MODEL ANSWERS

AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname ______________________

Forename(s) ______________________

Candidate signature ______________________

A-level
MATHEMATICS
Paper 1

Wednesday 5 June 2019 Morning Time allowed: 2 hours

Materials
- You must have the AQA Formulae for A-level Mathematics booklet.
- You should have a graphical or scientific calculator that meets the requirements of the specification.

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
TOTAL	

J U N 1 9 7 3 5 7 1 0 1

PB/Jun19/E4

7357/1

Do not write outside the box

Answer **all** questions in the spaces provided.

1 Given that $a > 0$, determine which of these expressions is **not** equivalent to the others.

Circle your answer.

[1 mark]

$-2\log_{10}\left(\frac{1}{a}\right)$ $\quad$ $2\log_{10}(a)$ $\quad$ $\log_{10}(a^2)$ $\quad$ $-4\log_{10}(\sqrt{a})$

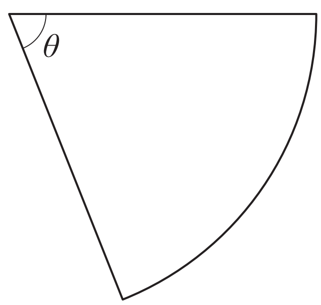
2 Given $y = e^{kx}$, where k is a constant, find $\frac{dy}{dx}$

Circle your answer.

[1 mark]

$\frac{dy}{dx} = e^{kx}$ $\quad$ $\frac{dy}{dx} = ke^{kx}$ $\quad$ $\frac{dy}{dx} = kxe^{kx-1}$ $\quad$ $\frac{dy}{dx} = \frac{e^{kx}}{k}$

3 The diagram below shows a sector of a circle.



The radius of the circle is 4 cm and $\theta = 0.8$ radians.

Find the area of the sector.

Circle your answer.

[1 mark]

$1.28\,\text{cm}^2$ $\quad$ $3.2\,\text{cm}^2$ $\quad$ $6.4\,\text{cm}^2$ $\quad$ $12.8\,\text{cm}^2$

Area $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times 0.8 = 6.4\,\text{cm}^2$

Do not write outside the box

4 The point A has coordinates $(-1, a)$ and the point B has coordinates $(3, b)$

The line AB has equation $5x + 4y = 17$

Find the equation of the perpendicular bisector of the points A and B.

[4 marks]

Line AB: $5x + 4y = 17$

$y = -\frac{5}{4}x + \frac{17}{4}$

Gradient of line AB: $-\frac{5}{4}$

Gradient of perpendicular bisector: $-1 \times -\frac{4}{5} = \frac{4}{5}$

Midpoint of AB $= \left(\frac{3-1}{2}, \frac{a+b}{2}\right) = \left(1, \frac{a+b}{2}\right)$

When $x=1$, $y = -\frac{5}{4}(1) + \frac{17}{4} = 3$ so midpoint is $(1, 3)$.

Perpendicular bisector: $y - 3 = \frac{4}{5}(x-1)$

$y = \frac{4}{5}x - \frac{4}{5} + 3$

$y = \frac{4}{5}x + \frac{11}{5}$

Turn over for the next question

5 An arithmetic sequence has first term a and common difference d.

The sum of the first 16 terms of the sequence is 260

5 (a) Show that $4a + 30d = 65$

[2 marks]

$S_n = \frac{n}{2}(2a + (n-1)d)$

$260 = \frac{16}{2}(2a + (16-1)d)$

$260 = 8(2a + 15d)$

$65 = 2(2a + 15d)$

$4a + 30d = 65$

5 (b) Given that the sum of the first 60 terms is 315, find the sum of the first 41 terms.

[3 marks]

$S_n = \frac{n}{2}(2a + (n-1)d)$

$315 = 30(2a + 59d)$

$105 = 10(2a + 59d)$

$105 = 20a + 590d$ ①

From (a): $65 = 4a + 30d$ ②

① − 5 × ②: $105 - 5(65) = 590d - 5(30d)$

$-220 = 440d$

$d = -0.5$

Substitute $d = -2$ into ①: $105 = 20a + 590(-0.5)$

$20a = 400$

$a = 20$

$S_{41} = \frac{41}{2}(2a + 40d)$

$= \frac{41}{2}(40 - 40 \times 0.5)$

$= \frac{41}{2}(20) = 410$

Do not write outside the box

5 (c) S_n is the sum of the first n terms of the sequence.

