MODEL ANSWERS

AQA

SPECIMEN MATERIAL

Please write clearly, in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname ____________________

Forename(s) ____________________

Candidate signature ____________________

A-level
MATHEMATICS
Paper 2

Exam Date — Morning — Time allowed: 2 hours

Materials

For this paper you must have:
- The AQA booklet of formulae and statistical tables.
- You may use a graphics calculator.

Instructions
- Use black ink or black ball-point pen. Pencil should be used for drawing.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice

Unless stated otherwise, you may quote formulae, without proof, from the booklet.
You do not necessarily need to use all the space provided.

Section A

Answer **all** questions in the spaces provided.

1 State the values of $|x|$ for which the binomial expansion of $(3+2x)^{-4}$ is valid.

Circle your answer.

[1 mark]

$|x| < \frac{2}{3}$ $\qquad$ $|x| < 1$ $\qquad$ $|x| < \frac{3}{2}$ $\qquad$ $|x| < 3$

(since $(3+2x)^{-4} = 3^{-4}(1+\frac{2}{3}x)^{-4}$, so $|\frac{2}{3}x| < 1$ which gives $|x| < \frac{3}{2}$)

2 A zoologist is investigating the growth of a population of red squirrels in a forest.

She uses the equation $N = \dfrac{200}{1+9e^{-\frac{t}{5}}}$ as a model to predict the number of squirrels, N, in the population t weeks after the start of the investigation.

What is the size of the squirrel population at the start of the investigation?

Circle your answer.

[1 mark]

5 $\qquad$ 20 $\qquad$ 40 $\qquad$ 200

(since at $t=0$, $N = \frac{200}{1+9e^0} = \frac{200}{10} = 20$)

3 A curve is defined by the parametric equations

$$x = t^3 + 2 \,, \qquad y = t^2 - 1$$

3 (a) Find the gradient of the curve at the point where $t = -2$

[4 marks]

$x = t^3 + 2$ $\qquad$ $y = t^2 - 1$

$\frac{dx}{dt} = 3t^2$ $\qquad$ $\frac{dy}{dt} = 2t$

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2t \times \frac{1}{3t^2} = \frac{2}{3t}.$$

At $t = -2$, $\frac{dy}{dx} = \frac{2}{3(-2)} = \frac{2}{-6} = -\frac{1}{3}$

3 (b) Find a Cartesian equation of the curve.

[2 marks]

We want to eliminate t:

$t^3 = x - 2$ $\qquad$ $t^2 = y + 1$

$t^6 = (x-2)^2$ $\qquad$ $t^6 = (y+1)^3$

$\Rightarrow \quad (x-2)^2 = (y+1)^3$

4 The equation $x^3 - 3x + 1 = 0$ has three real roots.

4 (a) Show that one of the roots lies between -2 and -1

[2 marks]

Let $f(x) = x^3 - 3x + 1$.

$f(-2) = (-2)^3 - 3(-2) + 1 = -1 < 0$

$f(-1) = (-1)^3 - 3(-1) + 1 = 3 > 0$

$f(x)$ is continuous and the change of sign indicates a root must lie between $x = -2$ and $x = -1$.

4 (b) Taking $x_1 = -2$ as the first approximation to one of the roots, use the Newton-Raphson method to find x_2, the second approximation.

[3 marks]

$f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$$x_{n+1} = x_n - \left(\frac{(x_n)^3 - 3x_n + 1}{3(x_n)^2 - 3}\right)$$

so $x_2 = -2 - \left(\frac{(-2)^3 - 3(-2) + 1}{3(-2)^2 - 3}\right)$

$$= -2 - \left(\frac{-1}{9}\right) = \frac{-17}{9}$$

4 (c) Explain why the Newton-Raphson method fails in the case when the first approximation is $x_1 = -1$

[1 mark]

$$x_2 = -2 - \left(\frac{(-1)^3 - 3(-1) + 1}{3(-1)^2 - 3}\right) = -2 - \left(\frac{3}{3-3}\right) = -2 - \frac{3}{0}$$

The division by 0 is undefined so the method fails.

Turn over for the next question

5 (a) Determine a sequence of transformations which maps the graph of $y = \cos\theta$ onto the graph of $y = 3\cos\theta + 3\sin\theta$

Fully justify your answer.

