UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/12**

Paper 1 **May/June 2010**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY *of* CAMBRIDGE International Examinations

1 The variables x and y are such that $y = -1$ when $x = 1$ and

$$x^2 + y^2 + \left(\frac{dy}{dx}\right)^3 = 29.$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 1$. [5]

2 The curve C has polar equation

$$r = a(1 - e^{-\theta}),$$

where a is a positive constant and $0 \leqslant \theta < 2\pi$.

(i) Draw a sketch of C. [3]

(ii) Show that the area of the region bounded by C and the lines $\theta = \ln 2$ and $\theta = \ln 4$ is

$$\tfrac{1}{2}a^2\left(\ln 2 - \tfrac{13}{32}\right).$$ [4]

3 At any point (x, y) on the curve C,

$$\frac{dx}{dt} = t\sqrt{(t^2 + 4)} \quad \text{and} \quad \frac{dy}{dt} = -t\sqrt{(4 - t^2)},$$

where the parameter t is such that $0 \leqslant t \leqslant 2$. Show that the length of C is $4\sqrt{2}$. [3]

Given that $y = 0$ when $t = 2$, determine the area of the surface generated when C is rotated through one complete revolution about the x-axis, leaving your answer in an exact form. [4]

4 The sum S_N is defined by $S_N = \sum_{n=1}^{N} n^5$. Using the identity

$$\left(n + \tfrac{1}{2}\right)^6 - \left(n - \tfrac{1}{2}\right)^6 \equiv 6n^5 + 5n^3 + \tfrac{3}{8}n,$$

find S_N in terms of N. [You need not simplify your result.] [4]

Hence find $\lim_{N\to\infty} N^{-\lambda} S_N$, for each of the two cases

(i) $\lambda = 6$,

(ii) $\lambda > 6$.

[3]

5 Let

$$I_n = \int_1^e x(\ln x)^n \, dx,$$

where $n \geqslant 1$. Show that

$$I_{n+1} = \tfrac{1}{2}e^2 - \tfrac{1}{2}(n + 1)I_n.$$ [3]

Hence prove by induction that, for all positive integers n, I_n is of the form $A_n e^2 + B_n$, where A_n and B_n are rational numbers. [6]

6 The equation

$$x^3 + x - 1 = 0$$

has roots α, β, γ. Use the relation $x = \sqrt{y}$ to show that the equation

$$y^3 + 2y^2 + y - 1 = 0$$

has roots α^2, β^2, γ^2. [2]

Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(i) Write down the value of S_2 and show that $S_4 = 2$. [3]

(ii) Find the values of S_6 and S_8. [4]

7 The lines l_1 and l_2 have vector equations

$$\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 4\mathbf{k}) \qquad \text{and} \qquad \mathbf{r} = 4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} + \mu(\mathbf{i} - \mathbf{j} - \mathbf{k})$$

respectively.

(i) Show that l_1 and l_2 intersect. [3]

(ii) Find the perpendicular distance from the point P whose position vector is $3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ to the plane containing l_1 and l_2. [3]

(iii) Find the perpendicular distance from P to l_1. [4]

8 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -1 \\ -4 & -1 & 4 \\ 0 & -1 & 5 \end{pmatrix}.$$

Given that one eigenvector of $\mathbf{A}$ is $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, find the corresponding eigenvalue. [2]

Given also that another eigenvalue of $\mathbf{A}$ is 4, find a corresponding eigenvector. [2]

Given further that $\begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, with corresponding eigenvalue 1, find matrices $\mathbf{P}$ and $\mathbf{Q}$, together with a diagonal matrix $\mathbf{D}$, such that $\mathbf{A}^5 = \mathbf{PDQ}$. [6]

9 **(i)** Write down the five fifth roots of unity. [2]

(ii) Hence find all the roots of the equation

$$z^5 + 16 + (16\sqrt{3})\mathrm{i} = 0,$$

giving answers in the form $r\mathrm{e}^{\mathrm{i}q\pi}$, where $r > 0$ and q is a rational number. Show these roots on an Argand diagram. [4]

Let w be a root of the equation in part **(ii)**.

(iii) Show that

$$\sum_{k=0}^{4}\left(\frac{w}{2}\right)^k = \frac{3 + \mathrm{i}\sqrt{3}}{2 - w}.$$ [3]

(iv) Identify the root for which $|2 - w|$ is least. [2]

10 Find the set of values of a for which the system of equations

$$\begin{aligned} x + 4y + 12z &= 5, \\ 2x + ay + 12z &= a - 1, \\ 3x + 12y + 2az &= 10, \end{aligned}$$

has a unique solution. [4]

Show that the system does not have any solution in the case $a = 18$. [2]

Given that $a = 8$, show that the number of solutions is infinite and find the solution for which $x + y + z = 1$. [5]

11 Answer only **one** of the following two alternatives.

EITHER

The variables z and x are related by the differential equation

$$3z^2\frac{d^2z}{dx^2} + 6z^2\frac{dz}{dx} + 6z\left(\frac{dz}{dx}\right)^2 + 5z^3 = 5x + 2.$$

Use the substitution $y = z^3$ to show that y and x are related by the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 5x + 2.$$ [3]

Given that $z = 1$ and $\frac{dz}{dx} = -\frac{2}{3}$ when $x = 0$, find z in terms of x. [9]

Deduce that, for large positive values of x, $z \approx x^{\frac{1}{3}}$. [2]

OR

The curve C has equation

$$y = \frac{x(x+1)}{(x-1)^2}.$$

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Show that there is exactly one point of intersection of C with the asymptotes and find its coordinates. [2]

(iii) Find $\frac{dy}{dx}$ and hence

(a) find the coordinates of any stationary points of C,

(b) state the set of values of x for which the gradient of C is negative.

[6]

(iv) Draw a sketch of C. [3]

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