Mark Scheme (Result)

November 2021

Pearson Edexcel GCE Further Mathematics
Advanced Level in
Core Pure Mathematics Paper 2
9FM0/02

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General Marking Guidance

- All candidates must receive the same treatment. Examiners must mark the first candidate in exactly the same way as they mark the last.
- Mark schemes should be applied positively. Candidates must be rewarded for what they have shown they can do rather than penalised for omissions.
- Examiners should mark according to the mark scheme not according to their perception of where the grade boundaries may lie.
- There is no ceiling on achievement. All marks on the mark scheme should be used appropriately.
- All the marks on the mark scheme are designed to be awarded. Examiners should always award full marks if deserved, i.e. if the answer matches the mark scheme. Examiners should also be prepared to award zero marks if the candidate's response is not worthy of credit according to the mark scheme.
- Where some judgement is required, mark schemes will provide the principles by which marks will be awarded and exemplification may be limited.
- When examiners are in doubt regarding the application of the mark scheme to a candidate's response, the team leader must be consulted.
- Crossed out work should be marked UNLESS the candidate has replaced it with an alternative response.

EDEXCEL GCE MATHEMATICS

General Instructions for Marking

1. The total number of marks for the paper is 75.

2. The Edexcel Mathematics mark schemes use the following types of marks:

- **M** marks: method marks are awarded for 'knowing a method and attempting to apply it', unless otherwise indicated.
- **A** marks: Accuracy marks can only be awarded if the relevant method (M) marks have been earned.
- **B** marks are unconditional accuracy marks (independent of M marks)
- Marks should not be subdivided.

3. Abbreviations

These are some of the traditional marking abbreviations that will appear in the mark schemes.

- bod – benefit of doubt
- ft – follow through
- the symbol $\sqrt{}$ will be used for correct ft
- cao – correct answer only
- cso - correct solution only. There must be no errors in this part of the question to obtain this mark
- isw – ignore subsequent working
- awrt – answers which round to
- SC: special case
- oe – or equivalent (and appropriate)
- dep – dependent
- indep – independent
- dp decimal places
- sf significant figures
- ∗ The answer is printed on the paper
- ☐ The second mark is dependent on gaining the first mark

4. All A marks are 'correct answer only' (cao.), unless shown, for example, as A1 ft to indicate that previous wrong working is to be followed through. After a misread however, the subsequent A marks affected are treated as A ft, but manifestly absurd answers should never be awarded A marks.

Question	Scheme	Marks	AOs
1(a) (i)	$\|z_1z_2\| = 3\sqrt{2}$	B1	1.1b
(ii)	$\arg(z_1z_2) = \frac{\pi}{3} + \left(-\frac{\pi}{12}\right) = \frac{\pi}{4}$ o.e.	B1	1.1b
		(2)	
(b) (i)	$n = 8$	B1ft	2.2a
(ii)	$\|w^n\| = (\text{'their}\|z_1z_2\|\text{'})^{\text{their } n}$	M1	1.1b
	$\|w^n\| = 104\,976$	A1	1.1b
		(3)	
			(5 marks)

Notes:

(a)

(i)

B1: Deduces $|z_1z_2| = 3\sqrt{2}$

(ii)

B1: Deduces $\arg(z_1z_2) = \frac{\pi}{4}$ o.e

These marks may be awarded for $z_1z_2 = 3\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

(b)

(i)

B1ft: 2π divided by their $\arg(z_1z_2)$ found in part (a) (ii) to give an integer

Alternatively smallest positive integer multiple required to make their argument a multiple of 2π

(ii)

M1: Their answer to (a) (i) to the power of their *n*.

A1: 104 976

Question	Scheme		Marks	AOs
2	$\begin{pmatrix}4 & -2\\5 & 3\end{pmatrix}\begin{pmatrix}x\\mx+c\end{pmatrix}=\begin{pmatrix}X\\mX+c\end{pmatrix}$ leading to an equation in x, m, c and X		M1	3.1a
	$4x-2(mx+c)=X$ and $5x+3(mx+c)=mX+c$		A1	1.1b
	$5x+3(mx+c)=m\left(4x-2(mx+c)\right)+c$ leading to $5+3m=4m-2m^2$ $\qquad$ ($3c=-2mc+c$)		M1	2.1
	$2m^2-m+5=0\Rightarrow b^2-4ac=$ $(-1)^2-4(2)(5)=\ldots$	Solves $3c=-2mc+c\Rightarrow m=\ldots$	dM1	1.1b
	Correct expression for the discriminant = $\{-39\}<0$ therefore there are no invariant lines.	$m=-1$ and shows a contradiction in $5+3m=4m-2m^2$ therefore there are no invariant lines.	A1	2.4
	Alternative $\begin{pmatrix}4 & -2\\5 & 3\end{pmatrix}\begin{pmatrix}x\\mx\end{pmatrix}=\begin{pmatrix}X\\mX\end{pmatrix}$ leading to an equation in x, m and X		M1	3.1a
	$4x-2(mx)=X$ and $5x+3(mx)=mX$		A1	1.1b
	$5x+3(mx)=m\left(4x-2(mx)\right)$ leading to $5+3m=4m-2m^2$		M1	2.1
	$2m^2-m+5=0\Rightarrow b^2-4ac=(-1)^2-4(2)(5)=\ldots$		dM1	1.1b
	Correct expression for the discriminant = $\{-39\}<0$ therefore there are no invariant lines that pass through the origin no invariant lines.		A1	2.4
			(5)	
			(5 marks)	