[6 marks]

$R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos(\theta - \alpha) = 3\cos\theta + 3\sin\theta$

Compare: $R\cos\alpha = 3$

$R\sin\alpha = 3$

So, $\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = 1 \Rightarrow \alpha = \frac{\pi}{4}$

$R = \frac{3}{\cos\alpha} = \frac{3}{\cos\frac{\pi}{4}} = \frac{3}{(\sqrt{2}/2)} = \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}.$

So the second equation, $y = 3\cos\theta + 3\sin\theta$, becomes

$$y = 3\sqrt{2}\cos(\theta - \pi/4).$$

This is a translation of $\pi/4$ along the positive x axis and a stretch in the y-direction of scale factor $3\sqrt{2}$.

5 (b) Hence or otherwise find the least value and greatest value of

$$4 + (3\cos\theta + 3\sin\theta)^2$$

Fully justify your answer.

[3 marks]

$4 + (3\cos\theta + 3\sin\theta)^2$

$= 4 + \left(3\sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)\right)^2 = 4 + 18\cos^2\left(\theta - \frac{\pi}{4}\right).$

The minimum of this is when $\cos^2\left(\theta - \frac{\pi}{4}\right) = 0$

so the smallest value is $4 + 0 = 4$.

The maximum is when $\cos^2\left(\theta - \frac{\pi}{4}\right) = 1$, so

the largest value is $4 + 18 = 22$.

Turn over for the next question

6 A curve *C*, has equation $y = x^2 - 4x + k$, where k is a constant.

It crosses the x-axis at the points $\left(2+\sqrt{5},0\right)$ and $\left(2-\sqrt{5},0\right)$

6 (a) Find the value of k .

[2 marks]

Solutions are found by $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

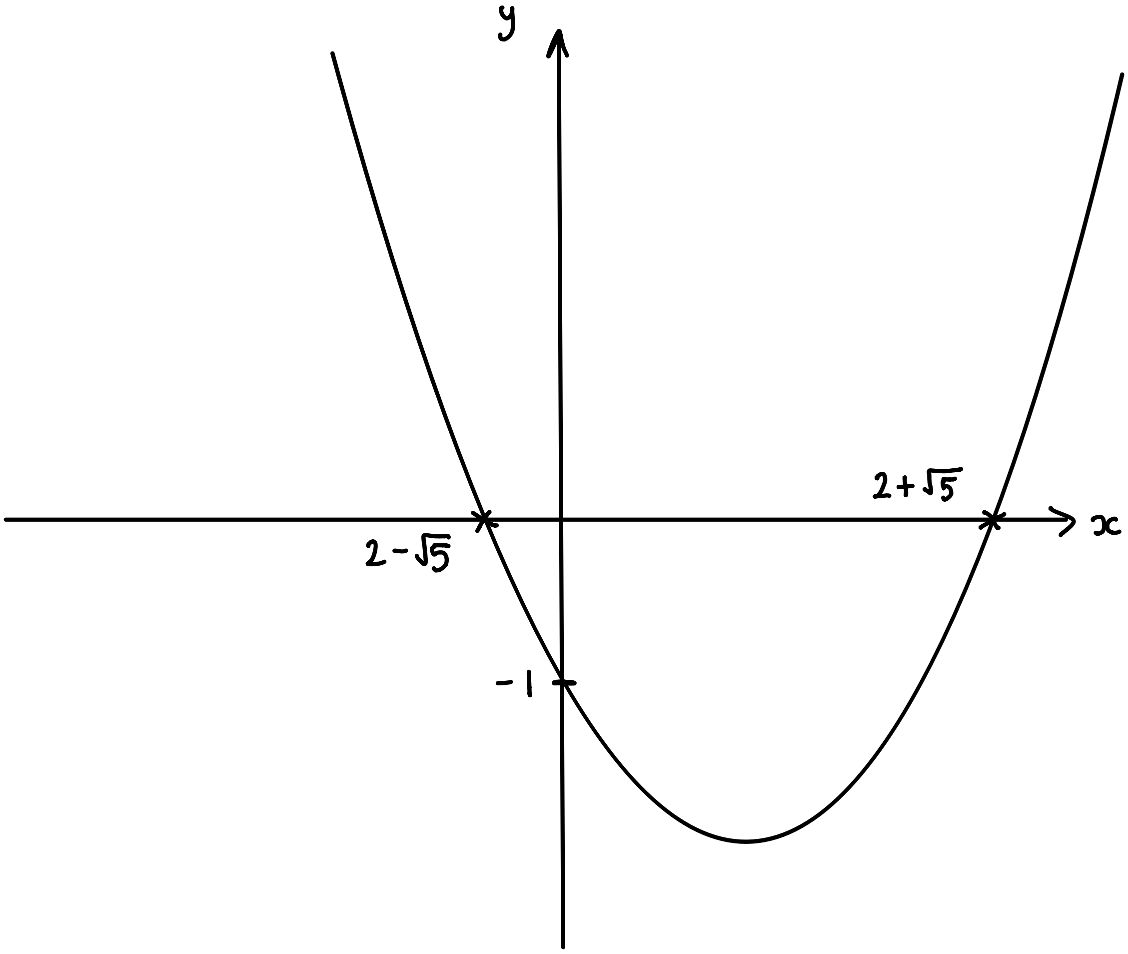
$\frac{4 \pm \sqrt{4^2 - 4k}}{2} = 2 \pm \sqrt{5} \Rightarrow \frac{\sqrt{16-4k}}{2} = \sqrt{5}$

