Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Time 1 hour 30 minutes

Paper reference **9FM0/02**

Further Mathematics

Advanced
PAPER 2: Core Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

1. A student was asked to answer the following:

For the complex numbers $z_1 = 3 - 3\mathrm{i}$ and $z_2 = \sqrt{3} + \mathrm{i}$, find the value of $\arg\left(\frac{z_1}{z_2}\right)$

The student's attempt is shown below.

Line 1 → $\arg(z_1) = \tan^{-1}\left(\frac{3}{3}\right) = \frac{\pi}{4}$

Line 2 → $\arg(z_2) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Line 3 → $\arg\left(\frac{z_1}{z_2}\right) = \frac{\arg(z_1)}{\arg(z_2)}$

Line 4 → $= \left(\frac{\pi}{4}\right) \Big/ \left(\frac{\pi}{6}\right) = \frac{3}{2}$

The student made errors in line 1 and line 3

Correct the error that the student made in

(a) (i) line 1

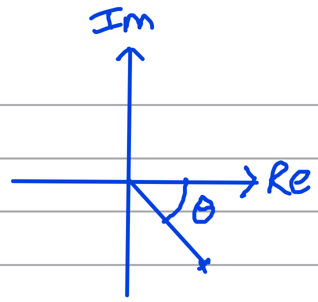
(ii) line 3

(2)

(b) Write down the correct value of $\arg\left(\frac{z_1}{z_2}\right)$

(1)

a) (i)



$\arg z_1 = -\tan\left(\frac{3}{3}\right)$ 1

below real axis so must be negative

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$ 1

b) $\arg\left(\frac{z_1}{z_2}\right) = -\frac{\pi}{4} - \frac{\pi}{6} = -\frac{5\pi}{12}$ 1

Question 1 continued

(Total for Question 1 is 3 marks)

2. **In this question you must show all stages of your working.**

A college offers only three courses: Construction, Design and Hospitality.

Each student enrols on just one of these courses.

In 2019, there was a total of 1110 students at this college.

There were 370 more students enrolled on Construction than Hospitality.

In 2020 the number of students enrolled on

- Construction **increased** by 1.25%
- Design **increased** by 2.5%
- Hospitality **decreased** by 2%

In 2020, the total number of students at the college increased by 0.27% to 2 significant figures.

(a) (i) Define, for each course, a variable for the number of students enrolled on that course in 2019.

(ii) Using your variables from part (a)(i), write down **three** equations that model this situation.

(4)

(b) By forming and solving a matrix equation, determine how many students were enrolled on each of the three courses in 2019.

(4)

a) (i) let C = number of construction students in 2019
D = number of Design students in 2019
H = number of Hospitality students in 2019 1

(ii) in 2019: $C + D + H = 1110$
number of students in 2020: 1110×1.0027
$= 1113$ (4sf) 1

$C = H + 370$
$\Rightarrow C - H = 370$ 1

$1.0125C + 1.025D + 0.98H = 1113$ 1

0.27% increase

1.25% increase

2.5% increase

100% + 1.25%

2% decrease

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Question 2 continued

b) ① ①

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1.0125 & 1.025 & 0.98 \end{pmatrix}^{-1}\begin{pmatrix} 1110 \\ 370 \\ 1113 \end{pmatrix}$$ ①

$$\begin{pmatrix} C \\ D \\ H \end{pmatrix} = \begin{pmatrix} 720 \\ 40 \\ 350 \end{pmatrix}$$

so in 2019, 720 students studied construction, 40 students studied Design and 350 students studied Hospitality. ①

Question 2 continued

Question 2 continued

(Total for Question 2 is 8 marks)

3. $\mathbf{M} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$ where a is a constant

(a) Prove by mathematical induction that, for $n \in \mathbb{N}$

$$\mathbf{M}^n = \begin{pmatrix} 3^n & \frac{a}{2}(3^n - 1) \\ 0 & 1 \end{pmatrix}$$

(6)

Triangle T has vertices A, B and C.

Triangle T is transformed to triangle T' by the transformation represented by $\mathbf{M}^n$ where $n \in \mathbb{N}$

Given that

- triangle T has an area of $5\,\text{cm}^2$
- triangle T' has an area of $1215\,\text{cm}^2$
- vertex $A(2, -2)$ is transformed to vertex $A'(123, -2)$

(b) determine

(i) the value of n

(ii) the value of a

(5)

a) Base case $n=1$

$M^1 = \begin{pmatrix} 3^1 & \frac{a}{2}(3^1-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}$

so the result is true when $n=1$. 1

Assume true for $n=k$, i.e.