Explain why the value you found in part **(b)** is the maximum value of S_n

[2 marks]

41^{st} term: $a + 40d = 20 - 0.5(40) = 0$

$20 - 0.5(n-1) > 0 \Rightarrow n-1 < 40 \Rightarrow n < 41$

This shows the terms before the 41^{st} term are all positive.

$20 - 0.5(n-1) < 0 \Rightarrow n-1 > 40 \Rightarrow n > 41$

This shows the terms after the 41^{st} term are all negative.

Since the terms before the 41^{st} term are all positive and the terms after the 41^{st} term are all negative, the sum of the first 41 terms gives the maximum value.

Turn over for the next question

Do not write outside the box

6 The function f is defined by

$$\mathrm{f}(x) = \frac{1}{2}(x^2 + 1),\ x \geq 0$$

6 (a) Find the range of f.

[1 mark]

$\{y : y \geq \frac{1}{2}\}$

6 (b) (i) Find $\mathrm{f}^{-1}(x)$

[3 marks]

$y = \frac{1}{2}(x^2 + 1)$

$2y = x^2 + 1$

$x^2 = 2y - 1$

$x = \sqrt{2y - 1}$

$f^{-1}(x) = \sqrt{2x - 1}$ Domain: $x \geq \frac{1}{2}$

6 (b) (ii) State the range of $\mathrm{f}^{-1}(x)$

[1 mark]

$\{y : y \geq 0\}$

6 (c) State the transformation which maps the graph of $y = f(x)$ onto the graph of $y = f^{-1}(x)$

[1 mark]

Reflection in the line $y=x$.

6 (d) Find the coordinates of the point of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$

[2 marks]

They intersect on the line $y=x$.

$x = \frac{1}{2}(x^2+1)$

$2x = x^2 + 1$

$x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$x = 1$

When $x=1$, $y = \frac{1}{2}(x^2+1) = 1$.

Coordinates of intersection: $(1,1)$

Turn over for the next question

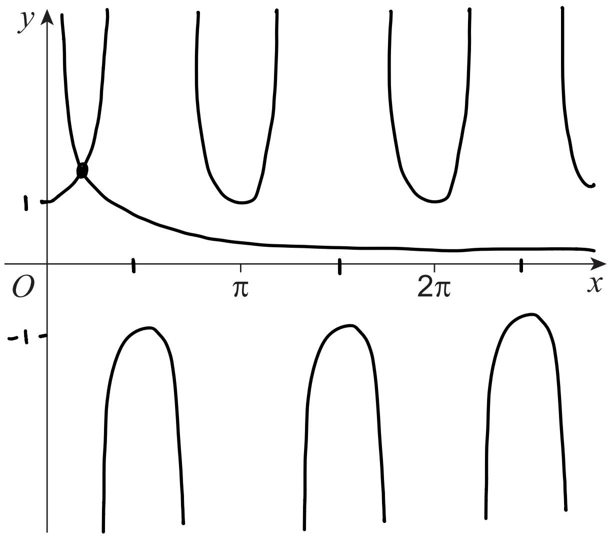
Do not write outside the box

7 (a) By sketching the graphs of $y = \dfrac{1}{x}$ and $y = \sec 2x$ on the axes below, show that the equation

$$\frac{1}{x} = \sec 2x$$

has exactly one solution for $x > 0$

[3 marks]



7 (b) By considering a suitable change of sign, show that the solution to the equation lies between 0.4 and 0.6

[2 marks]

$\frac{1}{x} = \sec 2x$

$\frac{1}{x} - \sec 2x = 0$

$f(x) = \frac{1}{x} - \sec 2x$

$f(0.4) = \frac{1}{0.4} - \sec(0.8) = 1.064\ldots > 0$

$f(0.6) = \frac{1}{0.6} - \sec(1.2) = -1.0930\ldots < 0$

The change of sign indicates the solution lies between 0.4 and 0.6.