$\Rightarrow \frac{16-4k}{4} = 5 \Rightarrow 16-4k = 20 \Rightarrow 4k = -4$

$\Rightarrow k = -1$

6 (b) Sketch the curve *C*, labelling the exact values of all intersections with the axes.

[3 marks]



Turn over for the next question

7 A student notices that when he adds two consecutive odd numbers together the answer always seems to be the difference between two square numbers.

He claims that this will always be true.

He attempts to prove his claim as follows:

Step 1: Check first few cases

$3+5=8$ and $8=3^2-1^2$

$5+7=12$ and $12=4^2-2^2$

$7+9=16$ and $16=5^2-3^2$

Step 2: Use pattern to predict and check a large example

$101+103=204$

subtract 1 and divide by 2 for the first number

Add 1 and divide by two for the second number

$52^2-50^2=204$ it works!

Step 3: Conclusion

The first few cases work and there is a pattern, which can be used to predict larger numbers.

Therefore, it must be true for all consecutive odd numbers.

7 (a) Explain what is wrong with the student's "proof".

[1 mark]

They have only proven it for those cases, may not apply to all numbers.

7 (b) Prove that the student's claim is correct.

[3 marks]

Let $2n+1$ and $2n+3$ be two consecutive odd numbers.

$(2n+1) + (2n+3) = 4n+4$

Let n^2 and $(n+2)^2$ be the two square numbers. Their difference is

$(n+2)^2 - n^2 = n^2 + 4n + 4 - n^2 = 4n+4$

and we know that $4n+4$ can be written as the sum of two consecutive odd numbers.

Turn over for the next question

8 A curve has equation $y = 2x\cos 3x + \left(3x^2 - 4\right)\sin 3x$

8 (a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, giving your answer in the form $\left(mx^2 + n\right)\cos 3x$, where m and n are integers.

[4 marks]

$y = 2x\cos 3x + (3x^2 - 4)\sin 3x$

$\dfrac{dy}{dx} = 2x(-3\sin 3x) + 2\cos 3x + 6x\sin 3x + 3(3x^2-4)\cos 3x$

$= \cancel{-6x\sin 3x} + 2\cos 3x + \cancel{6x\sin 3x} + 3(3x^2-4)\cos 3x$

$= (2 + 9x^2 - 12)\cos 3x$

$= (9x^2 - 10)\cos 3x$

8 (b) Show that the x-coordinates of the points of inflection of the curve satisfy the equation

$$\cot 3x = \frac{9x^2 - 10}{6x}$$

[4 marks]

A point of inflection is when $\frac{d^2y}{dx^2} = 0$.

$$\frac{d^2y}{dx^2} = -3(9x^2 - 10)\sin 3x + 18x\cos 3x = 0$$

$$18x\cos 3x = 3(9x^2 - 10)\sin 3x$$

$$\frac{\cos 3x}{\sin 3x} = \frac{3(9x^2 - 10)}{18x}$$

$$\cot 3x = \frac{3(9x^2 - 10)}{18x}$$

9 (a) Three consecutive terms in an arithmetic sequence are $3e^{-p}, 5, 3e^{p}$

Find the possible values of p. Give your answers in an exact form.

