Pearson Edexcel

Mark Scheme (Results)

Summer 2019

Pearson Edexcel GCE Further Mathematics
AS Further Core Pure Paper 8FM0_01

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS
General Instructions for Marking

1. The total number of marks for the paper is 80.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\surd$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- □ The second mark is dependent on gaining the first mark

4. For misreading which does not alter the character of a question or materially simplify it, deduct two from any A or B marks gained, in that part of the question affected.

5. Where a candidate has made multiple responses <u>and indicates which response they wish to submit</u>, examiners should mark this response.
If there are several attempts at a question <u>which have not been crossed out</u>, examiners should mark the final answer which is the answer that is the <u>most complete</u>.

6. Ignore wrong working or incorrect statements following a correct answer.

7. Mark schemes will firstly show the solution judged to be the most common response expected from candidates. Where appropriate, alternatives answers are provided in the notes. If examiners are not sure if an answer is acceptable, they will check the mark scheme to see if an alternative answer is given for the method used.

Question	Scheme	Marks	AOs
1. (a)	$(\det(\mathbf{M}) =)\ (4)(-7)-(2)(-5)$	M1	1.1a
	$\mathbf{M}$ is non-singular because $\det(\mathbf{M}) = -18$ and so $\det(\mathbf{M}) \neq 0$	A1	2.4
		(2)	
(b)	$\text{Area } R = \dfrac{\text{Area } S}{(\pm)\lvert\det \mathbf{M}\rvert} = \ldots$	M1	1.2
	$\text{Area}(R) = \dfrac{63}{\lvert -18\rvert} = \dfrac{7}{2}$ oe	A1ft	1.1b
		(2)	
(c)	$\begin{pmatrix}4 & -5\\ 2 & -7\end{pmatrix}\begin{pmatrix}x\\ 2x\end{pmatrix} = \begin{pmatrix}4x-10x\\ 2x-14x\end{pmatrix}$	M1	1.1b
	$= \begin{pmatrix}-6x\\ -12x\end{pmatrix}$ and so all points on $y = 2x$ map to points on $y = 2x$, hence the line is invariant. OR $= -6\begin{pmatrix}x\\ 2x\end{pmatrix}$ hence $y = 2x$ is invariant.	A1	2.1
		(2)	
	(6 marks)		

Notes

(a)	M1	An attempt to find $\det(\mathbf{M})$. Just the calculation is sufficient. Site of −18 implies this mark, which may be embedded in an attempt at the inverse..
	A1	$\det(\mathbf{M}) = -18$ **and** reference to zero, e.g. $-18 \neq 0$ **and** conclusion. The conclusion may precede finding the determinant (e.g. "Non-singular if $\det(\mathbf{M}) \neq 0$, $\det(\mathbf{M}) = -18 \neq 0$" is sufficient or accept "Non-singular if $\det(\mathbf{M}) \neq 0$, $\det(\mathbf{M}) = -18$, therefore non-singular" or some other indication of conclusion.) Need not mention "det(**M**)" to gain both marks here, a correct calculation, statement $-18\neq 0$, and conclusion hence **M** is non-singular can gain M1A1.
(b)	M1	Recalls determinant is needed for area scale factor by dividing 63 by ±their determinant.
	A1ft	$\dfrac{7}{2}$ or follow through $\dfrac{63}{\lvert \text{their det}\rvert}$. Must be positive and should be simplified to single fraction or exact decimal. (Allow if made positive following division by a negative determinant.)
(c)	M1	Attempts the matrix multiplication shown or with equivalent, e.g $\begin{pmatrix}\frac{1}{2}y\\ y\end{pmatrix}$. May use $\begin{pmatrix}x\\ y\end{pmatrix}$ **and** substitute $y = 2x$ later and this is fine for the method.
	A1	Correct multiplication and working leading to conclusion that the line is invariant. If the −6 is not extracted, they must make reference to image points being on line $y = 2x$. If the −6 is extracted to show it is a multiple of $\begin{pmatrix}x\\ 2x\end{pmatrix}$ followed by a conclusion "invariant" as minimum.

Notes Continued				
Alt for (c)	$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{-18}\begin{pmatrix} -7 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \frac{-1}{18}\begin{pmatrix} -7x+10x \\ -2x+8x \end{pmatrix}$		**M1**	1.1b
	$= \frac{-1}{18}\begin{pmatrix} 3x \\ 6x \end{pmatrix}\left(= \frac{-1}{6}\begin{pmatrix} x \\ 2x \end{pmatrix}\right) \Rightarrow b = 2a$ so points on line *y* = 2*x* map to points on *y*= 2*x*, hence it is invariant.		**A1**	2.1
	Marks as per main scheme,			
Alt 2	(Since linear transformations map straight lines to straight lines…) E.g. $(1, 2)$ is on line *y* = 2*x*, and $\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4-10 \\ 2-14 \end{pmatrix}$		**M1**	1.1b
	$= \begin{pmatrix} -6 \\ -12 \end{pmatrix}$, which is also on the line *y*=2*x*, hence as (0,0) and (1,2) both map to points on *y* = 2*x* (and transformation is linear) then *y* =2*x* is invariant.		**A1**	2.1
	Notes			
	M1	Identifies a point on the line *y* = 2*x* and finds its image under *T*. If (0,0) is used there must be a clear statement it is because this is on the line, but for other points accept with any line on *y* = 2*x* without statement.		
	A1	Shows the image and another point, which may be (0,0), on *y*=2*x* both map to points on *y* = 2*x* concludes line is invariant. Need not reference transformation being linear for either mark here.		
Alt 3	$\begin{pmatrix} 4 & -5 \\ 2 & -7 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix} \Rightarrow \begin{matrix} 4x-5(mx+c) = X \\ 2x-7(mx+c) = mX+c \end{matrix}$ $\Rightarrow 2x-7(mx+c) = m\left(4x-5(mx+c)\right)+c$ $\Rightarrow \left(5m^2-11m+2\right)x+\left(5m-8\right)c = 0$ $\Rightarrow (5m-1)(m-2) = 0 \Rightarrow m = ...$ Or similar work with *c* = 0 throughout.		**M1**	2.1
	$\left(5m-8 \neq 0 \Rightarrow c = 0\right)$ Hence *m* = 2 gives an invariant line (with *c* = 0), so *y* = 2*x* is invariant.		**A1**	1.1b
	Notes			
	M1	Attempts to find the equation of a general invariant line, or general invariant line through the origin (so may have *c* = 0 throughout). To gain the method mark they must progress from finding the simultaneous equations to forming a quadratic in *m* and solving to a value of *m*.		
	A1	Correct quadratic in *m* found, with *m* = 2 as solution (ignore the other) and deduction that hence *y* = 2*x* is an invariant line. Ignore errors in the (5*m* – 8) here as *c* = 0 is always a possible solution. No need to see *c* = 0 derived.		