7 (c) Show that the equation can be rearranged to give

$$x = \frac{1}{2}\cos^{-1} x$$

[2 marks]

$\frac{1}{x} = \sec 2x$

$\frac{1}{x} = \frac{1}{\cos 2x}$

$x = \cos 2x$

Do not write outside the box

$\cos^{-1}(x) = 2x$

$x = \frac{1}{2}\cos^{-1}(x)$

7 (d) (i) Use the iterative formula

$$x_{n+1} = \frac{1}{2}\cos^{-1} x_n$$

with $x_1 = 0.4$, to find x_2, x_3 and x_4, giving your answers to four decimal places.

[2 marks]

$x_1 = 0.4$

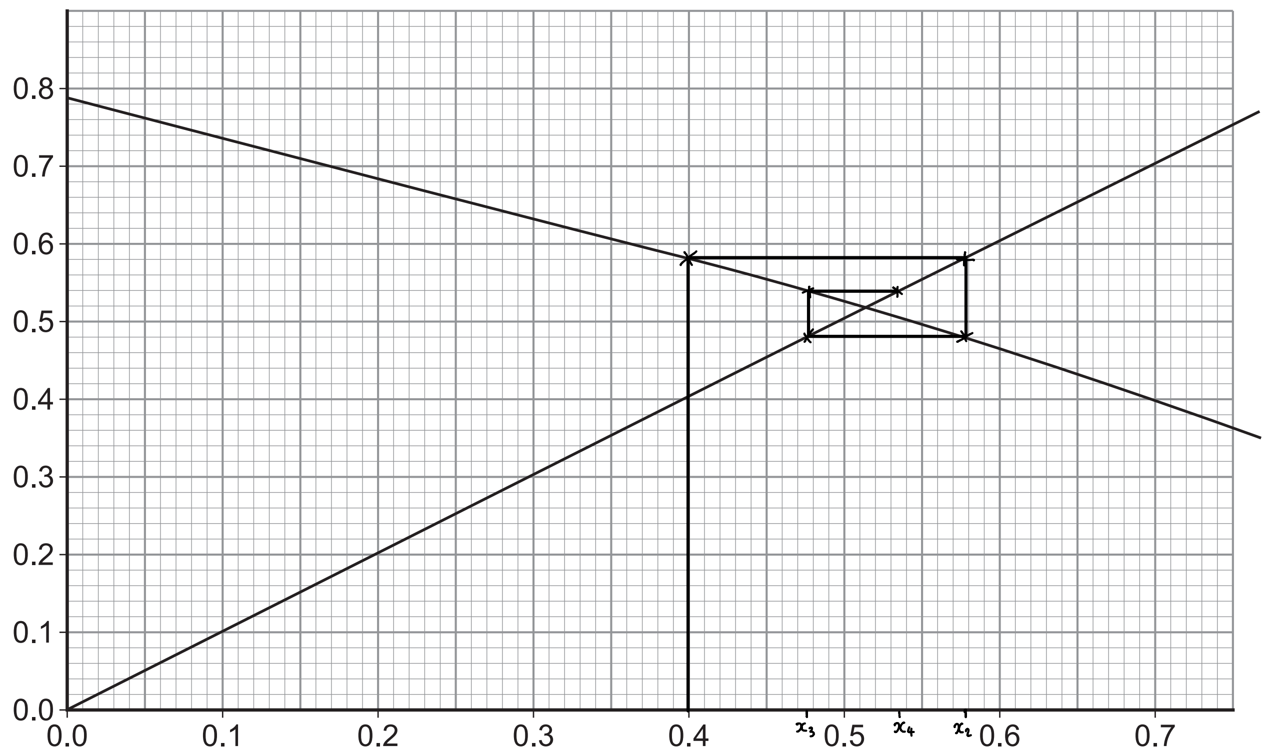
$x_2 = \frac{1}{2}\cos^{-1}(0.4) = 0.57963\ldots = 0.5796$

$x_3 = \frac{1}{2}\cos^{-1}(0.57963\ldots) = 0.47625\ldots = 0.4763$

$x_4 = \frac{1}{2}\cos^{-1}(0.47625\ldots) = 0.53720\ldots = 0.5372$

7 (d) (ii) On the graph below, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2, x_3 and x_4.