Notes:

M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation.

A1: Correct equations.

M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m.

dM1: Dependent on the previous method, finds the value of the discriminant, this can be seen in an attempt to solve the quadratic using the formula.

Alternatively solves $3c=-2mc+c$ and finds a value for m

Note: If the quadratic equation in m is solved on a calculator and complex roots given this is M0 as they are not showing why there are no real roots.

A1: Correct expression for the discriminant, states < 0 and draws the required conclusion.
Alternatively, correct value for m, shows a contradiction in $5+3m=4m-2m^2$ and draws the required conclusion.

Alternative

M1: Sets up a matrix equation in an attempt to find a fixed line and extract at least one equation.

A1: Correct equations.

M1: Eliminates X from the simultaneous equations and equates the coefficients of x leading to a quadratic equation in terms of m.

dM1: Dependent on the previous method, finds the value of the discriminant.

A1: Correct expression for the discriminant, states < 0 and draws the required conclusion.

Question	Scheme	Marks	AOs
3(a)	$\mathrm{f}'(x)=A\left(1-x^2\right)^{-\frac{1}{2}}$ $\quad \mathrm{f}''(x)=Bx\left(1-x^2\right)^{-\frac{3}{2}}$ and $\mathrm{f}'''(x)=C\left(1-x^2\right)^{-\frac{3}{2}}+Dx^2\left(1-x^2\right)^{-\frac{5}{2}}$ or $\dfrac{C\left(1-x^2\right)^{\frac{3}{2}}+Dx^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$	M1	2.1
	$\mathrm{f}'(x)=\left(1-x^2\right)^{-\frac{1}{2}}$ or $\dfrac{1}{\sqrt{1-x^2}}$ $\quad \mathrm{f}''(x)=x\left(1-x^2\right)^{-\frac{3}{2}}$ or $\dfrac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$ and $\mathrm{f}'''(x)=\left(1-x^2\right)^{-\frac{3}{2}}+3x^2\left(1-x^2\right)^{-\frac{5}{2}}$ or $\dfrac{1}{\left(1-x^2\right)^{\frac{3}{2}}}+\dfrac{3x^2}{\left(1-x^2\right)^{\frac{5}{2}}}$ from quotient rule $\dfrac{\left(1-x^2\right)^{\frac{3}{2}}+3x^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$	A1	1.1b
	Finds $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$ and applies the formula $\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\mathrm{f}''(0)\dfrac{x^2}{2}+\mathrm{f}'''(0)\dfrac{x^3}{6}$ $\{\mathrm{f}(0)=0,\ \mathrm{f}'(0)=1,\ \mathrm{f}''(0)=0,\ \mathrm{f}'''(0)=1\}$	M1	1.1b
	$\mathrm{f}(x)=x+\dfrac{x^3}{6}$ cso	A1	1.1b
		(4)	
(b)	$\arcsin\left(\dfrac{1}{2}\right)=\dfrac{1}{2}+\dfrac{\left(\frac{1}{2}\right)^3}{6}=\dfrac{\pi}{6}\Rightarrow \pi=\ldots$	M1	1.1b
	$\pi=\dfrac{25}{8}$ o.e.	A1ft	2.2b
		(2)	
		(6 marks)	

Notes:

(a)

M1: Finds the correct form of the first three derivatives, may be unsimplified – the third may come later.

A1: Correct first three derivatives, may be unsimplified – the third may come later.

M1: Finds $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$ and applies to the correct formula, needs to go up to x^3.