[6 marks]

Terms of an arithmetic sequence are a, $a+d$, $a+2d$.

So, $3e^{-p} = a$ ①

$5 = a+d$ ②

$3e^{p} = a+2d$ ③

From ③, substitute in ① and ②:

$3e^{p} = a+2d = 3e^{-p} + 2(5-a)$

$3e^{p} = 3e^{-p} + 10 - 2a$

$3e^{p} = 3e^{-p} + 10 - 6e^{-p}$

$0 = 3e^{p} - 10 + 3e^{-p}$

$0 = 3e^{2p} - 10e^{p} + 3$

$0 = (3e^{p} - 1)(e^{p} - 3)$

So, $3e^{p} - 1 = 0$ or $e^{p} - 3 = 0$

$e^{p} = \frac{1}{3}$ or $e^{p} = 3$

$p = \ln\left(\frac{1}{3}\right)$ or $p = \ln(3)$

9 (b) Prove that there is no possible value of q for which $3e^{-q}$, 5, $3e^{q}$ are consecutive terms of a geometric sequence.

[4 marks]

Assume it is possible and work towards a contradiction.

Terms of a geometric series are a, ar, ar^2.

So, $3e^{-q} = a$ (1)

$5 = ar$ (2)

$3e^{q} = ar^2$ (3)

Substitute (1) and (2) into (3):

$$3e^{q} = ar^2 = a\left(\frac{5}{a}\right)^2 = \frac{25}{a} = \frac{25}{3e^{-q}}$$

$$3e^{q} = \frac{25}{3e^{-q}} \Rightarrow 9 = 25$$ Contradiction.

Therefore, these terms cannot form a geometric series.

END OF SECTION A
TURN OVER FOR SECTION B

Section B

Answer **all** questions in the spaces provided.

10 A single force of magnitude 4 newtons acts on a particle of mass 50 grams.

Find the magnitude of the acceleration of the particle.

Circle your answer.

[1 mark]

12.5 m s^{-2} 0.08 m s^{-2} 0.0125 m s^{-2} 80 m s^{-2}

$50g = 0.05\text{ kg}$

$F = ma$

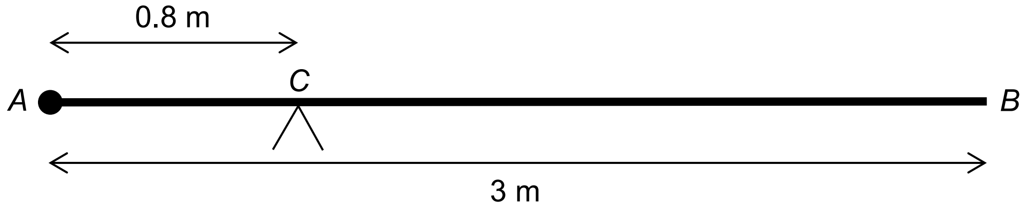
$4 = 0.05a$

$a = \frac{4}{0.05} = 80\text{ ms}^{-2}$

11 A uniform rod, AB, has length 3 metres and mass 24 kg.

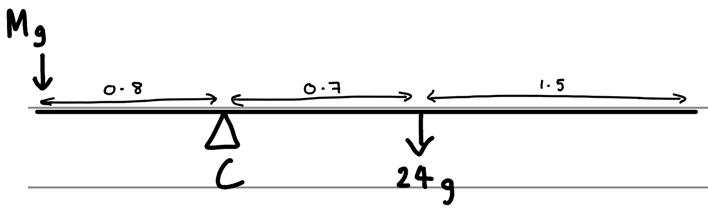
A particle of mass M kg is attached to the rod at A.

The rod is balanced in equilibrium on a support at C, which is 0.8 metres from A.



Find the value of M.