[2 marks]



Do not write outside the box

8 $\mathrm{P}(n) = \sum_{k=0}^{n} k^3 - \sum_{k=0}^{n-1} k^3$ where n is a positive integer.

8 (a) Find $\mathrm{P}(3)$ and $\mathrm{P}(10)$

[2 marks]

$P(3) = \sum_{k=0}^{3} k^3 - \sum_{k=0}^{2} k^3$

$= 0^3 + 1^3 + 2^3 + 3^3 - 0^3 - 1^3 - 2^3 = 3^3 = 27$

$P(10) = \sum_{k=0}^{10} k^3 - \sum_{k=0}^{9} k^3$

$= 0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 9^3 + 10^3 - 0^3 - 1^3 - 2^3 - 3^3 - 4^3 - 5^3 - 6^3 - 7^3 - 8^3 - 9^3$

$= 10^3 = 1000$

8 (b) Solve the equation $\mathrm{P}(n) = 1.25 \times 10^8$

[2 marks]

From (a) we see that $P(n) = n^3$.

$n^3 = 1.25 \times 10^8$

$n = \sqrt[3]{1.25 \times 10^8}$

$n = 500$

Do not write outside the box

9 Prove that the sum of a rational number and an irrational number is always irrational.
[5 marks]

Assume p is rational and q is irrational and $p+q$ is rational.

Since p is rational, $p = \frac{a}{b}$, where a, b are integers.

Since $p+q$ is rational, $p+q = \frac{c}{d}$, where c, d are integers.

So $\frac{a}{b} + q = \frac{c}{d}$

$\Rightarrow q = \frac{c}{d} - \frac{a}{b}$

$q = \frac{bc - ad}{bd}$

Therefore q is rational, which contradicts our original assumption.

Therefore the original statement is false and the sum of a rational and irrational must be irrational.

Turn over for the next question

Do not write outside the box

10 The volume of a spherical bubble is increasing at a constant rate.

Show that the rate of increase of the radius, r, of the bubble is inversely proportional to r^2

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

[4 marks]

The volume is increasing at a constant rate so $\frac{dV}{dt} = k$.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$k = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{k}{4\pi r^2}$

Therefore $\frac{dr}{dt} \propto \frac{1}{r^2}$.

Turn over for the next question

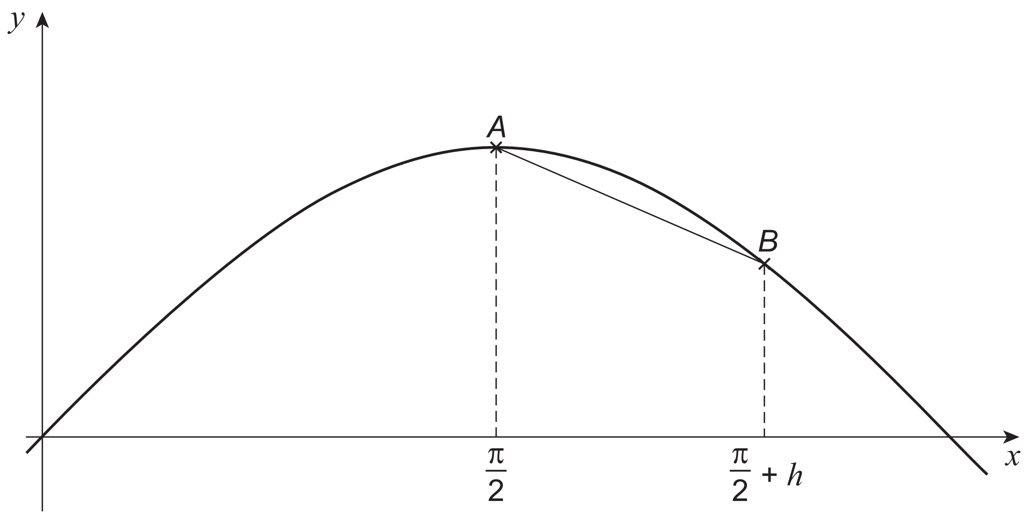
DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Turn over ▶

Do not write outside the box

11 Jodie is attempting to use differentiation from first principles to prove that the gradient of $y = \sin x$ is zero when $x = \dfrac{\pi}{2}$

Jodie's teacher tells her that she has made mistakes starting in Step 4 of her working. Her working is shown below.