A1: $x+\dfrac{x^3}{6}$ cso ignore any higher terms whether correct or not

Special case: If they think that their $\mathrm{f}''(0)\neq 0$ then maximum score M1 A0 M1 A0

M1 for correct form of the first two derivatives

M1 Correctly uses their $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and applies to the correct formula

Note: If candidates do not find the first three derivatives but use $f(0)=0,\ f'(0)=1,\ f''(0)=0,\ f'''(0)=1$ and use these correctly in the formula this can score M0 A0 M1 A0
(b) **M1:** Substitutes $x=\frac{1}{2}$ into both sides and rearranges to find $\pi = \ldots$ **A1ft:** Infers that $\pi=\frac{25}{8}$ o.e. Follow through their $6f\left(\frac{1}{2}\right)$

Question	Scheme	Marks	AOs
4(a)	A complete attempt to find the sum of the cubes of the first *n* odd numbers using three of the standard summation formulae. Attempts to find $\sum(2r+1)^3$ or $\sum(2r-1)^3$ by expanding and using summation formulae	M1	3.1a
	$\sum_{r=1}^{n}(2r-1)^3=\sum_{r=1}^{n}(8r^3-12r^2+6r-1)=8\sum_{r=1}^{n}r^3-12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$ or $\sum_{r=0}^{n-1}(2r+1)^3=\sum_{r=0}^{n-1}(8r^3+12r^2+6r+1)=8\sum_{r=0}^{n-1}r^3+12\sum_{r=0}^{n-1}r^2+6\sum_{r=0}^{n-1}r+\sum_{r=0}^{n-1}1$	M1	1.1b
	$=8\frac{n^2}{4}(n+1)^2-12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)-n$ or $=8\frac{(n-1)^2}{4}(n)^2+12\frac{(n-1)}{6}(n)(2n-1)+6\frac{(n-1)}{2}(n)+n$	M1 A1	1.1b 1.1b
	Multiplies out to achieve a correct intermediate line for example $n\ \ n+1\ \ 2n^2-2n+1\ \ -n=2n^4-2n^3+n^2+2n^3-2n^2+n-n$ $2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$ leading to $=n^2(2n^2-1)$ cso *	A1 *	2.1
		(5)	
(b)	$\sum_{r=n}^{n+9}(2r-1)^3=\sum_{r=1}^{n+9}(2r-1)^3-\sum_{r=1}^{n-1}(2r-1)^3$ $=(n+9)^2(2(n+9)^2-1)-(n-1)^2(2(n-1)^2-1)=99800$ or $\sum_{r=n+1}^{n+10}(2r-1)^3=\sum_{r=1}^{n+10}(2r-1)^3-\sum_{r=1}^{n}(2r-1)^3$ $=(n+10)^2(2(n+10)^2-1)-(n)^2(2n^2-1)=99800$ or $\sum_{r=n-9}^{n}(2r-1)^3=\sum_{r=1}^{n}(2r-1)^3-\sum_{r=1}^{n-10}(2r-1)^3$ $=(n)^2(2(n)^2-1)-(n-10)^2(2(n-10)^2-1)=99800$	M1	3.1a
	$80n^3+960n^2+5820n-86760=0$ or $80n^3+1200n^2+7980n-79900=0$ or $80n^3-1200n^2+7980n-119700=0$	A1	1.1b
	Solves cubic equation	dM1	1.1b

	Achieves $n=6$ and the smallest number as 11 or Achieves $n=5$ and the smallest number as 11 or Achieves $n=15$ and the smallest number as 11	A1	2.3
		(4)	
			(9 marks)

Notes:

(a)

M1: A complete attempt to find the sum of the cubes of *n* odd numbers using three of the standard summation formulae.

M1: Expands $\sum_{r=1}^{n}(2r-1)^3$ or $\sum_{r=0}^{n-1}(2r+1)^3$ and splits into fours appropriate sums.

M1: Applies the result for at least three summations $\sum_{r=0}^{n-1}r^3, \sum_{r=0}^{n-1}r^2, \sum_{r=0}^{n-1}r$ and $\sum_{r=0}^{n-1}1$ or $\sum_{r=1}^{n}r^3, \sum_{r=1}^{n}r^2, \sum_{r=1}^{n}r$ and $\sum_{r=1}^{n}1$ as appropriate to their expansion provided that there is an attempt at cubing some values.

A1: Correct unsimplified expression.

A1 *: Multiplies out to achieve a correct intermediate expression which clearly leads to the correct expression. cso

Special case: If uses $\sum_{r=1}^{n}(2r+1)^3$ leading to $=8\frac{n^2}{4}(n+1)^2+12\frac{n}{6}(n+1)(2n+1)+6\frac{n}{2}(n+1)+n$ max score is M1 M0 M1 A1 A0

(b)

M1: Uses the answer to part (a) to find the sum of the cubes of the first $N+10$ odd numbers minus the sum of the first *N* odd numbers and sets equal to 99800 or equivalent.

A1: Correct simplified cubic equation.

dM1: Uses their calculator to solve their cubic equation, dependent on previous method mark.