Question	Scheme	Marks	AOs
2.	$\{w = x+3 \Rightarrow\}\ x = w-3$	B1	3.1a
	$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (=0)$	M1	1.1b
	$2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (=0)$		
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	M1 A1 A1	3.1a 1.1b 1.1b
		(5)	
ALT 1	$\alpha + \beta + \gamma = -\frac{6}{2} = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -\frac{3}{2},\ \alpha\beta\gamma = -\frac{12}{2} = -6$	B1	3.1a
	sum roots $= \alpha + 3 + \beta + 3 + \gamma + 3$	M1	3.1a
	$= \alpha + \beta + \gamma + 9 = -3 + 9 = 6$		
	pair sum $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$		
	$= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha+\beta+\gamma) + 27$		
	$= -\frac{3}{2} + 6\times -3 + 27 = \frac{15}{2}$		
	product $= (\alpha+3)(\beta+3)(\gamma+3)$		
	$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma) + 9(\alpha+\beta+\gamma) + 27$		
	$= -6 + 3\times -\frac{3}{2} + 9\times -3 + 27 = -\frac{21}{2}$		
	$w^3 - 6w^2 + \frac{15}{2}w - \left(-\frac{21}{2}\right)\ (=0)$	M1	1.1b
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	A1 A1	1.1b 1.1b
		(5)	
	(5 marks)		

Notes

	B1	Selects the method of making a connection between x and w by writing $x = w-3$
	M1	Applies the process of substituting their $x = aw \pm b$ into $2x^3 + 6x^2 - 3x + 12\ (=0)$ So accept e.g. if $x = \frac{w}{3}$ is used.
	M1	Depends on having attempted substituting either $x = w-3$ or $x = w+3$ into the equation. This mark is for manipulating their resulting equation into the form $pw^3 + qw^2 + rw + s(=0)$ ($p \neq 0$). The "= 0" may be implied for this.
See note	**A1**	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "= 0")
	A1	Correct final equation, including "=0". Accept integer multiples.
ALT 1	**B1**	Selects the method of giving three correct equations each containing α, β and γ.
	M1	Applies the process of finding sum roots, pair sum and product.
	M1	Applies $w^3 -$ (their sum roots) $w^2 +$ (their pair sum) $w -$ (their product) (= 0) Must be correct identities, but if quoted allow slips in substitution, but the "=0" may be implied.
See note	**A1**	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "=0")
	A1	Correct final equation, including "=0". Accept multiples with integer coefficients.

Note: may use another variable than w for the first four marks, but the final equation must be in terms of w

Notes: Do not isw the final two A marks – if subsequent division by 2 occurs then mark the final answer.

Question	Scheme	Marks	AOs
3	$n=1,\ \sum_{r=1}^{1}\frac{1}{(2r-1)(2r+1)}=\frac{1}{1\times3}=\frac{1}{3}$ and $\frac{n}{2n+1}=\frac{1}{2\times1+1}=\frac{1}{3}$ (true for n=1)	B1	2.2a
	Assume general statement is true for $n=k$. So assume $\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}=\frac{k}{2k+1}$ is true.	M1	2.4
	$\left(\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\right)$ "$\frac{k}{2k+1}$" $+\frac{1}{(2k+1)(2k+3)}$	M1	2.1
	$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	dM1	1.1b
	$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$	A1	1.1b
	As $\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\frac{(k+1)}{2(k+1)+1}$ then the general result is true for $n=k+1$ As the general result has been shown to be true for $n=1$, and true for $n=k$ implies true for $n=k+1$, so the result is true for all $n\in\mathbb{N}$	A1cso	2.4
		(6)	
	(6 marks)		

Notes

B1	Substitutes $n=1$ into both sides of the statement to show they are eqaul. As a minimum expect to see $\frac{1}{1\times3}$ and $\frac{1}{2+1}$ for the substitutions. (No need to state true for n = 1 for this mark.)
M1	Assumes (general result) true for $n=k$. (Assume (true for) $n=k$ is sufficient – note that this may be recovered in their conclusion if they say e.g. if true for $n=k$ then ... etc.)
M1	Attempts to add $(k+1)$th term to their sum of k terms. Must be adding the (k+1)th term but allow slips with the sum.
dM1	Depends on previous M. Combines their two fractions over a correct common denominator for their fractions, which may be $(2k+1)^2(2k+3)$ (allow a slip in the numerator).
A1	Correct algebraic work leading to $\frac{(k+1)}{2(k+1)+1}$ or $\frac{k+1}{2k+3}$
A1	**cso** Depends on all except the **B** mark being scored (but must have an attempt to show the n = 1 case). Demonstrates the expression is the correct for $n=k+1$ (both sides must have been seen somewhere) and gives a correct induction statement with **all** three underlined statements (or equivalents) seen at some stage during their solution (so true for n = 1 may be seen at the start). For demonstrating the correct expression, accept giving in the form $\frac{(k+1)}{2(k+1)+1}$, or reaching $\frac{k+1}{2k+3}$ and stating "which is the correct form with $n=k+1$" or similar – but some indication is needed. Note: if mixed variables are used in working (r's and k's mixed up) then withhold the final A. Note: If n is used throughout instead of k allow all marks if earned.

Question	Scheme	Marks	AOs
4.	$(\mathbf{r}=)\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}$ **or** $\begin{pmatrix}-2\\5\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-3\end{pmatrix}$ (oe)	M1	1.1b
	So meet if $\begin{pmatrix}-2+\lambda\\5-\lambda\\4-3\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=-7\Rightarrow(-2+\lambda)\times1+(5-\lambda)\times-2+(4-3\lambda)\times1=-7$	M1 A1	3.1a 1.1b
	$\Rightarrow 0\lambda-8=-7\Rightarrow-8=-7$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
	(5 marks)		

Notes

M1 Forms a parametric form for the line. Allow one slip.

M1 Substitutes into the equation of the plane to an equation in λ. May use Cartesian form of plane to substitute into.