[2 marks]



Take moments about C:

$$0.8 \times Mg = 0.7 \times 24g$$

$$M = \frac{0.7 \times 24}{0.8} \qquad M = 21$$

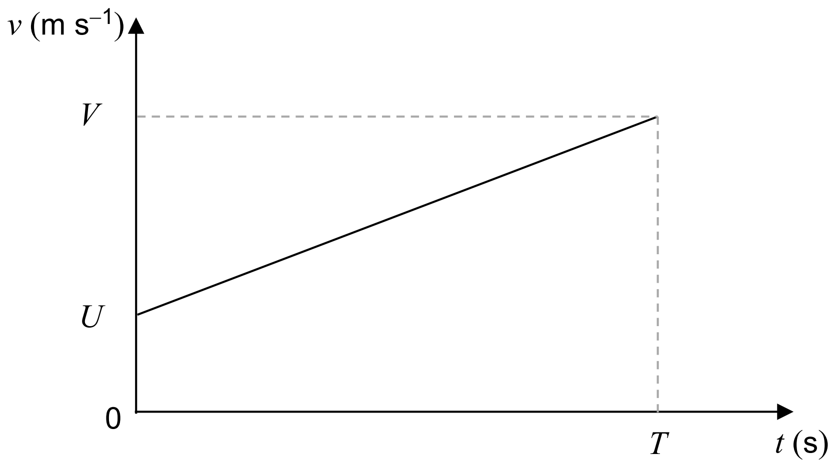
Turn over for the next question

12 A particle moves on a straight line with a constant acceleration, a m s^{-2}.

The initial velocity of the particle is U m s^{-1}.

After T seconds the particle has velocity V m s^{-1}.

This information is shown on the velocity-time graph.



The displacement, S metres, of the particle from its initial position at time T seconds is given by the formula

$$S = \frac{1}{2}(U + V)T$$

12 (a) By considering the gradient of the graph, or otherwise, write down a formula for a in terms of U, V and T.

[1 mark]

acceleration = gradient

$a = \dfrac{V - U}{T}$

12 (b) Hence show that $V^2 = U^2 + 2aS$

[3 marks]

$$T = \frac{V - U}{a}$$

From the formula we are given, $T = \frac{2S}{U+V}$.

$$\frac{2S}{U+V} = \frac{V-U}{a}$$

$$2as = (V-U)(U+V)$$

$$2as = V^2 - U^2$$

$$V^2 = U^2 + 2as$$

Turn over for the next question

13 The three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are acting on a particle.

$$\mathbf{F}_1 = (25\mathbf{i} + 12\mathbf{j})\text{ N}$$
$$\mathbf{F}_2 = (-7\mathbf{i} + 5\mathbf{j})\text{ N}$$
$$\mathbf{F}_3 = (15\mathbf{i} - 28\mathbf{j})\text{ N}$$

The unit vectors **i** and **j** are horizontal and vertical respectively.

The resultant of these three forces is **F** newtons.

13 (a) (i) Find the magnitude of **F**, giving your answer to three significant figures.

[2 marks]

Total force: $\underline{F} = \underline{F_1} + \underline{F_2} + \underline{F_3}$

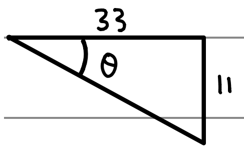
$= 25\underline{i} + 12\underline{j} - 7\underline{i} + 5\underline{j} + 15\underline{i} - 28\underline{j} = 33\underline{i} - 11\underline{j}$

Magnitude: $|F| = \sqrt{33^2 + (-11)^2} = \sqrt{1089 + 121} = 34.785...$

$= 34.8$ N

13 (a) (ii) Find the acute angle that **F** makes with the horizontal, giving your answer to the nearest 0.1°

[2 marks]



$$\tan\theta = \frac{11}{33}$$

$$\theta = \tan^{-1}\left(\frac{11}{33}\right)$$

$$= 18.4349...$$

$$= 18.4°$$

13 (b) The fourth force, $\mathbf{F}_4$, is applied to the particle so that the four forces are in equilibrium.

Find $\mathbf{F}_4$, giving your answer in terms of **i** and **j**.

[1 mark]

Let $\underline{F}_4 = x\underline{i} + y\underline{j}$. $\underline{F} + \underline{F}_4 = 0$.