Step 1 Gradient of chord $AB = \dfrac{\sin\left(\frac{\pi}{2} + h\right) - \sin\left(\frac{\pi}{2}\right)}{h}$

Step 2 $= \dfrac{\sin\left(\frac{\pi}{2}\right)\cos(h) + \cos\left(\frac{\pi}{2}\right)\sin(h) - \sin\left(\frac{\pi}{2}\right)}{h}$

Step 3 $= \sin\left(\frac{\pi}{2}\right)\left(\dfrac{\cos(h) - 1}{h}\right) + \cos\left(\frac{\pi}{2}\right)\dfrac{\sin(h)}{h}$

Step 4 For gradient of curve at A,

let $h = 0$ then

$\dfrac{\cos(h) - 1}{h} = 0$ and $\dfrac{\sin(h)}{h} = 0$

Step 5 Hence the gradient of the curve at A is given by

$\sin\left(\frac{\pi}{2}\right) \times 0 + \cos\left(\frac{\pi}{2}\right) \times 0 = 0$

Do not write outside the box

Complete Steps 4 and 5 of Jodie's working below, to correct her proof.

[4 marks]

Step 4 For gradient of curve at *A*,

let $h \to 0$ then $\frac{\cos(h)-1}{h} \to 0$ and $\frac{\sin(h)}{h} \to 1$.

Step 5 Hence the gradient of the curve at *A* is given by

$\sin\left(\frac{\pi}{2}\right) \times 0 + \cos\left(\frac{\pi}{2}\right) \times 1 = 0$

Turn over for the next question

Do not write outside the box

12 (a) Show that the equation

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

can be written in the form

$$a\operatorname{cosec}^2 x + b\operatorname{cosec} x + c = 0$$

[2 marks]

$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$

$2(\operatorname{cosec}^2 x - 1) + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

Do not write outside the box

12 (b) Hence, given x is obtuse and

$$2\cot^2 x + 2\operatorname{cosec}^2 x = 1 + 4\operatorname{cosec} x$$

find the exact value of $\tan x$

Fully justify your answer.

[5 marks]

$4\operatorname{cosec}^2 x - 4\operatorname{cosec} x - 3 = 0$

$(2\operatorname{cosec} x - 3)(2\operatorname{cosec} x + 1) = 0$

$\operatorname{cosec} x = \frac{3}{2}$ or $\operatorname{cosec} x = -\frac{1}{2}$

Reject $\operatorname{cosec} x = -\frac{1}{2}$ as $\operatorname{cosec} x \geq 1$ or $\operatorname{cosec} x \leq -1$.

So $\operatorname{cosec} x = \frac{3}{2}$.

$\cot^2 x = \operatorname{cosec}^2 x - 1$

$\cot^2 x = \left(\frac{3}{2}\right)^2 - 1 = \frac{5}{4}$

$\tan x = \frac{\pm 1}{\sqrt{5/4}} = \frac{\pm 2}{\sqrt{5}} = \pm\frac{2\sqrt{5}}{5}$

Since x is obtuse, $\tan x = -\frac{2\sqrt{5}}{5}$.

Turn over for the next question

Do not write outside the box

13 A curve, C, has equation

$$y = \frac{e^{3x-5}}{x^2}$$

Show that C has exactly one stationary point.

Fully justify your answer.

[7 marks]

$y = \frac{e^{3x-5}}{x^2}$

$\frac{dy}{dx} = \frac{3e^{3x-5}x^2 - 2x e^{3x-5}}{x^4}$

Turning points occur when $\frac{dy}{dx} = 0$

$\Rightarrow \frac{3e^{3x-5}x^2 - 2xe^{3x-5}}{x^4} = 0$

$\Rightarrow 3e^{3x-5}x^2 - 2xe^{3x-5} = 0$

$\Rightarrow x(3x-2)e^{3x-5} = 0$

As $e^{3x-5} > 0$, $x = 0$ or $x = \frac{2}{3}$. But $x \neq 0$ as y is undefined at $x = 0$ so there is only one stationary point at $x = \frac{2}{3}$.