A1 Correct equation in λ

A1ft Simplifies and derives a contradiction and deduces line and plane do not meet. Follow through in their initial equation in λ so
- contradiction so no intersection if λ disappears and constants unequal
- line lies in plane if a tautology is arrived at
- meet in a point if a solution for λ is found.

But do not allow for incorrect simplification from a correct initial equation in λ

Note that a miscopy/misread of 7 instead of −7 can therefore score a maximum of M1M1A0A1A0.

A1cso Correct deduction from correct working. This may be seen two separate statements in their working. You may see attempts at showing the line is parallel before/after deducing there is no intersection.

Alt 1

Note that some may a attempt a mix of the main scheme and Alt 1. Mark under main scheme unless Alt 1 would score higher.

	Scheme	Marks	AOs
Alt 1	$\begin{pmatrix}1\\-1\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=1\times1+(-1)\times(-2)+(-3)\times1=0$	M1	3.1a
	Hence l is parallel to Π	A1	1.1b
	$(-2,5,4)$ on l, but $(1)(-2)+(-2)(5)+1(4)=-8$	M1	1.1b
	$-8\neq-7$ so $(-2,5,4)$ is not on the plane.	A1ft	2.3
	Hence l is (parallel to Π but) not in the plane.	A1cso	3.2a
		(5)	
	(5 marks)		

Alt 1 Notes

M1 Attempts the dot product between the two direction vectors.

A1 Shows dot product is zero and makes the correct deduction that line is parallel to plane.

M1 Finds a point on l and substitutes into the equation of Π (vector or Cartesian)

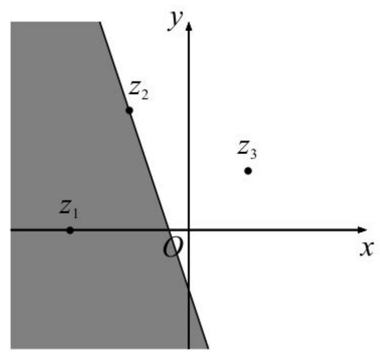
A1ft Simplifies and derives a contradiction – follow through their equation, so if arrive at a tautology, they should deduce the line is in the plane.

A1cso Correct deduction from correct working but may be split across working.

Question	Scheme	Marks	AOs
Alt 2	Attempts to solve $\frac{x+2}{1}=\frac{y-5}{-1}=\frac{z-4}{-3}$ and $x-2y+z=-7$ simultaneously – eliminates one variable for M mark.	M1	3.1a
	e.g. $y=-(x+2)+5=-x+3 \Rightarrow x-2(-x+3)+z=-7 \Rightarrow 3x+z=-1$ (oe)	A1	1.1b
	Solves reduced equations, e.g. $-3(x+2)=z-4 \Rightarrow 3x+z=-2$ and $3x+z=-1 \Rightarrow (3x+z)-(3x+z)=-2-(-1)$	M1	1.1b
	$\Rightarrow 0=-1$ a contradiction so no intersection	A1ft	2.3
	Hence l is parallel to Π but not in it.	A1cso	3.2a
		(5)	
	(5 marks)		

Alt 2 notes

M1	Attempts to solve the Cartesian equation of the line and plane, using the plane equation to eliminate one variable for the M.
A1	Correct elimination of their chosen variable. (E.g may see $3-3y+z=-7$ or $-2x-2y-2=-7$ etc)
M1	Solves the reduced equations in two variables…
A1ft	… and derives a contradiction/line and plane do not meet. Follow through their result, so may reach a tautology and deduce lies in plane, or find single solution and deduce meet in a point.
A1cso	Correct deduction from correct working.

Question	Scheme		Marks	AOs
5 (a)	Complex roots of a real polynomial occur in conjugate pairs		M1	1.2
	so a polynomial with z_1, z_2 and z_3 as roots also needs z_2* and z_3* as roots, so 5 roots in total, but a quartic has at most 4 roots, so no quartic can have z_1, z_2 and z_3 as roots.		A1	2.4
			(2)	
(b)	$\frac{z_2 - z_1}{z_3 - z_1} = \frac{-1+2\text{i}-(-2)}{1+\text{i}-(-2)} = \frac{1+2\text{i}}{3+\text{i}} \times \frac{3-\text{i}}{3-\text{i}} = \ldots$		M1	1.1b
	$= \frac{3-\text{i}+6\text{i}+2}{9+1} = \frac{5+5\text{i}}{10} = \frac{1}{2}+\frac{1}{2}\text{i}$ oe		A1	1.1b
	As $\frac{1}{2}+\frac{1}{2}\text{i}$ is in the first quadrant (may be shown by diagram), hence $\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arctan\left(\frac{1/2}{1/2}\right)\left(=\arctan(1)\right) = \frac{\pi}{4}$ *		A1*	2.1
			(3)	
(c)	$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_1) = \arg(1+2\text{i}) - \arg(3+\text{i})$		M1	1.1b
	Hence $\arctan(2) - \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$ *		A1*	2.1
			(2)	
(d)		Line passing through z_2 and the negative imaginary axis drawn.	B1	1.1b
		Area below and left of their line shaded, where the line must have negative gradient passing through negative imaginary axis but need not pass through z_2	B1	1.1b
	Unless otherwise indicated by the student mark Diagram 1(if used) if there are multiple attempts.			
			(2)	
				(9 marks)

Notes		
(a)	**M1**	Some evidence that complex roots occur as conjugate pairs shown, e.g. stated as in scheme, or e.g. identifying if $-1+2\text{i}$ is a root then so is $-1-2\text{i}$. Mere mention of complex conjugates is sufficient for this mark.
	A1	A complete argument, referencing that a quartic has at most 4 roots, but would need at least 5 for all of z_1, z_2 and z_3 as roots. There should be a clear statement about the number of roots of a quartic (e.g a quartic has four roots), and that this is not enough for the two conjugate pairs and real root.
(b)	**M1**	Substitutes the numbers in expression and attempts multiplication of numerator and denominator by the conjugate of their denominator or uses calculator to find the quotient. (May be implied.) NB Applying the difference of arguments and using decimals is M0 here.
	A1	Obtains $\frac{1}{2}+\frac{1}{2}\text{i}$. (May be from calculator.) Accepted equivalent Cartesian forms.
	A1*	Uses arctan on their quotient and makes reference to first quadrant or draws diagram to show they are in the first quadrant. to justify the argument.
(c)	**M1**	Applies the formula for the argument of a difference of complex numbers and substitutes values (may go directly to arctans if the arguments have already been established). If used in (b) it must be seen or referred to in (c) for this mark to be awarded. Allow for $\arg(z_2-z_1)-\arg(z_3-z_1)$ if z_2-z_1 and z_3-z_1 have been clearly identified in earlier work.
	A1*	Completes the proof clearly by identifying the required arguments and using the result of (b). Use of decimal approximations is A0.
(d)	**B1**	Draws a line through z_2 and passing through negative imaginary axis.
	B1	Correct side of bisector shaded. Allow this mark if the line does not pass through z_2. But it should be an attempt at the perpendicular bisector of the other two points – so have negative gradient and pass through the negative real axis. Ignore any other lines drawn for these two marks.