$33\underline{i} - 11\underline{j} + x\underline{i} + y\underline{j} = 0\underline{i} + 0\underline{j}$

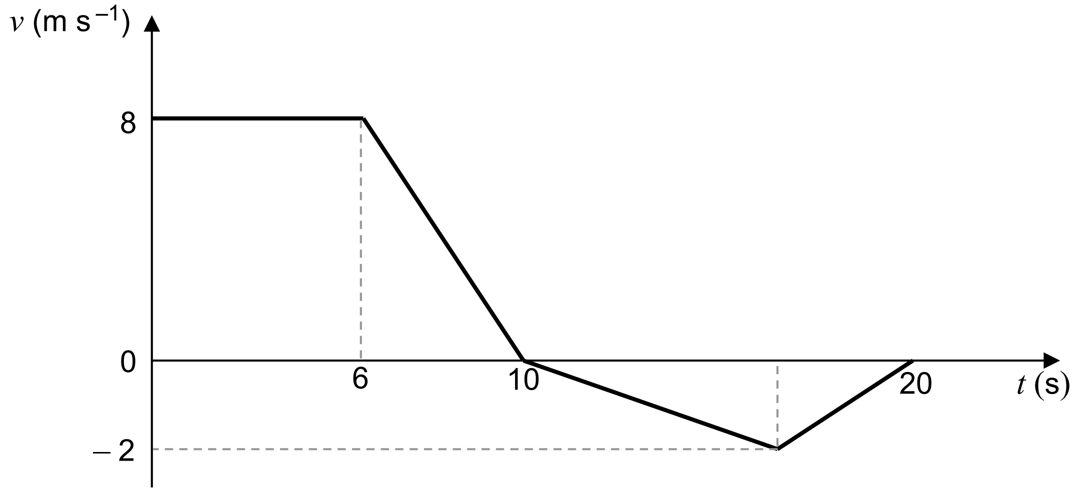
$x\underline{i} + y\underline{j} = -33\underline{i} + 11\underline{j}$ so $x = -33$, $y = 11$ and $\underline{F}_4 = -33\underline{i} + 11\underline{j}$

Turn over for the next question

14 The graph below models the velocity of a small train as it moves on a straight track for 20 seconds.

The front of the train is at the point *A* when $t = 0$

The mass of the train is 800kg.



14 (a) Find the total distance travelled in the 20 seconds.

[3 marks]

In the first 6 seconds: $6 \times 8 = 48$ m

At $6 < t < 10$, $\frac{(10-6)\times 8}{2} = 16$m

At $10 < t < 20$, $\frac{(20-10)\times 2}{2} = 10$ m

Total: $48 + 16 + 10 = 74$m

14 (b) Find the distance of the front of the train from the point *A* at the end of the 20 seconds.

[1 mark]

The part of the graph underneath the x axis is when the train is heading back to A, so it decreases the distance from A:

$$(48 + 16) - 10 = 54 \text{ m}$$

14 (c) Find the maximum magnitude of the resultant force acting on the train.

[2 marks]

The steepest part of the graph is from 6 - 10 seconds, so the acceleration has greatest magnitude here :

$$a = \frac{8}{10-6} = 2$$

$$F = ma = 800 \times 2 = 1600 \text{ N}$$

14 (d) Explain why, in reality, the graph may not be an accurate model of the motion of the train.

[1 mark]

Changes in velocity don't happen suddenly, you would expect a gradual change so the graph would have curves.

15 At time $t = 0$, a parachutist jumps out of an airplane that is travelling horizontally.

The velocity, $\mathbf{v}$ m s^{-1}, of the parachutist at time t seconds is given by:

$$\mathbf{v} = \left(40e^{-0.2t}\right)\mathbf{i} + 50\left(e^{-0.2t} - 1\right)\mathbf{j}$$

The unit vectors $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical respectively.

Assume that the parachutist is at the origin when $t = 0$

Model the parachutist as a particle.

15 (a) Find an expression for the position vector of the parachutist at time t.

[4 marks]

$\underline{r} = \int \underline{v}\ dt = \left(\int 40e^{-0.2t}\,dt\right)\underline{i} + \left(\int 50(e^{-0.2t} - 1)\,dt\right)\underline{j}$

$= (-200e^{-0.2t} + c_1)\underline{i} + 50(-5e^{-0.2t} - t + c_2)\underline{j}$

At $t = 0$, $\underline{r} = \underline{0}$: $-200e^{-0.2(0)} + c_1 = 0$

$c_1 = 200$

$50(-5e^{-0.2(0)} - 0 + c_2) = 0$

$c_2 = 250$

So, $\underline{r} = (-200e^{-0.2t} + 200)\underline{i} + (-250e^{-0.2t} - 50t + 250)\underline{j}$

15 (b) The parachutist opens her parachute when she has travelled 100 metres horizontally.

Find the vertical displacement of the parachutist from the origin when she opens her parachute.