A1: cao

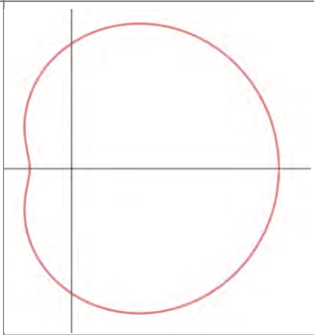
Question	Scheme	Marks	AOs
5(a)	$\dfrac{dy}{dx}=\dfrac{-\lambda}{\sqrt{1-\beta x^2}}$ where $\lambda>0$ and $\beta>0$ and $\beta\neq 1$ Alternatively $2\cos y = x \Rightarrow \dfrac{dx}{dy}=\alpha\sin y \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\alpha\sin y}$	M1	1.1b
	$\dfrac{dy}{dx}=\dfrac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}x^2}}$ or $\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{1-\frac{1}{4}x^2}}$ o.e. or $\dfrac{dy}{dx}=-\dfrac{1}{2\sin y}$ or	A1	1.1b
	States that $\dfrac{dy}{dx}\neq 0$ therefore C has no stationary points. Tries to solve $\dfrac{dy}{dx}=0$ and ends up with a contradiction e.g. -1 = 0 therefore C has no stationary points. As cosec $y>1$ therefore C has no stationary points.	A1	2.4
		(3)	
(b)	$\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{1-\frac{1}{4}\times 1^2}}=\left\{-\dfrac{1}{\sqrt{3}}\right\}$	M1	1.1b
	Normal gradient $=-\dfrac{1}{m}$ and $y-\dfrac{\pi}{3}=m_n(x-1)$ Alternatively $\dfrac{\pi}{3}=m_n(1)+c \Rightarrow c=...\left\{\dfrac{\pi}{3}-\sqrt{3}\right\}$ and then $y=m_n x+c$	M1	1.1b
	$y=0\Rightarrow 0-\dfrac{\pi}{3}=\sqrt{3}(x_A-1)\Rightarrow x_A=...\left\{1-\dfrac{\pi}{3\sqrt{3}}\text{ or }1-\dfrac{\pi\sqrt{3}}{9}\right\}$ and $x=0\Rightarrow y_B-\dfrac{\pi}{3}=\sqrt{3}(0-1)\Rightarrow y_B=...\left\{\dfrac{\pi}{3}-\sqrt{3}\right\}$	M1	3.1a
	Area $=\dfrac{1}{2}\times x_A\times -y_B=\dfrac{1}{2}\left(1-\dfrac{\pi}{3\sqrt{3}}\right)\left(\sqrt{3}-\dfrac{\pi}{3}\right)$	M1	1.1b
	Area $\dfrac{1}{54}\left(27\sqrt{3}-18\pi+\sqrt{3}\pi^2\right)$ $(p=27, q=-18, r=1)$	A1	2.1
		(5)	
			(8 marks)

Notes:

(a)

M1: Finds the correct form for $\dfrac{dy}{dx}$

A1: Correct $\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1:** States or shows that $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ and draws the required conclusion. This mark can be scored as long as the M mark has been awarded.
(b) **M1:** Substitutes $x=1$ into their $\frac{\mathrm{d}y}{\mathrm{d}x}$ **M1:** Finds the normal gradient and finds the equation of the normal using $y-\frac{\pi}{3}=m_n(x-1)$ **M1:** Finds where their normal cuts the x-axis and the y-axis. **M1:** Finds the area of the triangle OAB $=\frac{1}{2}\times x_A \times -y_B$. **A1:** Correct area Special case: If finds the tangent to the curve, the x and y intercepts and the area of the triangle max score M1 M0 M1 M0 A0 Note common error $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{\sqrt{1-\frac{1}{4}x^2}}$ In part (b) this leads to $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2}{\sqrt{3}}$ leading to normal gradient $\frac{\sqrt{3}}{2}$ and $y=\frac{\sqrt{3}}{2}x-\frac{\sqrt{3}}{2}+\frac{\pi}{3}$ and $\left(0,\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$ and $\left(1-\frac{2\pi}{3\sqrt{3}},0\right)$ therefore area $=\frac{1}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3\sqrt{3}}-1\right)$ This can score M1 M1 M1 M1 A0