Question	Scheme	Marks	AOs
6. (a)	(mean $=\bar{x}=$) $\frac{1}{n}\sum_{r=1}^{n}(7+3r)$	M1	1.1a
	$\sum_{r=1}^{n}(7+3r)=\left(7\sum_{r=1}^{n}1+3\sum_{r=1}^{n}r=\right)7n+3\frac{n}{2}(n+1)$	M1	1.1b
	$\bar{x}=7+\frac{3}{2}(n+1)=\frac{14+3n+3}{2}=\frac{1}{2}(3n+17)$ *	A1*	2.1
		(3)	
(b)	Correct overall strategy to find the variance or standard deviation. This must include: • An attempt to find the mean • An attempt at $\sum(7+3r)^2$ as part of their formula (however poor, or if stated and followed by a value or if used with incorrect limits). • An attempt at either variance formula with their mean (allow slips in the formula)	M1	3.1a
(Mean)	mean $(=\bar{x})=136$	B1	1.1b
(Sum)	Way1: $\sum_{r=1}^{n}(7+3r)^2=\sum_{r=1}^{n}(49+42r+9r^2)$ $=\underline{\underline{49n}}+\underline{42\times\frac{1}{2}n(n+1)}+\underline{9\times\frac{1}{6}n(n+1)(2n+1)}$ Way 2: $\sum_{r=1}^{n}(x_i-\bar{x})^2=\sum_{r=1}^{n}(7+3r-\text{"136"})^2=a\sum_{r=1}^{n}r^2+b\sum_{r=1}^{n}r+c\sum_{r=1}^{n}1$ $=\underline{9\times\frac{1}{6}n(n+1)(2n+1)}-\text{"774"}\times\underline{\frac{1}{2}n(n+1)}+\underline{\underline{\text{"16641"}n}}$	<u>M1</u> B1	1.1b 1.1b
(Variance/standard deviation)	Way 1: $=\frac{\text{"2032690"}}{85}-136^2=\ldots$ or $\frac{\text{"2032690"}}{84}-\frac{85}{84}\times136^2=\ldots$ Way 2: $=\frac{\text{"460530"}}{85}=\ldots$ or $\frac{\text{"460530"}}{84}=\ldots$ (using sample standard deviation).	M1	1.1b
	So s.d $=\sqrt{5418}=73.6$ (g) Accept 74.0 (g) if sample s.d. used	A1	1.1b
		(6)	
	(9 marks)		

Notes		
(a)	**M1**	Selects the correct procedure for finding the mean ($\overline{x}$), attempting sum and dividing by n.
	M1	Splits the sum and applies the formulae for $\sum r$ (accept $7+3\frac{n}{2}(n+1)$ here) Or uses arithmetic series formula $\frac{1}{2}n(a+l)$ with a = 10 and l an attempt at 7 + 3×n, or $\frac{n}{2}(2a+(n-1)d)$ with a = 10 and d = 3..
	A1*	Correct work proceeding to the answer with an intermediate step shown. **Special case:** Award M0M1A0 for candidates who use $\frac{1}{2}(a+l)$ or equivalent without justification of the division by n.
(b)	**M1**	Correct overall strategy to get as far as the variance of marbles in the collection. The attempt at variance should be recognisable (though allow e.g sign slips in the formula for this mark) and an attempt, however poor, at $\sum(7+3r)^2$ must have been made
	B1	Correct value for the mean for 85 marbles (accept as a single fraction, $\frac{272}{2}$). If a student works algebraically until the last step, a correct final answer will imply this mark.
	M1	Expands brackets and applies summation formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to their expression, either in terms of n or with n = 85 **but must have correct limits**. Allow for obtaining an expression of the correct form for Way 2 if the mean is kept in terms of "n". This mark is for correct application of these two summation formula on an attempt at $\sum_{r=1}^{n}(7+3r)^2$ so accept even if this is not part of an attempt at the variance.
	B1	Correct use of $\sum_{r=1}^{n} 1 = n$ in their expression (must be correct limits).
	M1	**Correctly applies** variance or standard deviation formula with n = 85, their attempt at $\sum x^2$ (which need not be using 7 + 3r or correct limits) and their mean. Accept use of the sample variance/standard deviation is used (dividing by $n-1$) For reference the variance formula is $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2 = \left(\frac{1}{n}\sum_{i=1}^{n}x_i^2\right)-\overline{x}^2$ where $x_r = 7+3r$ here, or accept for sample variance $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2 = \left(\frac{1}{n-1}\sum_{i=1}^{n}x_i^2\right)-\frac{n\overline{x}^2}{n-1}$
	A1	Correct standard deviation to 1 decimal place. If sample standard deviation is used, the answer will be 74.0 g to 1 d.p. (74.04…)
	Note: Question specifies use of summation formula and so these must be seen for the 2nd M and 2nd B mark. However, if just 2032690 appears from a calculator all other marks are available.	