$\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}$ 1

Show true for $n=k+1$

$\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^1$

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Question 3 continued

$$\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^k & \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a \\ 0 & 1 \end{pmatrix} \quad 1$$

$$= \begin{pmatrix} 3(3^k) & a(3^k) + \frac{a}{2}(3^k-1) \\ 0 & 1 \end{pmatrix} \quad 1$$

$$= \begin{pmatrix} 3^{k+1} & \frac{a}{2}\left[2(3^k) + (3^k-1)\right] \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & \frac{a}{2}\left[3(3^k) - 1\right] \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3^{k+1} & \frac{a}{2}(3^{k+1}-1) \\ 0 & 1 \end{pmatrix} \quad 1$$

so it is true for $n=k+1$.

If true for $n=k$ then true for $n=k+1$, and as it is true for $n=1$ it is true for all positive integers n. 1

b) (i) using $\det(M^n) = \det(M)^n$ and $\det(M) = 3$

$\det(M^n) = 3^n$ 1

Area $T \times \det M^n$ = Area T'

$5(3^n) = 1215$ 1

$3^n = 243$

$n = 5$ 1

(ii) $M^5 = \begin{pmatrix} 3^5 & \frac{a}{2}(3^5-1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 243 & 121a \\ 0 & 1 \end{pmatrix}$

Question 3 continued

$$\begin{pmatrix} 243 & 121a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 123 \\ -2 \end{pmatrix}$$

$$2(243) - 2(121a) = 123 \quad 1$$

$$486 - 242a = 123$$

$$a = \frac{486 - 123}{242}$$

$$a = 1.5 \quad 1$$

Question 3 continued

(Total for Question 3 is 11 marks)

4. (i) Given that

$$z_1 = 6e^{\frac{\pi}{3}i} \text{ and } z_2 = 6\sqrt{3}e^{\frac{5\pi}{6}i}$$

show that

$$z_1 + z_2 = 12e^{\frac{2\pi}{3}i}$$

(3)

(ii) Given that

$$\arg(z - 5) = \frac{2\pi}{3}$$

determine the least value of $|z|$ as z varies.

(3)

(i) $z_1 = 6\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 3 + 3\sqrt{3}\,i$

$z_2 = 6\sqrt{3}\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right] = -9 + 3\sqrt{3}\,i$

$z_1 + z_2 = 3 + 3\sqrt{3}\,i - 9 + 3\sqrt{3}\,i$

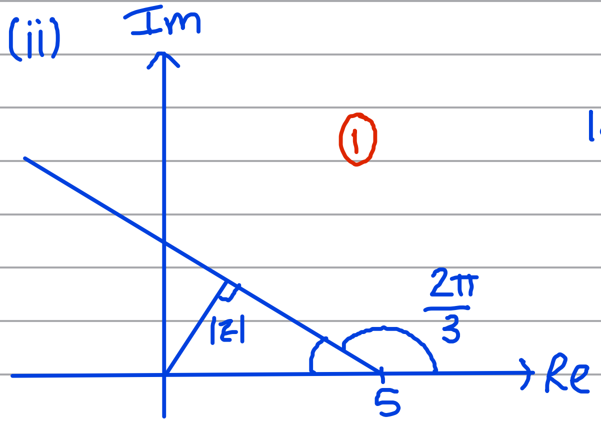
$= -6 + 6\sqrt{3}\,i$ (1)

$|z_1 + z_2| = \sqrt{(-6)^2 + (6\sqrt{3})^2} = 12$

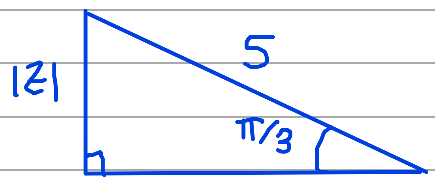
$\arg(z_1 + z_2) = \tan^{-1}\left(\frac{6\sqrt{3}}{-6}\right) = \frac{2\pi}{3}$ (1)

$z_1 + z_2 = 12e^{\frac{2\pi i}{3}}$ (1)

(ii)



(1)



(1) (1)

$\sin\frac{\pi}{3} = \frac{|z|}{5} \Rightarrow |z| = \frac{5\sqrt{3}}{2}$

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Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 6 marks)

PMT

5. (a) Given that

$$y = \arcsin x \qquad -1 \leqslant x \leqslant 1$$

show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

(3)

(b) $$f(x) = \arcsin(e^x) \qquad x \leqslant 0$$

Prove that f(x) has no stationary points.