Question	Scheme	Marks	AOs
6(a)	$x = r\cos\theta = a\left(p + 2\cos\theta\right)\cos\theta$ Leading to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta\left(p + 2\cos\theta\right)$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin\theta\cos\theta + \beta\sin\theta$ or $x = a\left(p\cos\theta + 2\cos^2\theta\right) = a\left(\cos 2\theta + p\cos\theta + 1\right)$ leading to $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \alpha\sin 2\theta + \beta\sin\theta$	M1	3.1a
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\left[-2\sin\theta\cos\theta - \sin\theta\left(p + 2\cos\theta\right)\right]$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -4a\sin\theta\cos\theta - ap\sin\theta$ or $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2a\sin 2\theta - ap\sin\theta$	A1	1.1b
	$a\left[-2\sin\theta\cos\theta - \sin\theta\left(p + 2\cos\theta\right)\right] = 0$ $\pm a\left(4\sin\theta\cos\theta + p\sin\theta\right) = 0$ $a\sin\theta\left(4\cos\theta + p\right) = 0$ Either $\sin\theta = 0$ or $\cos\theta = -\frac{p}{4}$	M1	3.1a
	$\sin\theta = 0$ implies 2 solutions (tangents which are perpendicular to the initial line) e.g. $\theta = 0, \pi$	B1	2.2a
	Therefore two solutions to $\cos\theta = -\frac{p}{4}$ are required $-\frac{p}{4} > -1 \Rightarrow p < 4$ as p is a positive constant $2 < p < 4$ *	A1*	2.4
		(5)	
(b)	 Correct shape and position. Condone cusp	B1	2.2a
		(1)	
(c)	Area = $2 \times \frac{1}{2}\int_0^{\pi}\left[20\left(3 + 2\cos\theta\right)\right]^2 \mathrm{d}\theta = 400\int_0^{\pi}\left(9 + 12\cos\theta + 4\cos^2\theta\right)\mathrm{d}\theta$ or $= \int_0^{\pi}\left(3600 + 4800\cos\theta + 1600\cos^2\theta\right)\mathrm{d}\theta$	M1	3.4

	or $\frac{1}{2}\int_0^{2\pi}\left[20(3+2\cos\theta)\right]^2\,d\theta = 200\int_0^{2\pi}(9+12\cos\theta+4\cos^2\theta)\,d\theta$ or $=\int_0^{2\pi}(1800+2400\cos\theta+800\cos^2\theta)\,d\theta$		
	$\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta \Rightarrow$ $A=...\int(9+12\cos\theta+2+2\cos 2\theta)\,d\theta=\alpha\theta\pm\beta\sin\theta\pm\lambda\sin 2\theta$	M1	3.1a
	$=400[11\theta+12\sin\theta+\sin 2\theta]$ or $=200[11\theta+12\sin\theta+\sin 2\theta]$	A1	1.1b
	Using limits $\theta=0$ and $\theta=\pi$ or $\theta=0$ and $\theta=2\pi$ as appropriate and subtracts the correct way round provided there is an attempt at integration $=400[11\pi-0]=4400\pi=13823.0\,(\text{cm}^2)$ or $=200[11(2\pi)-0]=4400\pi=13823.0\,(\text{cm}^2)$	M1	1.1b
	Volume $=$ area $\times 90=396\,000\pi=1\,244\,070.691\,(\text{cm}^3)$	M1	3.4
	time $=\frac{1\,244\,070.691}{50\,000}=...$ or volume = 1244 litres therefore time = $\frac{1244}{50}=...$	M1	2.2b
	25 (minutes)	A1	3.2a
		(7)	
(d)	For example Polar equation is not likely to be accurate. Some comment that the sides will not be smooth and draws an appropriate conclusion. The hole may not be uniform depth The pond may leak/ ground may absorb some water	B1	3.5b
		(1)	
			(14 marks)

Notes:

(a)

M1: Complete method to find the correct form for $\frac{dx}{d\theta}$

A1: Correct $\frac{dx}{d\theta}$

M1: Sets $\frac{dx}{d\theta}=0$ and factorises to find values for either $\sin\theta$ or $\cos\theta$.

B1: Deduces that as $\sin\theta=0$ this provides two tangents. This can be implied by 2 values for θ

A1*: Concludes that as $\cos\theta=-\frac{p}{4}>-1\Rightarrow p<4$ and p is a positive constant $\therefore\ 0<p<4$

(b) **B1:** Correct shape and position.
(c) **M1:** Uses the model to find the area of the cross section $2\times\frac{1}{2}\int_0^{\pi}\left[20\left(3+2\cos\theta\right)\right]^2 \mathrm{d}\theta$ or $\frac{1}{2}\int_0^{2\pi}\left[20\left(3+2\cos\theta\right)\right]^2 \mathrm{d}\theta$ **M1:** Uses the identity $\cos 2\theta = 2\cos^2\theta - 1$ to integrate to the required form. **A1:** Correct integration. **M1:** Uses limits $\theta = 0$ and $\theta = \pi$ or $\theta = 0$ and $\theta = 2\pi$ as appropriate and subtracts the correct way around provided there is an attempt at integration. Note if first M1 is not awarded for incorrect limits then award this mark for their limits used. **M1:** Multiplies their area by 90 (cm). **M1**: Divides their volume by 50000 **A1:** 25 (minutes)
(d) **B1:** See scheme for examples. Any reference to the flow of water is B0