[4 marks]

Horizontal distance: $-200e^{-0.2t} + 200 = 100$

$200e^{-0.2t} = 100$

$e^{-0.2t} = \frac{1}{2}$

$-0.2t = \ln\left(\frac{1}{2}\right)$

$t = -5\ln\left(\frac{1}{2}\right)$

$t = 5\ln(2)$.

At $t = 5\ln 2$, the vertical displacement is

$-250e^{-0.2(5\ln 2)} - 50(5\ln 2) + 250 = -48.286...$

$= -48.3$

So the parachutist is 50m below the origin.

15 (c) Carefully, explaining the steps that you take, deduce the value of g used in the formulation of this model.

[3 marks]

g is the acceleration in the vertical direction so we need to differentiate the vertical velocity component:

$\frac{d}{dt}\left(50e^{-0.2t} - 50\right) = -10e^{-0.2t}$

Initially, $t=0$ so the acceleration is $-10e^{-0.2(0)} = -10$.

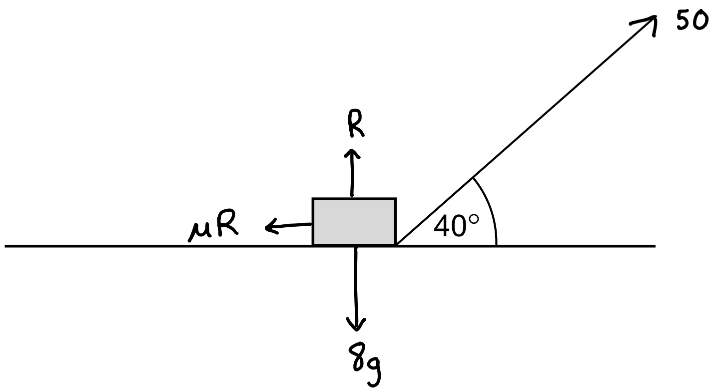
At $t=0$, there is no vertical component of velocity so the initial acceleration is all due to gravity so $g = 10ms^{-2}$

16 **In this question use g = 9.8 m s^{-2}.**

The diagram shows a box, of mass 8.0 kg, being pulled by a string so that the box moves at a constant speed along a rough horizontal wooden board.

The string is at an angle of 40° to the horizontal.

The tension in the string is 50 newtons.



The coefficient of friction between the box and the board is μ

Model the box as a particle.

16 (a) Show that $\mu = 0.83$

[4 marks]

$R(\uparrow)$: $R + 50\sin 40 = 8g$

$R = 8g - 50\sin 40$

$R = (8 \times 9.8) - 50\sin 40 = 46.2606$

$R(\rightarrow)$: $\mu R = 50\cos 40$

$\mu = \frac{50\cos 40}{R} = \frac{50\cos 40}{46.2606}$

$\mu = 0.827966\ldots$

$\mu = 0.83$

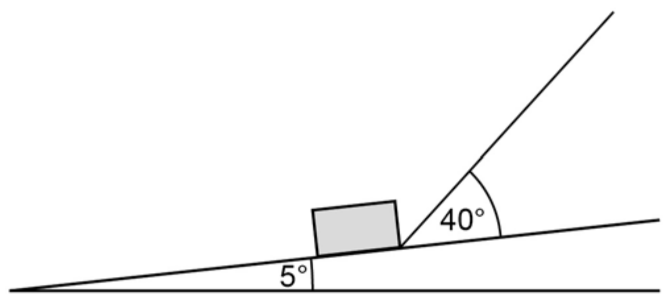
Question 16 continues on the next page

16 (b) One end of the board is lifted up so that the board is now inclined at an angle of 5° to the horizontal.

The box is pulled up the inclined board.