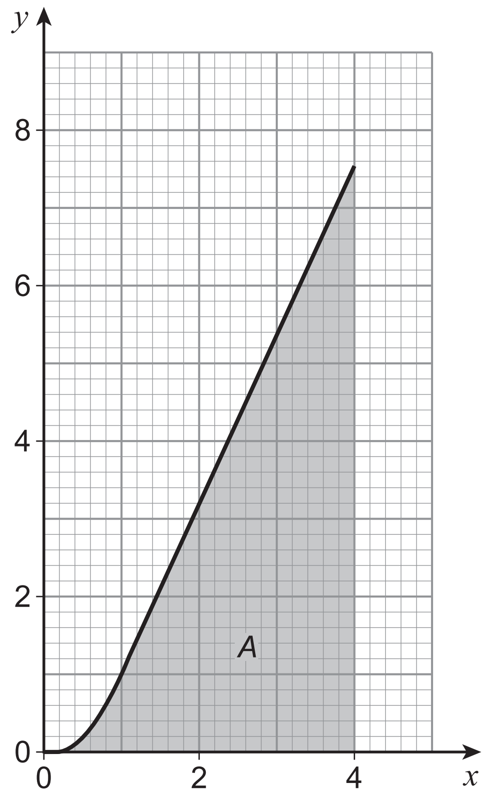
Turn over for the next question

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Turn over ▶

Do not write outside the box

14 The graph of $y = \dfrac{2x^3}{x^2+1}$ is shown for $0 \le x \le 4$



Caroline is attempting to approximate the shaded area, *A*, under the curve using the trapezium rule by splitting the area into n trapezia.

14 (a) When $n = 4$

14 (a) (i) State the number of ordinates that Caroline uses.

[1 mark]

5

14 (a) (ii) Calculate the area that Caroline should obtain using this method.

Give your answer correct to two decimal places.

[3 marks]

$y = \dfrac{2x^3}{x^2+1}$

When $x=0$: $y=0$

When $x=1$: $y=1$

When $x=2$: $y=3.2$

Do not write outside the box

When $x=3$: $y=5.4$

When $x=4$: $y=7.52941$

Area $\approx \frac{1}{2} \times 1 \times (0 + 2(1 + 3.2 + 5.4) + 7.52941)$

$= 13.364705 = 13.36$ (2. d.p)

14 (b) Show that the exact area of *A* is

$$16 - \ln 17$$

Fully justify your answer.

[5 marks]

$\int_0^4 \frac{2x^3}{x^2+1} \, dx$

Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{1}{2x} du$

When $x=0$, $u=1$.

When $x=4$, $u=17$.

$\int_1^{17} \frac{2x^3}{u} \times \frac{1}{2x} du = \int_1^{17} \frac{x^2}{u} du = \int_1^{17} \frac{u-1}{u} du = \int_1^{17} (1 - \frac{1}{u}) du$

$= \left[u - \ln u\right]_1^{17}$

$= (17 - \ln 17) - (1 - \ln 1)$

$= 16 - \ln 17$

Question 14 continues on the next page

Do not write outside the box

14 (c) Explain what would happen to Caroline's answer to part **(a)(ii)** as $n \to \infty$

[1 mark]

As n increases, Caroline's answer will tend to $16 - \ln 17$.

Turn over for the next question

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

15 (a) At time t hours **after a high tide**, the height, h metres, of the tide and the velocity, v knots, of the tidal flow can be modelled using the parametric equations

$$v = 4 - \left(\frac{2t}{3} - 2\right)^2$$

$$h = 3 - 2\sqrt[3]{t-3}$$

High tides and low tides occur alternately when the velocity of the tidal flow is zero.

A high tide occurs at 2 am.

15 (a) (i) Use the model to find the height of this high tide.

[1 mark]

At $t=0$, $h = 3 - 2\sqrt[3]{0-3} = 5.884499141$

$= 5.88$ metres

15 (a) (ii) Find the time of the first **low** tide after 2 am.