Question	Scheme		Marks	AOs
7(a)	Using $\operatorname{arsinh}\alpha = \frac{1}{2}\ln 3$ $\alpha = \dfrac{e^{\frac{1}{2}\ln 3} - e^{-\frac{1}{2}\ln 3}}{2}$	$\ln\left(\alpha + \sqrt{\alpha^2+1}\right) = \frac{1}{2}\ln 3$	B1	1.2
	$\alpha = \dfrac{\sqrt{3} - \frac{1}{\sqrt{3}}}{2} \Rightarrow \alpha = \ldots$	$\alpha + \sqrt{\alpha^2+1} = \sqrt{3}$ $\sqrt{\alpha^2+1} = \sqrt{3} - \alpha$ $\alpha^2 + 1 = 3 - 2\sqrt{3}\alpha + \alpha^2 \Rightarrow \alpha = \ldots$	M1	1.1b
	$\alpha = \dfrac{\sqrt{3}}{3}$ or $\dfrac{1}{\sqrt{3}}$		A1	2.2a
			(3)	
(b)	Volume = $\pi\int_0^{\frac{1}{2}\ln 3} \sinh^2 y \, dy$		B1	2.5
	$\{\pi\}\int\left(\dfrac{e^y - e^{-y}}{2}\right)^2 dy = \{\pi\}\int\left(\dfrac{e^{2y} - 2 + e^{-2y}}{4}\right) dy$ or $\{\pi\}\int \frac{1}{2}\cosh 2y - \frac{1}{2} \, dy$		M1	3.1a
	$\frac{1}{4}\left(\frac{1}{2}e^{2y} - 2y - \frac{1}{2}e^{-2y}\right)$ or $\frac{1}{4}\sinh 2y - \frac{1}{2}y$		dM1 A1	1.1b 1.1b
	Use limits $y = 0$ and $y = \frac{1}{2}\ln 3$ and subtracts the correct way round		M1	1.1b
	$\dfrac{\pi}{4}\left(\dfrac{4}{3} - \ln 3\right)$ or exact equivalent		A1	1.1b
			(6)	
			(9 marks)	

Notes:

(a)

B1: Recalls the definition for $\sinh\left(\frac{1}{2}\ln 3\right)$ or forms an equation for arcsinh x

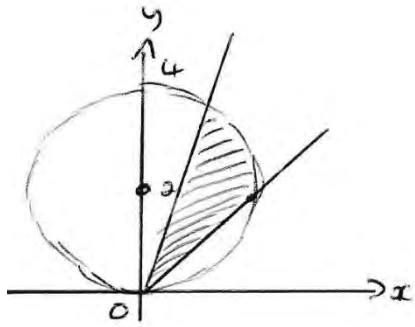
M1: Uses logarithms to find a value for α or forms and solves a correct equation without log

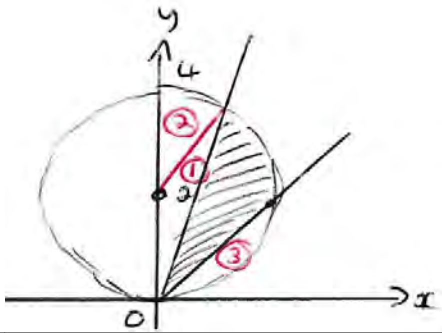
A1: Deduces the correct exact value for α

Note using the result

$$\ln\left(\frac{1}{\sqrt{3}} + \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}\right) = \ln\left(\frac{1}{\sqrt{3}} + \sqrt{\frac{4}{3}}\right) = \ln\sqrt{3} = \frac{1}{2}\ln 3 \text{ therefore } \operatorname{arsinh}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2}\ln 3$$

B1 for substituting in α into arcsinh$x$, M1 for rearranging to show $\frac{1}{2}\ln 3$, A1 for conclusion
(b) **B1:** Correct expression for the volume $\pi\int_0^{\frac{1}{2}\ln 3} \sinh^2 y \, dy$ requires integration signs, dy and correct limits. **M1:** Uses the exponential formula for sinh y or the identity $\cosh 2y = \pm 1 \pm 2\sinh^2 y$ to write in a form which can be integrated at least one term **dM1:** Dependent of previous method mark, integrates. **A1:** Correct integration. **M1:** Correct use of the limits $y = 0$ and $y = \frac{1}{2}\ln 3$ **A1**: Correct exact volume.