Question	Scheme	Marks	AOs
7. (a)	$\alpha+\beta+\left(\alpha+\frac{12}{\alpha}-\beta\right)=8$ so $2\alpha+\frac{12}{\alpha}=8$	M1	1.1b
		A1	1.1b
	$\Rightarrow 2\alpha^2-8\alpha+12=0$ or $\alpha^2-4\alpha+6=0$ $\Rightarrow \alpha=\frac{4\pm\sqrt{(-4)^2-4(1)(6)}}{2(1)}$ or $(\alpha-2)^2-4+6=0 \Rightarrow \alpha=...$	M1	1.1b
	$\Rightarrow \alpha=2\pm i\sqrt{2}$ are the two complex roots	A1	1.1b
	A correct full method to find the third root. Common methods are: Sum of roots = 8 $\Rightarrow$ third root $=8-\left(2+i\sqrt{2}\right)-\left(2-i\sqrt{2}\right)=...$ third root $=2+i\sqrt{2}+\frac{12}{2+i\sqrt{2}}-\left(2-i\sqrt{2}\right)=...$ Product of roots = 24 $\Rightarrow$ third root $=\frac{24}{\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)}=...$ $(z-\alpha)(z-\beta)=z^2-4z+6\Rightarrow f(z)=\left(z^2-4z+6\right)(z-\gamma)\Rightarrow\gamma=...$ (or long division to find third factor).	M1	3.1a
	Hence the roots of $f(z)=0$ are $2\pm i\sqrt{2}$ and 4	A1	1.1b
		(6)	
(b)	E.g. f(4) = 0 $\Rightarrow 4^3-8\times4^2+4p-24=0\Rightarrow p=...$ Or $p=\left(2+i\sqrt{2}\right)\left(2-i\sqrt{2}\right)+4\left(2+i\sqrt{2}\right)+4\left(2-i\sqrt{2}\right)\Rightarrow p=...$ Or $f(z)=(z-4)(z^2-4z+6)\Rightarrow p=...$	M1	3.1a
	$\Rightarrow p=22$ **cso**	A1	1.1b
		(2)	
	(8 marks)		

Notes

(a)	**M1**	Equates sum of roots to 8 and obtains an equation in just α.
	A1	Obtains a correct equation in α.
	M1	Forms a three term quadratic equation in α and attempts to solve this equation by either completing the square or using the quadratic formula to give $\alpha=....$
	A1	$\alpha=2\pm i\sqrt{2}$
	M1	Any correct method for finding the remaining root. There are various routes possible. See scheme for common ones. Allow this mark if –24 is used as the product. See note below for a less common approach.
	A1	Third root found with all three roots correct. Note α and β need not be identified.
(b)	**M1**	Any correct method of finding p. For example, applies the factor theorem, process of finding the pair sum of roots, or uses the roots to form f(z).
	A1	$p=22$ by correct solution only. Note: this can be found using only their complex roots from (a) (e.g. by factor theorem)

Note for (a) final M – it is possible to find the second and third roots using only one initial root (e.g. if second root forgotten or error leads to only one initial root being found).

Product of roots = $\alpha\beta\left(\alpha+\frac{12}{\alpha}-\beta\right)=24\Rightarrow\alpha\beta^2-\left(\alpha^2+12\right)\beta+24=0$, substitutes in α and attempts to solve the quadratic in β to achieve remaining roots. The final M can be gained once three roots in total have been obtained. (This is unlikely to be seen as part of a correct answer.) Allow if −24 has been used for the product.

PMT

Question	Scheme	Marks	AOs
8. (a)	**Note:** Allow alternative vector forms throughout, e.g row vectors, **i**, **j**, **k** notation $\mathbf{b}=\pm\left[\begin{pmatrix}300\\300\\-50\end{pmatrix}-\begin{pmatrix}-300\\400\\-150\end{pmatrix}\right]=\pm\begin{pmatrix}600\\-100\\100\end{pmatrix}$	M1	1.1b
	So $\mathbf{r}=\begin{pmatrix}-300\\400\\-150\end{pmatrix}+\lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}$ oe $\left(\text{e.g. } \mathbf{r}=\begin{pmatrix}300\\300\\-50\end{pmatrix}+\lambda\begin{pmatrix}6\\-1\\1\end{pmatrix}\right)$	A1	2.5
		(2)	
(b)(i)	$k = 200$	B1	2.2a
	If M is the point on mountain, and X a general point on the line then eg. $\overrightarrow{MX}=\begin{pmatrix}-300\\400\\-150\end{pmatrix}+\lambda\begin{pmatrix}600\\-100\\100\end{pmatrix}-\begin{pmatrix}100\\k\\100\end{pmatrix}=\begin{pmatrix}-400+600\lambda\\400-k-100\lambda\\-250+100\lambda\end{pmatrix}=\begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix}$ May be in terms of k or with k = 200 used.	M1	3.1b
	e.g. $\begin{pmatrix}-400+600\lambda\\200-100\lambda\\-250+100\lambda\end{pmatrix}\bullet\begin{pmatrix}600\\-100\\100\end{pmatrix}=0\Rightarrow\lambda=...$	dM1	1.1b
	So e.g. $\overrightarrow{OX}=\begin{pmatrix}-300\\400\\-150\end{pmatrix}+\frac{3}{4}\begin{pmatrix}600\\-100\\100\end{pmatrix}=...$	M1	3.4
	So coordinates of X are (150, 325, −75) Accept as $\begin{pmatrix}150\\325\\-75\end{pmatrix}$	A1	1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{(150-100)^2+(325-200)^2+(-75-100)^2}=...$	M1	1.1b
	Awrt 221m from correct working, so λ must have been correct. (Must include units)	A1	1.1b
		(2)	
(c)	$\left\lvert\overrightarrow{OP}\right\rvert=\sqrt{(-300)^2+400^2+(-150)^2}\approx 522$ $\left\lvert\overrightarrow{OQ}\right\rvert=\sqrt{300^2+300^2+50^2}\approx 427$	M1	1.1b
	New tunnel length is signficantly shorter than these values so it is likely that the company will decide to build the accessway. Reason and conclusion needed.	A1ft	2.2b
		(2)	
(d)	E.g. The mountainside is not likely to be flat so a plane may not be a good model. The tunnel and/or pipeline will not have negligible thickness so modelling as lines may not be appropriate. A shortest length tunnel may not be possible, or most practical, as the strata of the rock in the mountain have not been considered by the model.	B1	3.5b
		(1)	
	(12 marks)		