(3)

a) $y = \arcsin x \Rightarrow x = \sin y$

$\frac{dx}{dy} = \cos y$ (1) $\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$ (1)

??

$\cos^2 y + \sin^2 y = 1$

$\cos^2 y = 1 - x^2$

$\cos y = \sqrt{1-x^2}$ (1)

b) $f(x) = \arcsin(e^x)$

$f'(x) = \frac{1}{\sqrt{1-e^{2x}}}$ (1) $\times e^x$ (1) $= \frac{e^x}{\sqrt{1-e^{2x}}} = 0$

$e^x = 0$

no solutions as $e^x \neq 0$

so there are no stationary points. (1)

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Question 5 continued

(Total for Question 5 is 6 marks)

6. The cubic equation

$$4x^3 + px^2 - 14x + q = 0$$

where p and q are real positive constants, has roots α, β and γ

Given that $\alpha^2 + \beta^2 + \gamma^2 = 16$

(a) show that $p = 12$

(3)

Given that $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{14}{3}$

(b) determine the value of q

(3)

Without solving the cubic equation,

(c) determine the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(4)

a) $4x^3 + px^2 - 14x + q = 0 \Rightarrow x^3 + \frac{p}{4}x^2 - \frac{14}{4}x + \frac{q}{4} = 0$

$\alpha + \beta + \gamma = -\frac{p}{4} \qquad \alpha\beta + \beta\gamma + \gamma\alpha = -\frac{14}{4}$ 1

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\left(\frac{-p}{4}\right)^2 = 16 + 2\left(\frac{-14}{4}\right)$ 1 $\Rightarrow p = 12$ 1

b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \beta\alpha}{\alpha\beta\gamma}$ 1 $= \frac{(-7/2)}{(-q/4)}$

$\frac{-7/2}{-q/4} = \frac{14}{3}$ 1

$q = 3$ 1

c) $(\alpha - 1)(\beta - 1)(\gamma - 1) = (\alpha - 1)(\beta\gamma - \beta - \gamma + 1)$ 1

$= \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) + (\alpha + \beta + \gamma) - 1$ 1

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Question 6 continued

$$= \left(-\frac{3}{4}\right) - \left(\frac{-7}{2}\right) + \left(\frac{-12}{4}\right) - 1 \quad 1$$

$$= -\frac{5}{4} \quad 1$$

Question 6 continued

Question 6 continued

(Total for Question 6 is 10 marks)

7.

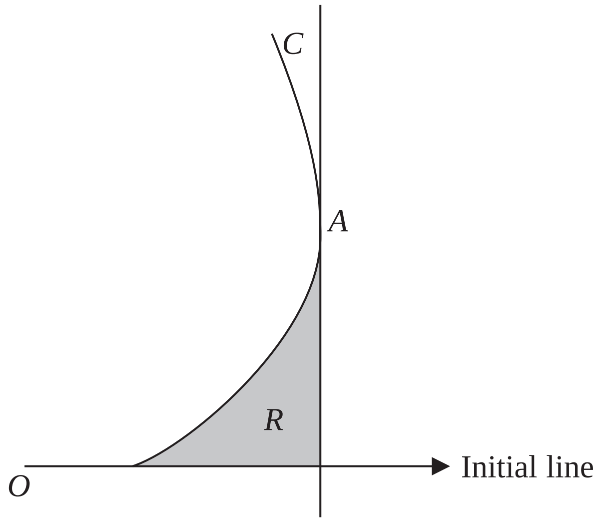

Figure 1

Figure 1 shows a sketch of the curve C with equation

$$r = 1 + \tan\theta \qquad\qquad 0 \leqslant \theta < \frac{\pi}{3}$$

Figure 1 also shows the tangent to C at the point A.
This tangent is perpendicular to the initial line.

(a) Use differentiation to prove that the polar coordinates of A are $\left(2, \frac{\pi}{4}\right)$

(4)

The finite region R, shown shaded in Figure 1, is bounded by C, the tangent at A and the initial line.