Question	Scheme	Marks	AOs
8(i)	$\lvert z\rvert = \sqrt{6^2+6^2} = \ldots\ 6\sqrt{2}$ or $\sqrt{72}$ and $\arg z = \tan^{-1}\left(\frac{6}{6}\right) = \ldots\left\{\frac{\pi}{4}\right\}$ Can be implied by $r = 6\sqrt{2}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{5}$ to their argument $z = 6\sqrt{2}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} \times \mathrm{e}^{\frac{2\pi k}{5}\mathrm{i}}$ or $z = 6\sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+\frac{2\pi k}{5}\right)\right]$	M1	1.1b
	$z = r\mathrm{e}^{\left(\theta+\frac{2\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta+\frac{4\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta+\frac{6\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta+\frac{8\pi}{5}\right)\mathrm{i}}$ o.e. or $z = r\mathrm{e}^{\left(\theta+\frac{2\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta-\frac{2\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta-\frac{6\pi}{5}\right)\mathrm{i}},\ r\mathrm{e}^{\left(\theta-\frac{8\pi}{5}\right)\mathrm{i}}$ o.e.	A1ft	1.1b
	$z = 6\sqrt{2}\mathrm{e}^{\frac{13\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{\frac{21\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{\frac{29\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{\frac{37\pi}{20}\mathrm{i}}$ o.e. or $z = 6\sqrt{2}\mathrm{e}^{\frac{13\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{-\frac{19\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{-\frac{11\pi}{20}\mathrm{i}},\ 6\sqrt{2}\mathrm{e}^{-\frac{3\pi}{20}\mathrm{i}}$ o.e.	A1	1.1b
		(5)	
(ii)(a)	Circle centre (0, 2) and radius 2 or  with the point on the origin	B1	1.1b
	Fully correct 	B1	1.1b
		(2)	
(ii)(b)	area $= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(4\sin\theta\right)^2 \mathrm{d}\theta$ or area $= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\alpha\sin\theta\right)^2 \mathrm{d}\theta$	M1	3.1a
	Uses $\sin^2\theta = \frac{1}{2}-\frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta + B\sin 2\theta$ area $= 8\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2\theta\ \mathrm{d}\theta = 4\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1-\cos 2\theta\ \mathrm{d}\theta = 4\theta - 2\sin 2\theta$	M1	3.1a
	Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around $\left[4\left(\frac{\pi}{3}\right)-2\sin\left(\frac{2\pi}{3}\right)\right]-\left[4\left(\frac{\pi}{4}\right)-2\sin\left(\frac{2\pi}{4}\right)\right]$	M1	1.1b

	Area $= \frac{\pi}{3} - \sqrt{3} + 2$	A1	1.1b
		(4)	
	Alternative 		
	Finds either the areas 1 or 2 Area 1 $= \frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right)\left\{= \sqrt{3}\right\}$ Area 2 $= \frac{1}{2} \times 2^2 \times \frac{\pi}{3}\left\{= \frac{2\pi}{3}\right\}$	M1	1.1b
	A complete method to find area 3 Area 3 $= \frac{1}{4}\pi \times 2^2 - \frac{1}{2} \times 2^2 \{= \pi - 2\}$	M1	3.1a
	A complete method to find the required area Shaded area = Area of semi circle − area 1 − area 2 − area 3 $= \left[\frac{1}{2}\pi \times 2^2\right] - \left[\frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right)\right] - \left[\frac{1}{2} \times 2^2 \times \frac{\pi}{3}\right] - \left[\frac{1}{4}\pi \times 2^2 - \frac{1}{2} \times 2^2\right]$ $= 2\pi - \sqrt{3} - \frac{2\pi}{3} - (\pi - 2)$ Or Shaded area = Area of sector − area 1 − area 3 $= \left[\frac{1}{2} \times 4 \times \left(\frac{2\pi}{3}\right)\right] - \left[\frac{1}{2} \times 2^2 \times \sin\left(\frac{2\pi}{3}\right)\right] - \left[\frac{1}{4}\pi \times 2^2 - \frac{1}{2} \times 2^2\right]$ $= \frac{4\pi}{3} - \sqrt{3} - (\pi - 2)$	M1	3.1a
	Area $= \frac{\pi}{3} - \sqrt{3} + 2$	A1	1.1b
		(4)	
			(11 marks)

Notes:

(i)
M1: Finds the modulus and argument of z
A1: Correct modulus and argument of z

M1: Uses a correct method to find to all the other 4 vertices of the pentagon. Must be doing the equivalent of adding/ subtracting multiples of $\frac{2\pi}{5}$ to the argument. **A1ft**: All 4 vertices following through on their modulus and argument. Does not need to be simplified for this mark. **A1:** All 4 vertices correct in the required form
(ii)(a) **B1:** Circle centre (0, 2) and radius 2 or  with the vertex on the origin. **B1:** Fully correct region shaded.
(ii) (b) **M1:** Writes the required area using polar coordinates **M1:** Uses $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and integrates to the form $A\theta + B\sin 2\theta$ **M1:** Uses the limits of $\frac{\pi}{4}$ and $\frac{\pi}{3}$ and subtracts the correct way around. Must be some attempt at $\text{area} = \frac{1}{2}\int \alpha\sin\theta^{\,2}\,\mathrm{d}\theta$ and integration. **A1:** Correct exact area $= \frac{\pi}{3} - \sqrt{3} + 2$
Alternative **M1:** Finds either area 1 or area 2 **M1:** A complete method to find the area 3 **M1:** A complete method to find the required area $=$ Area of semi circle $-$ area 1 $-$ area 2 $-$ area 3 or $=$ Area of sector $-$ area 1 $-$ area 3 **A1:** Correct exact area $= \frac{\pi}{3} - \sqrt{3} + 2$