Notes		
(a)	**M1**	Attempts the direction between positions *P* and *Q*. If no method shown, two correct entries imply the method.
	A1	A correct equation in the correct form. Any point on the line may used, and any non-zero multiple of the direction. Must begin **r** = …
(b)		**Note:** mark part (b) as a whole.
(i)	**B1**	Correct value of *k* deduced.
	M1	Realises the need to find the distance from the point on the mountain to a general point on the line.
	dM1	Takes the dot product with the direction vector of line and sets to zero and proceeds to find a value of λ. If working with *k* as well, allow for finding either λ in terms of *k* or *k* in terms of λ.
	M1	Substitutes their λ into their line equation. (This may not have come from correct work, but the method is for using the line equation here.) May be implied by two out of three correct coordinates for their λ **Note:** May omit this step and substitute λ into $\overrightarrow{MX}$. This gains M0 here, but can gain M1A1 in (ii) for finding the length of $\overrightarrow{MX}$.
	A1	Correct point.
(b)(ii)	**M1**	Uses the distance formula with their point and *M*, or with their $\overrightarrow{MX}$ from (i). (May be implied by two out of three correct coordinates for their λ)
	A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m
(c)	**M1**	Calculates the two distances *OP* and *OQ*.
	A1ft	Makes an appropriate conclusion for their tunnel length, but distances *OP* and *OQ* must be correct. A reason and a conclusion is needed. Accept for reason e.g "significantly shorter" or "tunnel is more than 100m less than either existing accessway", as these act as a comparative judgement. But do not accept just "shorter" or just inequalities given with no comparative evidence.
(d)	**B1**	Any appropriate criticism of the model given. The model must be referred to in some way – e.g. criticise the straightness/thickness of line, flatness of plane or lack of taking strata etc of mountain into account (as e.g this means line may not be straight). Note: reference to measurements not being correct is **NOT** a limitation of the model.

<u>For reference</u> Some of the other common equations/values of λ in (b)(i) are:

$$\overrightarrow{MX} = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} -400+6\lambda \\ 200-\lambda \\ -250+\lambda \end{pmatrix} \Rightarrow \lambda = 75$$

$$\overrightarrow{MX} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} + \lambda \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200+600\lambda \\ 100-100\lambda \\ -150+100\lambda \end{pmatrix} \Rightarrow \lambda = -\frac{1}{4}$$

$$\overrightarrow{MX} = \begin{pmatrix} 300 \\ 300 \\ -50 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200+6\lambda \\ 100-\lambda \\ -150+\lambda \end{pmatrix} \Rightarrow \lambda = -25$$

(If the negative direction vectors are used in any case, the value of λ is just the negative of the above.)
See Appendix for some alternatives to part (b)

Question	Scheme	Marks	AOs
9.	A correct overall strategy, an attempt at integrating y^2 with respect to x combine in some way with the volume of revolution formula (use of $\pi\int y^2\,\mathrm{d}x$ or $\alpha\int y^2\,\mathrm{d}x$ for any variable α is fine) followed by attempt to find an angle/form an equation in θ	M1	3.1a
	$y^2 = kx^{\frac{2}{3}} + \ldots + \dfrac{m}{x^{\frac{4}{3}}}$ or $y^2 = kx^{\frac{2}{3}} + \ldots + mx^{-\frac{4}{3}}$ where … is one or two more terms.	M1	1.1b
	$y^2 = 4x^{\frac{2}{3}} + 4x^{-\frac{1}{3}} + x^{-\frac{4}{3}}$ or $y^2 = 4x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + x^{-\frac{4}{3}} + 2x^{-\frac{1}{3}}$ (oe)	A1	1.1b
	$\int y^2\,\mathrm{d}x = \int 4x^{\frac{2}{3}} + \dfrac{4}{x^{\frac{1}{3}}} + \dfrac{1}{x^{\frac{4}{3}}}\,\mathrm{d}x = \alpha x^{\frac{5}{3}} + \beta x^{\frac{2}{3}} + \gamma x^{-\frac{1}{3}}$	M1	1.1b
	$= \dfrac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \dfrac{3}{x^{\frac{1}{3}}}$ (oe)	A1ft A1	1.1b 1.1b
	$\dfrac{\theta}{2}\left[\dfrac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \dfrac{3}{x^{\frac{1}{3}}}\right]_{\frac{1}{8}}^{8} = \dfrac{461}{2}$ $\Rightarrow \dfrac{\theta}{2}\left[\left(\dfrac{12\times 8^{\frac{5}{3}}}{5} + 6\times 8^{\frac{2}{3}} - \dfrac{3}{8^{\frac{1}{3}}}\right) - \left(\dfrac{12\times\left(\frac{1}{8}\right)^{\frac{5}{3}}}{5} + 6\times\left(\dfrac{1}{8}\right)^{\frac{2}{3}} - \dfrac{3}{\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right)\right] = \dfrac{461}{2} \Rightarrow \theta = \ldots$ OR $\pi\left[\dfrac{12x^{\frac{5}{3}}}{5} + 6x^{\frac{2}{3}} - \dfrac{3}{x^{\frac{1}{3}}}\right]_{\frac{1}{8}}^{8} = \pi\left[\left(\dfrac{12\times 8^{\frac{5}{3}}}{5} + 6\times 8^{\frac{2}{3}} - \dfrac{3}{8^{\frac{1}{3}}}\right) - \left(\dfrac{12\times\left(\frac{1}{8}\right)^{\frac{5}{3}}}{5} + 6\times\left(\dfrac{1}{8}\right)^{\frac{2}{3}} - \dfrac{3}{\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right)\right] = ..$ followed by $\dfrac{\theta}{2\pi}\times\ldots = \dfrac{461}{2} \Rightarrow \theta = \ldots$	M1	3.1a
	$\theta = \dfrac{40}{9}$ (radians)	A1	1.1b
		(8)	
	(8 marks)		

Notes		
	M1	A correct overall strategy, either finding full volume rotated by 2π first, then performing some kind of scaling, or using $\alpha\int y^2\mathrm{d}x$ for a variable α (ideally $\frac{\theta}{2}$, but for the strategy accept with any variable multiple), to form an equation in just the angle.
	M1	Attempting to square y to a three or four term expression. Look for correct powers on first and last term with some term(s) in the middle.
	A1	Correct expansion in three or four terms – award when first seen.
	M1	Integrates y^2 w.r.t. x. Must have at least two terms in their y^2 with fractional indices. Power to be increased by 1 in at least two terms.
	A1ft	Two terms of integral correct. Follow through on their expansion. Need not be simplified.
	A1	Fully correct integral. Need not be simplified. May still be four terms
	M1	Either : Substitutes limits and subtracts correct way round (must be seen or implied by the answer), and equates to $\frac{461}{2}$ if using $\frac{1}{2}\theta\int y^2\mathrm{d}x$ and proceeds to find θ. Or : Substitutes limits and subtracts correct way round (seen or implied) and multiplies by π to get the full volume AND then multiplies the result by $\frac{\theta}{2\pi}$ before equating to $\frac{461}{2}$. **The method must be correct for this mark – so they must be using $\frac{\theta}{2}\int y^2\mathrm{d}x$ directly or $\pi\int y^2\mathrm{d}x$ and scale by $\frac{\theta}{2\pi}$ when setting equal to $\frac{461}{2}$**
	A1	Correct angle found. Accept $\frac{40}{9}$, awrt 4.44 or awrt $255°$ (as long as the degrees units are made clear – do not accept just 255) isw once a correct value of θ is found.