(b) Use calculus to show that the exact area of R is $\frac{1}{2}(1 - \ln 2)$

(6)

a) Perpendicular to initial line: $\frac{dx}{d\theta} = 0$

$x = r\cos\theta$
$= (1 + \tan\theta)\cos\theta$
$= \cos\theta + \sin\theta$ 1

$\frac{dx}{d\theta} = -\sin\theta + \cos\theta = 0$ 1
$\sin\theta = \cos\theta$
$\tan\theta = 1$
$\theta = \frac{\pi}{4}$ 1

$r = 1 + \tan\frac{\pi}{4}$
$= 1 + 1 = 2$ 1

$\therefore A\left(2, \frac{\pi}{4}\right)$

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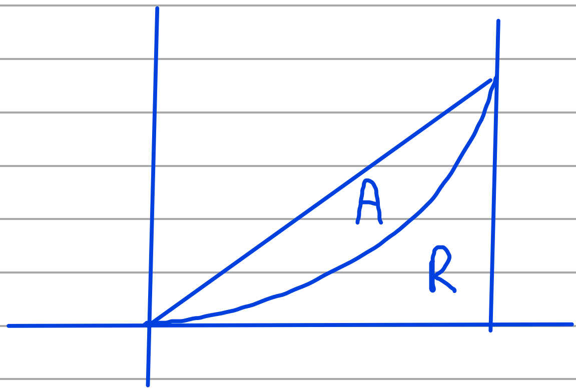
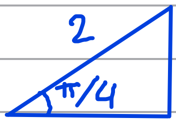
Question 7 continued

b)



R =  − A

$$A = \frac{1}{2}\int_0^{\pi/4} (1+\tan\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (1+2\tan\theta+\tan^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (1+2\tan\theta+(\sec^2\theta-1))\, d\theta \quad (1)$$

$$= \frac{1}{2}\int_0^{\pi/4} (2\tan\theta+\sec^2\theta)\, d\theta$$

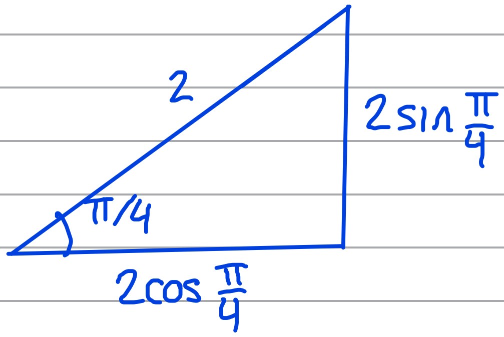
$$= \frac{1}{2}\Big[2\ln|\sec\theta| + \tan\theta\Big]_0^{\pi/4} \quad (1)$$

$$= \frac{1}{2}\Big[2\ln|\sec\tfrac{\pi}{4}| + \tan\tfrac{\pi}{4} - 2\ln|\sec 0| - \tan 0\Big]$$

Question 7 continued

$A = \ln\sqrt{2} + \frac{1}{2}$ 1

Area of triangle:



$\text{Area} = \frac{1}{2}\left(2\sin\frac{\pi}{4}\right)\left(2\cos\frac{\pi}{4}\right) = 1$ 1

$R = \text{triangle} - A$

$= 1 - \left(\ln\sqrt{2} + \frac{1}{2}\right)$ 1

$= \frac{1}{2}\left(1 - \ln 2\right)$ 1

Question 7 continued

(Total for Question 7 is 10 marks)

8. Two birds are flying towards their nest, which is in a tree.

Relative to a fixed origin, the flight path of each bird is modelled by a straight line.

In the model, the equation for the flight path of the first bird is

$$\mathbf{r}_1 = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix}$$

and the equation for the flight path of the second bird is

$$\mathbf{r}_2 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

where λ and μ are scalar parameters and a is a constant.

In the model, the angle between the birds' flight paths is 120°

(a) Determine the value of a.

(4)

(b) Verify that, according to the model, there is a common point on the flight paths of the two birds and find the coordinates of this common point.

(5)

The position of the nest is modelled as being at this common point.

The tree containing the nest is in a park.

The ground level of the park is modelled by the plane with equation

$$2x - 3y + z = 2$$

(c) Hence determine the shortest distance from the nest to the ground level of the park.

(3)

(d) By considering the model, comment on whether your answer to part (c) is reliable, giving a reason for your answer.