[3 marks]

Low tide occurs when $v=0$: $4 - \left(\frac{2t}{3} - 2\right)^2 = 0$

$4 = \left(\frac{2t}{3} - 2\right)^2$

$2 = \frac{2t}{3} - 2$

$\frac{2t}{3} = 4$

$t = 6$

6 hours after 2am is 8am.

15 (a) (iii) Find the height of this low tide.

[1 mark]

When $t=6$, $h = 3 - 2\sqrt[3]{6-3} = 0.1155008...$

$= 0.12$ metres

Do not write outside the box

15 (b) Use the model to find the height of the tide when it is flowing with maximum velocity.

[3 marks]

$v = 4 - \left(\frac{2t}{3} - 2\right)^2$

Maximum velocity occurs when $\left(\frac{2t}{3} - 2\right) = 0$

$\Rightarrow 2t = 6 \Rightarrow t = 3$

At $t = 3$, $h = 3 - 2\sqrt[3]{3-3}$

$= 3$ metres.

15 (c) Comment on the validity of the model.

[2 marks]

After one tide cycle, the model is no longer valid.

After 6 hours the model predicts that the height continues to decrease.

Turn over for the next question

Do not write outside the box

16 (a) $y = e^{-x}(\sin x + \cos x)$

Find $\frac{dy}{dx}$

Simplify your answer.

[3 marks]

$\frac{dy}{dx} = -e^{-x}(\sin x + \cos x) + e^{-x}(\cos x - \sin x)$

$= -e^{-x}\sin x - e^{-x}\cos x + e^{-x}\cos x - e^{-x}\sin x$

$= -2e^{-x}\sin x$

16 (b) Hence, show that

$$\int e^{-x}\sin x \, dx = ae^{-x}(\sin x + \cos x) + c$$

where a is a rational number.

[2 marks]

$\int (-2e^{-x}\sin x)\,dx = e^{-x}(\sin x + \cos x) + K$

So,

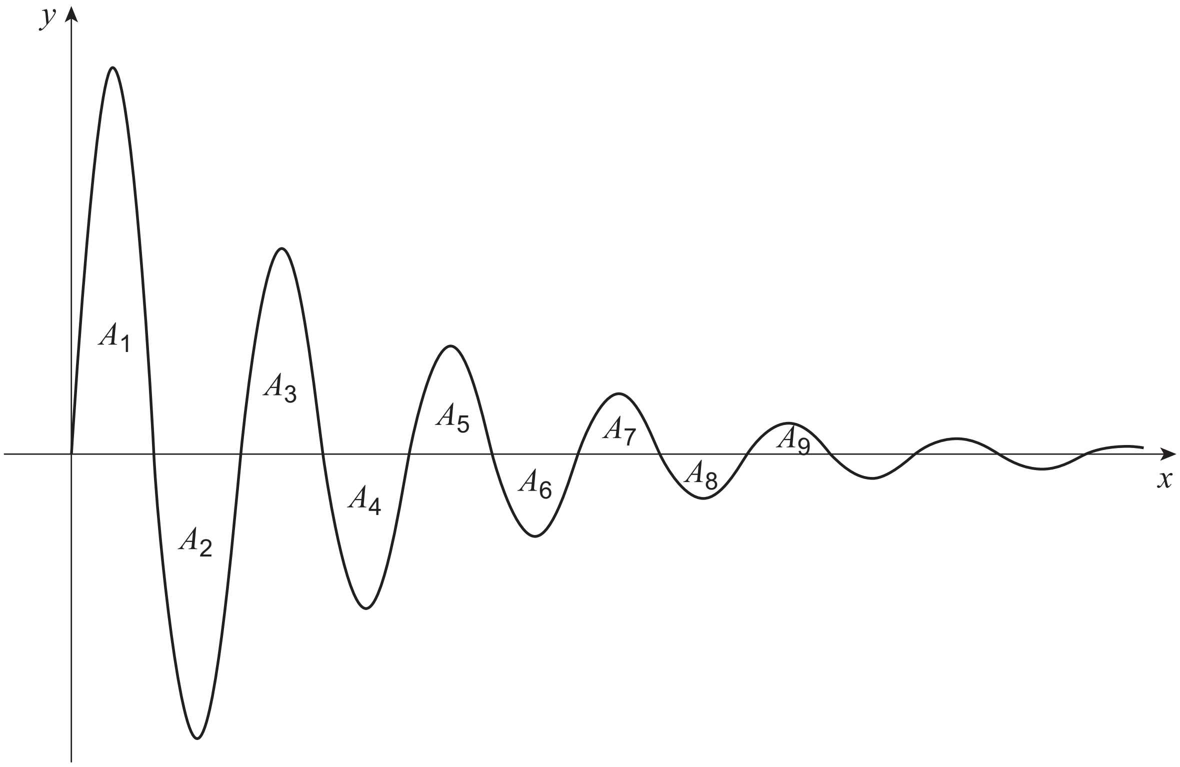
$\int e^{-x}\sin x \, dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x) + K\right]$