<u>Special case</u> The question specified that algebraic integration must be used, so use of a calculator to find the integral cannot score the marks for integration but may be allowed the strategy and answer marks. A maximum of M1M0A0M0A0A0M1A1 is available in such cases.
Expanding y^2 first but showing no integration can score the second M and first A (if earned) as well.

Note that $\int_{1/8}^{8}\left(2x^{1/3}+x^{-2/3}\right)^2\,\mathrm{d}x=\frac{4149}{40}=103.725$ but just this alone is worth **no marks**. There must be an attempt to incorporate this within a strategy to gain access to marks.

Question	Scheme	Marks	AOs
10. (a)	a represents the proportion of juvenile chimpanzees that (survive and) **remain** juvenile chimpanzees the next year.	B1	3.4
		(1)	
(b)(i)	Determinant $= 0.82a - 0.08\times 0.15$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \ldots\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	M1	1.1b
	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1} = \dfrac{1}{0.82a - 0.012}\begin{pmatrix} 0.82 & -0.15 \\ -0.08 & a \end{pmatrix}$	A1	1.1b
		(3)	
(ii)	$\begin{pmatrix} a & 0.15 \\ 0.08 & 0.82 \end{pmatrix}^{-1}\begin{pmatrix} 15360 \\ 43008 \end{pmatrix} = \dfrac{1}{0.82a - 0.012}\begin{pmatrix} 0.82\times 15360 - 0.15\times 43008 \\ (-0.08)\times 15360 + 43008a \end{pmatrix}$ OR forms equations $15360 = aJ_0 + 0.15\times A_0$ $43008 = 0.08\times J_0 + 0.82\times A_0$	M1	3.1a
	$\dfrac{1}{0.82a - 0.012}\left[6144 + \left(43008a - 1228.8\right)\right] = 64000$ $\Rightarrow 4915.2 + 43008a = 64000(0.82a - 0.012) \Rightarrow a = \ldots$ OR $A_0 = 64000 - J_0 \Rightarrow 43008 = 0.08\times J_0 + 0.82\times\left(64000 - J_0\right) = J_0 = \ldots$ $\Rightarrow a = \dfrac{15360 - \left(64000 - J_0\right)}{J_0} = \ldots$	M1	3.1a
	$a = \dfrac{5683.2}{9472} = 0.60$	A1	1.1b
		(3)	
(iii)	Initial juvenile population $= \dfrac{\text{"6144"}}{\text{"0.48"}} = 12800$	M1	3.4
	So change of 2560 juvenile chimpanzees	A1	1.1b
		(2)	
(c)	As the number of juveniles has increased, the model is not initially predicting a decline, so is not suitable in the short term. (Follow through their answer to (b) – but they must have made an attempt at it to find at least a value for J_0)	B1ft	3.5a
		(1)	
(d)	Third category needs to be introduced for chimpanzees aged 40 and above, mature chimpanzees M_n, and a matrix multiplication of increased dimension set up. Accept $3\times 3, 3\times 2$ or 2×3 matrices including all three categories in the column vector.	M1	3.5c
	The corresponding matrix model will have the form $\begin{pmatrix} J_{n+1} \\ A_{n+1} \\ M_{n+1} \end{pmatrix} = \begin{pmatrix} a & b & \underline{\mathbf{0}} \\ 0.08 & c & 0 \\ 0 & d & e \end{pmatrix}\begin{pmatrix} J_n \\ A_n \\ M_n \end{pmatrix}$ (The underlined zero must be correct but do not be concerned about any values used in the other entries.)	A1	3.3
		(2)	
			(12 marks)

Notes		
(a)	**B1**	Correct interpretation. Need not mention survival but must be clear it is the (proportion of) **juveniles that remain as juveniles** the next year (ie those that survive but don't progress to adulthood). E.g. accept "(number of) juveniles who do not become adults" but do not accept "surviving juveniles".
		Mark part (b) as a whole.
(b)(i)	**M1**	Attempts the determinant in terms of a Allow miscopies for the attempt. Allow $0.82a - 0.12$ as a slip.
	M1	Attempts the form of the inverse, swapped leading diagonals and sign changed on both off diagonals. Allow miscopies of the numbers but the signs must be correct.
	A1	Correct inverse matrix
(ii)	**M1**	Use the inverse matrix and attempts to find the initial juvenile and adult populations. . (May have determinant 1 for this mark.) Alternatively, sets up simultaneous equations from the original system, $15360 = aJ_0 + 0.15\times A_0$ and $43008 = 0.08\times J_0 + 0.82\times A_0$ Accept with J_n and A_n or other appropriate variables.
	M1	Uses the sum of initial populations equals 64000 in an attempt to find a. (May have determinant 1 for this mark.) If using alternative, use of e.g. $A_0 = 64000 - J_0$ in second equation to find J_0, followed by attempt to find a. Award for an attempt to solve the equations, but don't be too concerned with the algebraic process as long as they are attempting to use all three equations.
	A1	Correct value, $a = 0.6$ (or 0.60 or $\frac{3}{5}$).
(iii)	**M1**	Uses their a to find the value of J_0. This mark may be gained for work done in (ii) if the alternative has been used but must have come from a correct method.
	A1	Correct difference found, as long as there is no contradictory statement – so "decrease of 2560" is A0.
(c)	**B1ft**	Comments that the change is an increase so does not fit the model. Follow through their answer to (b) as long as at least a value for J_0 has been found. If a decrease has been found allow for commenting the model is suitable. If an answer is given to (b)(iii), follow through on whatever their answer is. If no answer has been given, but an initial population found, a comparison should be made between this value and 153600 with conclusion must be consistent with their answer for J_0
(d)	**M1**	Introduces a third category (may be *M*ature, *E*lderly or any suitable letter used) and sets up a matrix multiplication (the left hand side may be missing for this mark) with all three categories in the column vector. The dimension of the matrix should be 3 in at least either row or column, and there should be a 3×1 vector.
	A1	Sets up the new matrix equation, including both sides and making clear the zero (underlined) so that the correct progression that no new juveniles arise from the mature chimpanzees is clear. Overlook other values, though ideally the other two zeroes are shown too, to indicate mature chimpanzees do not regress to adulthood, and juveniles cannot proceed directly to mature chimpanzees.