(1)

a)

$$\frac{\begin{pmatrix} 2 \\ a \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{\sqrt{2^2+a^2}\sqrt{1^2+1^2}} = \cos 120 \quad 1$$

$$\frac{a}{\sqrt{4+a^2}\sqrt{2}} = -\frac{1}{2} \quad 1$$

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Question 8 continued

$2a = -\sqrt{4+a^2}\sqrt{2}$ (1)

$4a^2 = 2(4+a^2)$

$a^2 = 4$
$a = \pm 2$

check both results are valid, as squaring adds solutions:
if $a = 2$

$\frac{2}{\sqrt{4+(2)^2}\sqrt{2}} = \frac{1}{2} \neq -\frac{1}{2}$ so $a=2$ is not valid.

if $a=-2$:

$\frac{-2}{\sqrt{4+(-2)^2}\sqrt{2}} = -\frac{1}{2}$ so $a=-2$ is the only solution. (1)

b) common point: $\begin{pmatrix} -1+2\lambda \\ 5-2\lambda \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1+\mu \\ 3-\mu \end{pmatrix}$

first two rows:
$-1+2\lambda = 4 \Rightarrow \lambda = \frac{5}{2}$ (1)

$5-2\lambda = -1+\mu \Rightarrow 0 = -1+\mu \Rightarrow \mu = 1$ (1)

check with third equation:
$2 = 3-\mu \Rightarrow \mu = 1$ ✓ (1)

consistent values of λ and μ $\therefore$ there exists a common point:
$\begin{pmatrix} 4 \\ -1+1 \\ 3-1 \end{pmatrix}$ (1) $= \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$ (1)

Question 8 continued

c) shortest distance between nest and ground:

$$2x - 3y + z = 2 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2$$

$$\underline{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$$\text{shortest distance} = \frac{|2(4) - 3(0) + 1(2) - 2|}{\sqrt{2^2 + 3^2 + 1^2}} \quad (2)$$

$$= \frac{8}{\sqrt{14}} \quad (1)$$

d) Not reliable as the birds will not fly in a straight line. (1)

Question 8 continued

(Total for Question 8 is 13 marks)

9. $$y = \cosh^n x \qquad n \geqslant 5$$

(a) (i) Show that

$$\frac{d^2 y}{dx^2} = n^2 \cosh^n x - n(n-1)\cosh^{n-2} x$$

(4)

(ii) Determine an expression for $\frac{d^4 y}{dx^4}$

(2)

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form.

(2)

a) (i) $y = \cosh^n x$

$\frac{dy}{dx} = n\cosh^{n-1} x \sinh x$ 1

$\frac{d^2y}{dx^2} = n(n-1)\cosh^{n-2} x \sinh^2 x + n\cosh^{n-1} x \cosh x$

$= n(n-1)\cosh^{n-2}(\cosh^2 x - 1) + n\cosh^n x$ 1

$= n(n-1)\cosh^n x - n(n-1)\cosh^{n-2} x + n\cosh^n x$

$= n^2\cosh^n x - n(n-1)\cosh^{n-2} x$ as required 2

(ii) $\frac{d^3y}{dx^2} = n^3\cosh^{n-1} x \sinh x - n(n-1)(n-2)\cosh^{n-3} x \sinh x$ 1

$\frac{d^4y}{dx^2} = n^3(n-1)\cosh^{n-2} x \sinh^2 x + n^3\cosh^{n-1} x \cosh x$

$- n(n-1)(n-2)(n-3)\cosh^{n-4} x \sinh^2 x$

$- n(n-1)(n-2)\cosh^{n-3} x \cosh x$ 1

b) $y: \cosh^n 0 = 1$

$y': n\cosh^{n-1} 0 \sinh 0 = 0$

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Question 9 continued

$y'': n^2\cosh^n 0 - n(n-1)\cosh^{n-2} 0 = n^2 - n(n-1) = n$

$y''': n^3\cosh^{n-1} 0 \sinh 0 - n(n-1)(n-2)\cosh^{n-1} 0 \sinh 0 = 0$

$y^4: n^3(n-1)\cosh^{n-2} 0 \sinh 0 + n^3\cosh^n 0$
$- n(n-1)(n-2)(n-3)\cosh^{n-4} 0 \sinh 0$
$- n(n-1)(n-2)\cosh^{n-2} 0 \quad = n^3 - n(n-1)(n-2)$
$= n^3 - n^3 + 3n^2 - 2n$
$= 3n^2 - 2n$ (1)

$\therefore y = 1 + 0x + \frac{nx^2}{2!} + \frac{0x^3}{3!} + \frac{(3n^2-2n)x^4}{4!} + \ldots$

$y = 1 + \frac{nx^2}{2} + \frac{(3n^2-2n)x^4}{24}$ (1)

Question 9 continued

(Total for Question 9 is 8 marks)

TOTAL FOR PAPER IS 75 MARKS