$= -\frac{1}{2}e^{-x}(\sin x + \cos x) + c$

Do not write outside the box

16 (c) A sketch of the graph of $y = e^{-x} \sin x$ for $x \geq 0$ is shown below.

The areas of the finite regions bounded by the curve and the x-axis are denoted by $A_1, A_2, ..., A_n, ...$



16 (c) (i) Find the exact value of the area A_1

[3 marks]

$$\int_0^{\pi} (e^{-x} \sin x)\,dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_0^{\pi}$$

$$= -\frac{1}{2}\left[e^{-\pi}(\sin\pi + \cos\pi) - e^{-0}(\sin 0 + \cos 0)\right]$$

$$= -\frac{1}{2}\left[e^{-\pi}(-1) - (1)\right]$$

$$= \frac{e^{-\pi}+1}{2}$$

Question 16 continues on the next page

Turn over ▶

Do not write outside the box

16 (c) (ii) Show that

$$\frac{A_2}{A_1} = e^{-\pi}$$

[4 marks]

Area of A_2 : $\int_{\pi}^{2\pi} (e^{-x}\sin x)\,dx = -\frac{1}{2}\left[e^{-x}(\sin x + \cos x)\right]_{\pi}^{2\pi}$

$= -\frac{1}{2}\left[e^{-2\pi}(\sin 2\pi + \cos 2\pi) - e^{-\pi}(\sin\pi + \cos\pi)\right]$

$= -\frac{1}{2}\left[e^{-2\pi}(1) - e^{-\pi}(-1)\right]$

$= -\frac{1}{2}\left[e^{-2\pi} + e^{-\pi}\right] = \frac{-e^{-2\pi} - e^{-\pi}}{2}$

Area is positive so $A_2 = \frac{e^{-2\pi} + e^{-\pi}}{2}$.

$\frac{A_2}{A_1} = \frac{\frac{e^{-2\pi}+e^{-\pi}}{2}}{\frac{e^{-\pi}+1}{2}} = \frac{e^{-2\pi}+e^{-\pi}}{e^{-\pi}+1} = \frac{e^{-\pi}+1}{1+e^{\pi}} = \frac{e^{-\pi}(1+e^{\pi})}{1+e^{\pi}} = e^{-\pi}$

Do not write outside the box

16 (c) (iii) Given that

$$\frac{A_{n+1}}{A_n} = e^{-\pi}$$

show that the exact value of the total area enclosed between the curve and the x-axis is

$$\frac{1 + e^{\pi}}{2(e^{\pi} - 1)}$$

[4 marks]

The areas form a geometric series with $r = e^{-\pi}$ and $a = A_1$.

So,

$$\frac{a}{1-r} = \frac{e^{-\pi}+1}{2} \times \frac{1}{1-e^{-\pi}}$$

$$= \frac{e^{-\pi}+1}{2(1-e^{-\pi})}$$

$$= \frac{1+e^{\pi}}{2(e^{\pi}-1)}$$

END OF QUESTIONS

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

Do not write outside the box

There are no questions printed on this page

DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED

There are no questions printed on this page

**DO NOT WRITE ON THIS PAGE
ANSWER IN THE SPACES PROVIDED**