Appendix: Alternatives to 8(b)

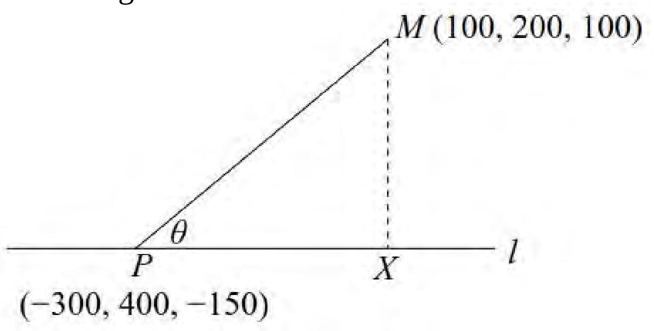
Note that variations may occur with the line equation chosen in part (a), but mark as follows:

Question	Scheme	Marks	AOs
Alt 1 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $=380000\left(\lambda - \frac{3}{4}\right)^2 + 48750 \Rightarrow \lambda = \ldots$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{\text{"48750"}} = \ldots$	M1	1.1b
	Awrt 221m from correct working, so completion of square must have been correct. (Must include units)	A1	1.1b
		(2)	
	Notes		
(i)	**B1M1** As per main scheme. **M1** Realises the need to find the distance from the point on the mountain to a general point on the line. **dM1** Attempts the distance or distance squared of $\overrightarrow{MX}$, expands and completes the square to find the value of λ for which distance is minimum. May obtain other forms for the completed square. Look for $A(B\lambda - C)^2 - D + \text{"262500"}$ where $A, B, C, D \neq 0$ but B may be 1. **M1A1** As per main scheme.		
(ii)	**M1** Correct method for the distance. May be as per main scheme, or via extracting from the completed square constant term. **A1** Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m		
Alt 2 (b)(i)	As per main scheme.	B1 M1	2.2a 3.1b
	$d^2 = (-400+600\lambda)^2 + (200-100\lambda)^2 + (-250+100\lambda)^2$ $= 380000\lambda^2 - 570000\lambda + 262500$ $\frac{\mathrm{d}}{\mathrm{d}x}(d^2) = 0 \Rightarrow 760000\lambda - 570000 = 0 \Rightarrow \lambda = \ldots$	dM1	1.1b
	As per main scheme.	M1 A1	3.4 1.1b
		(5)	
(ii)	Length of tunnel is $\sqrt{(150-100)^2 + (325-200)^2 + (-75-100)^2} = \ldots$	M1	1.1b
	Awrt 221m from correct working, differentiation etc must have been correct. (Must include units)	A1	1.1b
		(2)	

Notes				
		As per main scheme except for:		
(i)	**dM1**	Attempts the distance or distance squared of $\overrightarrow{MX}$, differentiates and set to zero to find λ for minimum distance.		
(ii)	**M1**	May substitute λ into the distance squared formula to find distance.		
Alt 3 (b)(i)		$k = 200$	B1	2.2a
		If M is the point on mountain, then e.g (may use Q rather than P) $\overrightarrow{MP} = \begin{pmatrix} -400 \\ 200 \\ -250 \end{pmatrix} \Rightarrow \cos\theta = \dfrac{\begin{pmatrix} -400 \\ 200 \\ -250 \end{pmatrix} \cdot \begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}}{\sqrt{(-400)^2 + 200^2 + (-250)^2}\sqrt{600^2 + (-100)^2 + 100^2}}$ $\Rightarrow \cos\theta = \ldots$ or $\theta = \ldots$ (where θ is the angle between the line and $\overrightarrow{MP}$)	M1	3.1b
		$\Rightarrow \lvert\overrightarrow{PX}\rvert = \lvert\overrightarrow{MP}\rvert \cos\theta = \ldots$	dM1	1.1b
		So e.g. $\overrightarrow{OX} = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \dfrac{\lvert\overrightarrow{PX}\rvert}{\left\lvert\begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix}\right\rvert}\begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} = \begin{pmatrix} -300 \\ 400 \\ -150 \end{pmatrix} + \dfrac{"75\sqrt{8}"}{100\sqrt{38}}\begin{pmatrix} 600 \\ -100 \\ 100 \end{pmatrix} = \ldots$	M1	3.4
		So coordinates of X are (150, 325, −75) Accept as $\begin{pmatrix} 150 \\ 325 \\ -75 \end{pmatrix}$	A1	1.1b
			(5)	
(ii)		Length of tunnel is $\lvert\overrightarrow{MP}\rvert \sin\theta = \ldots$ (oe)	M1	1.1b
		Awrt 221m from correct working. (Must include units)	A1	1.1b
			(2)	

Notes		
(i)	**B1**	Correct value of k deduced.
	M1	Finds $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$) and attempts scalar product formula with this and the direction of the line to find the angle or cosine of the angle between line and $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$)
	dM1	Uses their angle with the cosine to find the length of $\overrightarrow{PX}$ (or $\overrightarrow{QX}$). Accept equivalent trigonometric methods (e.g. finding opposite side first and using tangent or Pythagoras.
	M1	Uses the length of and $\overrightarrow{PX}$ (or $\overrightarrow{QX}$) to find the coordinates of the point on the line at shortest distance from M.
	A1	Correct point.
(ii)	**M1**	Correct method for the distance. May be as per main scheme, or use of sine ratio with their angle between the line and and $\overrightarrow{MP}$ (or $\overrightarrow{MQ}$). Accept equivalent trigonometric methods.
	A1	Correct distance, including units. Accept awrt 221 m or $25\sqrt{78}$ m

Useful diagram:



Note for P, $\cos\theta = \pm\dfrac{57}{\sqrt{38}\sqrt{105}}$,

$\theta = 25.5\ldots°$ and $\lvert\overrightarrow{PX}\rvert = 75\sqrt{38}$

For Q $\cos\theta = \pm\dfrac{19}{\sqrt{38}\sqrt{29}}$,

$\theta = 55.08\ldots°$, $\lvert\overrightarrow{QX}\rvert = 25\sqrt{38}$

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