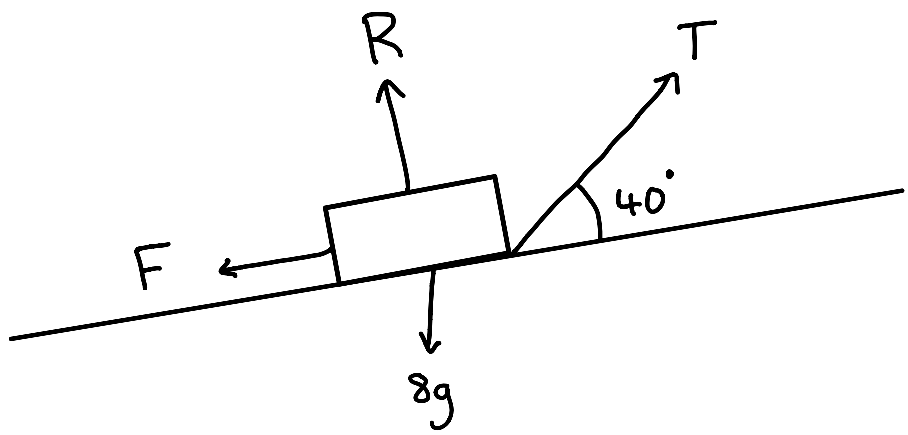
The string remains at an angle of 40° to the board.

The tension in the string is increased so that the box accelerates up the board at 3 m s^{-2}



16 (b) (i) Draw a diagram to show the forces acting on the box as it moves.

[1 mark]



16 (b) (ii) Find the tension in the string as the box accelerates up the slope at 3 m s^{-2}.

[7 marks]

$R(\uparrow)$: $8g\cos 5 = R + T\sin 40$

$R(\rightarrow)$: $F = ma$

$T\cos 40 - F - 8g\sin 5 = 8(3) = 24$

$T(\cos 40 + \mu\sin 40) = 24 + 8g\sin 5 + \mu 8g\cos 5$

$$T = \frac{24 + 8g\sin 5 + \mu 8g\cos 5}{\cos 40 + \mu\sin 40}$$

$$T = \frac{24 + 8(9.8)\sin 5 + 8(0.827966)(9.8)\cos 5}{\cos 40 + 0.827966\sin 40}$$

$T = 73.559\ldots$

$T = 74$

17 **In this question use $g = 9.81$ m s^{-2}.**

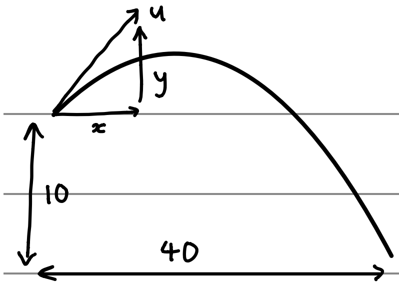
A ball is projected from the origin. After 2.5 seconds, the ball lands at the point with position vector (40**i** – 10**j**) metres.

The unit vectors **i** and **j** are horizontal and vertical respectively.

Assume that there are no resistance forces acting on the ball.

17 (a) Find the speed of the ball when it is at a height of 3 metres above its initial position.

[6 marks]



$R(\rightarrow)$: $x = \frac{s}{t}$

$x = \frac{40}{2.5}$

$x = 16$

$R(\uparrow)$: $s = -10$, $u = y$, $a = -9.81$, $t = 2.5$

$s = ut + \frac{1}{2}at^2$

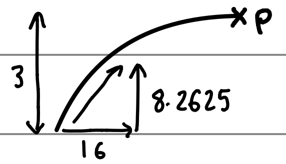
$-10 = 2.5y + \frac{1}{2}(-9.81)(2.5)^2$

$2.5y = \frac{1}{2}(9.81)(2.5)^2 - 10$

$y = \frac{2}{5}\left[\frac{1}{2}(9.81)(2.5)^2 - 10\right] = 8.2625$

So, the initial velocity is $16\underline{i} + 8.2625\underline{j}$

At the new position :



At P, horizontal velocity is still 16 as it is not accelerating this way. For the vertical velocity: $s=3$, $u = 8.2625$, $v = v$, $a = -9.81$

$v^2 = u^2 + 2as$

$v = \sqrt{8.2625^2 + 2(-9.81)(3)} = 3.067$

Velocity at P is : $16\underline{i} + 3.067\underline{j}$

Speed : $\sqrt{16^2 + 3.067^2} = 16.29 \text{ ms}^{-1}$

17 (b) State the speed of the ball when it is at its maximum height.

[1 mark]

vertical component vanishes at the peak, so it is only travelling horizontally. So speed is $16 ms^{-1}$.

17 (c) Explain why the answer you found in part **(b)** may not be the actual speed of the ball when it is at its maximum height.

[1 mark]

We assumed there was no air resistance which was not realistic. So horizontal speed would not be constant as it would decrease with air resistance.

END OF QUESTIONS

There are no questions printed on this page

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