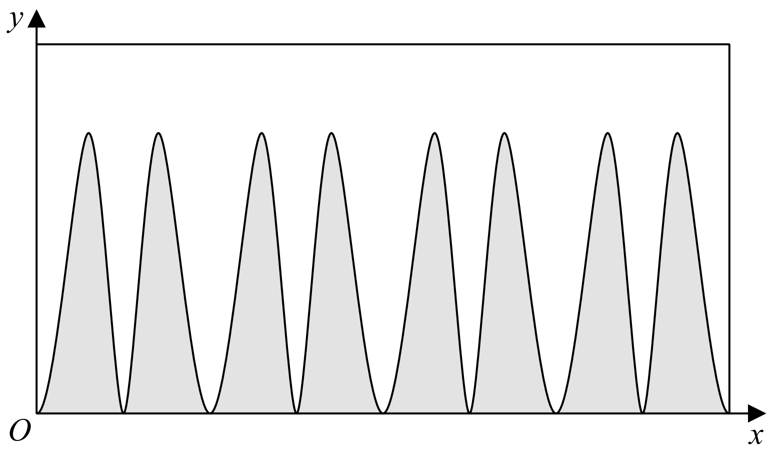


Figure 2

A designer is asked to produce a poster to completely cover the curved surface area of a solid cylinder which has diameter 1 m and height 0.7 m.

He uses a large sheet of paper with height 0.7 m and width of π m.

Figure 2 shows the first stage of the design, where the poster is divided into two sections by a curve.

The curve is given by the equation

$$y = \sin^2(4x) - \sin^{10}(4x)$$

relative to axes taken along the bottom and left hand edge of the paper.

The region of the poster below the curve is shaded and the region above the curve remains unshaded, as shown in Figure 2.

Find the exact area of the poster which is shaded.

(5)

a) $I_n = \int_0^{\pi/2} \sin^{n-1} x \; \sin x \, dx$

by parts

$\frac{dv}{dx} = \sin x \qquad u = (\sin x)^{n-1}$

$v = -\cos x \qquad \frac{du}{dx} = (n-1)(\sin^{n-2} x)(\cos x)$

$\Rightarrow \overbrace{\left[-\cos x (\sin x)^{n-1}\right]_0^{\pi/2}}^{0} - \int_0^{\pi/2} (-\cos x)(n-1)(\sin^{n-2} x)(\cos x) \, dx$

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Question 7 continued

$\Rightarrow (n-1)\int_0^{\pi/2} (\cos^2 x)(\sin^{n-2} x)\,dx$

$\Rightarrow (n-1)\int_0^{\pi/2} (1-\sin^2 x)(\sin^{n-2} x)\,dx = (n-1)\underbrace{\int_0^{\pi/2} \sin^{n-2} x\,dx}_{I_{n-2}} - (n-1)\underbrace{\int_0^{\pi/2} \sin^n x\,dx}_{I_n}$

$\therefore I_n = (n-1)I_{n-2} - (n-1)I_n$

$I_n(1+n-1) = (n-1)I_{n-2}$

hence $nI_n = (n-1)I_{n-2}$

b) $I_n = \frac{(n-1)}{n} I_{n-2}$

$I_{10} = \frac{9}{10} I_8$

$I_8 = \frac{7}{8} I_6$

$I_6 = \frac{5}{6} I_4$

$I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} (1)\,dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

so $I_{10} = \frac{9}{10}\left(\frac{7}{8}\left(\frac{5}{6}\left(\frac{3}{4}\left(\frac{1}{2}\left(\frac{\pi}{2}\right)\right)\right)\right)\right) = \frac{63\pi}{512}$

$I_2 = \frac{1}{2} I_0 = \frac{\pi}{2}$

I_n has limits 0 and $\frac{\pi}{2}$

$\text{Area} = \int_0^{\pi} \sin^2 4x - \sin^{10} 4x\,dx = 2(I_2 - I_{10})$

$= 2\left(\frac{\pi}{2} - \frac{63\pi}{512}\right)$

$= \boxed{\frac{65\pi}{256}}$

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Question 7 continued

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Question 7 continued

(Total for Question 7 is 9 marks)

8. A staircase has n steps. A tourist moves from the bottom (step zero) to the top (step n). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If u_n is the number of ways that the tourist can climb up a staircase with n steps,

(a) explain why u_n satisfies the recurrence relation

$$u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2$$

(3)

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

(1)

A staircase at a certain tourist attraction has 400 steps.

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$$

(5)

a) suppose first move is just one step ... then there are u_{n-1} ways to climb the remainder of steps, but if the first move is two steps ... then there are u_{n-2} ways to climb the other steps.

so total # of ways $= u_{n-1} + u_{n-2}$

$u_1 = 1$ since there is only 1 way to go up 1 step and similarly $u_2 = 2$ as there are 2 ways.

b) $u_3 = u_2 + u_1 = 3$

$u_4 = u_3 + u_2 = 3+2 = 5$

$u_5 = u_4 + u_3 = 5+3 = 8$

$u_6 = u_5 + u_4 = 8+5 = 13$

$u_7 = u_6 + u_5 = 13+8 = 21$

$u_8 = u_7 + u_6 = 21+13 = \boxed{34}$

c) (homogenous 2^{nd} order R.R)

auxiliary : $r^2 = r+1$

$r^2 - r - 1 = 0$

by quadratic equation : $r = \frac{1+\sqrt{5}}{2}$

$a = 1$

$b = -1$ $\qquad r = \frac{1-\sqrt{5}}{2}$

$c = -1$

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Question 8 continued

so $u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$ is our general solution

Finding A and B:

$u_1 = 1$: $1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$ — ①

$u_2 = 2$: $2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$

$2 = A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$ — ②

$2\times$① = ② $\quad A(1+\sqrt{5}) + B(1-\sqrt{5}) = A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$

$\times 2$: $\quad A(2+2\sqrt{5}) + B(2-2\sqrt{5}) = A(3+\sqrt{5}) + B(3-\sqrt{5})$

$A(1-\sqrt{5}) + B(1+\sqrt{5}) = 0$

$A = \frac{-(1+\sqrt{5})}{(1-\sqrt{5})}B$

so from ①: $1 = -B\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = -B\frac{(6+2\sqrt{5})}{2-2\sqrt{5}} + B\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = -B(-2-\sqrt{5}) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$1 = B\left(\frac{1}{2} - \frac{\sqrt{5}}{2} + 2 + \sqrt{5}\right)$

$= B\left(\frac{5+\sqrt{5}}{2}\right)$

so $B = \frac{2}{5+\sqrt{5}}$

$= \frac{5-\sqrt{5}}{10}$

and $A = \frac{-(1+\sqrt{5})}{(1-\sqrt{5})} \cdot \frac{(5-\sqrt{5})}{(10)}$

$= \frac{5+\sqrt{5}}{10}$

$\therefore u_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$

at $n = 400$, $u_{400} = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{400} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{400}$

but note $\frac{5+\sqrt{5}}{10} = \frac{1+\sqrt{5}}{2\sqrt{5}}$ $\quad$ $\times 5\sqrt{5}$ $\quad$ $\times 5\sqrt{5}$ $\quad$ this makes the simplification much easier

and $\frac{5-\sqrt{5}}{10} = \frac{-1+\sqrt{5}}{2\sqrt{5}}$ similarly

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Question 8 continued

$$\therefore u_{400} = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{400} + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{400}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{400} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{400}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{401}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$$

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Question 8 continued

(Total for Question 8 is 9 marks)

TOTAL FOR PAPER IS 75 MARKS