Question	Scheme	Marks	AOs
9(a)	$\dfrac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\text{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\text{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\text{i}\sin 3\theta)+...$	M1	3.1a
	$\dfrac{1}{1-z}=\dfrac{1}{1-\frac{1}{2}(\cos\theta+\text{i}\sin\theta)}\times\dfrac{1-\frac{1}{2}\cos\theta+\frac{1}{2}\text{i}\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}\text{i}\sin\theta}$ or $\dfrac{1}{1-z}=\dfrac{2}{2-(\cos\theta+\text{i}\sin\theta)}\times\dfrac{2-(\cos\theta-\text{i}\sin\theta)}{2-(\cos\theta-\text{i}\sin\theta)}$	M1	3.1a
	$\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+...\right\}=\dfrac{\frac{1}{2}\sin\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $\left\{\frac{1}{2}(\sin\theta)+\frac{1}{4}(\sin 2\theta)+\frac{1}{8}(\sin 3\theta)+...\right\}=\dfrac{2\sin\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+...=\dfrac{\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}=\dfrac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
	Alternative $1+z+z^2+z^3+...$ $=1+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+\text{i}\sin\theta)\right)^3+...$ $=1+\frac{1}{2}(\cos\theta+\text{i}\sin\theta)+\frac{1}{4}(\cos 2\theta+\text{i}\sin 2\theta)+\frac{1}{8}(\cos 3\theta+\text{i}\sin 3\theta)+...$	M1	3.1a

	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}e^{i\theta}}\times\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}}=\frac{4-2e^{-i\theta}}{5-2\left(e^{i\theta}+e^{-i\theta}\right)}=\frac{4-2(\cos\theta-i\sin\theta)}{5-2(2\cos\theta)}$	M1	2.1
	Select the imaginary part $\frac{2\sin\theta}{5-4\cos\theta}$	M1	1.1b
	$\frac{1}{2}\sin\theta+\frac{1}{4}\sin 2\theta+\frac{1}{8}\sin 3\theta+...=\frac{2\sin\theta}{5-4\cos\theta}$ *	A1*	1.1b
		(5)	
(b)(ii)	$\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}=0\Rightarrow\cos\theta=2$	M1	3.1a
	As $(-1\le)\cos\theta\le 1$ therefore there is no solution to $\cos\theta=2$ so there will also be a real part, hence the sum cannot be purely imaginary.	A1	2.4
	Alternative 1 Imaginary part is $\frac{4-2\cos\theta}{5-4\cos\theta}=\frac{1}{2}+\frac{3}{2(5-4\cos\theta)}$	M1	3.1a
	$-1\le\cos\theta\le 1$ therefore $\frac{1}{6}<\frac{3}{2(5-4\cos\theta)}<\frac{3}{2}$ so sum must contain real part	A1	2.4
	Alternative 2 $\frac{1}{1-z}=ki\Rightarrow z=1+\frac{i}{k}$	M1	3.1a
	mod $z>1$ contradiction hence cannot be purely imaginary	A1	2.4
		(2)	
			(8 marks)

Notes:

(a)
B1: See scheme

(b)(i)

M1: Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem.

M1: Substitutes $z=\frac{1}{2}(\cos\theta+i\sin\theta)$ into their answer to part (a) and rationalises the denominator.

M1: Equates the imaginary terms.

M1: Multiplies out the denominator and simplifies by using the identity $\cos^2\theta+\sin^2\theta=1$

A1*: cso. Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series. Alternative **M1:** Substitutes $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into at least 3 terms of the series and applies de Moivre's theorem. **M1:** Substitutes $z=\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ into their answer to part (a) and rationalises the denominator. **M1:** Uses $\mathrm{e}^{-\mathrm{i}\theta}=\cos\theta-\mathrm{i}\sin\theta$ and $\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}=2\cos\theta$ to express in terms of $\sin\theta$ and $\cos\theta$ **M1:** Select the imaginary terms. **A1*:** cso Achieves the printed answer having substituted $z=\frac{1}{2}(\cos\theta+\mathrm{i}\sin\theta)$ into 4 terms of the series.
(b)(ii) **M1:** Setting the real part of the series = 0 and rearranges to find $\cos\theta=...$ **A1:** See scheme
Alternative 1 **M1:** Rearranges imaginary part so that $\cos\theta$ only appears once **A1:** Uses $-1\le\cos\theta\le1$ to show that the sum must always be positive so must contain a real part
Alternative 2 **M1:** Sets sum as purely imaginary and rearranges to make *z* the subject **A1:** Shows a contradiction and draws an appropriate conclusion