A-LEVEL
Further Mathematics

F2

Mark scheme

Specimen

Version 1.1

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
dM	mark is dependent on one or more M marks and is for method
R	mark is for reasoning
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
‘their’	Indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, only the last complete attempt should be awarded marks.

Q	Marking Instructions	AO	Marks	Typical Solution
1	Circles correct answer	AO1.1b	B1	$\frac{\pi}{12}$
	Total		**1**	
2	Defines generalised z and z^* in Cartesian or polar form	AO1.2	B1	Let $z = a + bi$ then $z^* = a - bi$
	Expands and simplifies zz^* and $\|z\|^2$ (at least one correct)	AO1.1b	M1	$zz^* - \|z\|^2 = (a+bi)(a-bi) - \left(\sqrt{a^2+b^2}\right)^2$ $= a^2 + abi - abi - (bi)^2 - (a^2+b^2)$ $= a^2 + b^2 - (a^2+b^2)$ $= 0$
	Completes a well-structured argument to prove the required result. **AG** Mark awarded if they have a completely correct solution, which is clear, easy to follow and contains no slips	AO2.1	R1	ALT Let $z = re^{i\theta}$ then $z^* = re^{-i\theta}$ $zz^* - \|z\|^2 = re^{i\theta}re^{-i\theta} - r^2$ $= r^2e^{i\theta - i\theta} - r^2$ $= r^2 - r^2$ $= 0$
	Total		**3**	

Q	Marking Instructions	AO	Marks	Typical Solution
3	Commences a proof by correctly setting up an equation using the definition of an invariant point	AO2.1	R1	For an invariant point $\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
	Pre-multiplies by $\mathbf{M}^{-1}$.	AO2.1	R1	Pre-multiply both sides by $\mathbf{M}^{-1}$ $\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$
	Uses $\mathbf{M}^{-1}\mathbf{M} = \boldsymbol{I}$ and concludes their rigorous mathematical argument to deduce that (x, y) is invariant under S **AG**	AO2.2a	R1	$\mathbf{M}^{-1}\mathbf{M} = \boldsymbol{I}$ hence $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ Therefore $\begin{pmatrix} x \\ y \end{pmatrix}$ is invariant under S.
	Total		**3**	

Q	Marking Instructions	AO	Marks	Typical Solution
4	Expresses i or z in polar form	AO1.2	B1	$\mathrm{i} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$
	Uses de Moivre's Theorem	AO3.1a	M1	$z = \left[\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}+2n\pi)}\right]^{\frac{1}{3}} = \left[\mathrm{e}^{\mathrm{i}(\frac{\pi}{6}+\frac{2n\pi}{3})}\right]$
	Finds three consecutive values for θ	AO1.1a	A1	$\frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2} \left(\text{or } \frac{3\pi}{2} \text{ etc}\right)$ $z = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$
	Finds all three correct solutions for z	AO1.1b	A1	ALT $z = \mathrm{e}^{\mathrm{i}\theta} \Rightarrow z = \cos\theta + \mathrm{i}\sin\theta$ $z^3 = \mathrm{i} \Rightarrow \cos 3\theta + \mathrm{i}\sin 3\theta = \mathrm{i}$ $\therefore \cos 3\theta = 0$ and $\sin 3\theta = 1$ $\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2} \left(\text{or } \frac{3\pi}{2} \text{ etc}\right)$ $z = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}, \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}, \mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical Solution
5	Uses de Moivre's theorem	AO3.1a	M1	$(\cos\theta + \mathrm{i}\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-\mathrm{i})$ $\Rightarrow \cos 5\theta + \mathrm{i}\sin 5\theta = \frac{1}{\sqrt{2}}(1-\mathrm{i})$
	Equates real and imaginary parts and obtains two equations	AO1.1a	A1	$\cos 5\theta = \frac{1}{\sqrt{2}} \quad \sin 5\theta = -\frac{1}{\sqrt{2}}$
	Deduces that the smallest possible value of 5θ is $\frac{7\pi}{4}$ FT from 'their' equations provided M1 has been awarded	AO2.2a	A1F	$(5\theta =)\frac{7\pi}{4}$
	Obtains the smallest possible value of θ from fully correct reasoning FT from 'their' 5θ provided M1 has been awarded	AO1.1b	A1F	$\theta = \frac{7\pi}{20}$
	Total		**4**	

Q	Marking Instructions	AO	Marks	Typical Solution
6	Uses proof by induction and investigates the expression for $n = 0$ and $n = k$ (must see evidence of both $n = 0$ and $n = k$ being considered)	AO3.1a	B1	Let $\mathrm{f}(n) = 8^n - 7n + 6$ $\mathrm{f}(0) = 1 + 6 = 7$ $\Rightarrow \mathrm{f}(n)$ is divisible by 7 when $n = 0$ Consider $n = k$ Assume that $\mathrm{f}(k)$ is divisible by 7 $\mathrm{f}(k+1) = 8^{k+1} - 7(k+1) + 6$ $\mathrm{f}(k+1) - 8\mathrm{f}(k) = 56k - 7(k+1) + 6 - 48$ $\mathrm{f}(k+1) - 8\mathrm{f}(k) = 49k - 49$ $\mathrm{f}(k+1) = 8\mathrm{f}(k) + 49(k-1)$ $= 8\mathrm{f}(k) + 7(7k - 7)$ $\therefore \mathrm{f}(k+1)$ is divisible by 7 since $\mathrm{f}(k)$ is divisible by 7 Therefore $\mathrm{f}(k)$ is divisible by 7 $\Rightarrow$ $\mathrm{f}(k+1)$ is divisible by 7
	Shows that statement is true for $n = 0$	AO1.1b	B1	
	Commences argument by considering $\mathrm{f}(k + 1)$ in terms of $\mathrm{f}(k)$	AO2.1	R1	
	Makes correct deduction that if $\mathrm{f}(n)$ is divisible by 7 then $\mathrm{f}(n + 1)$ is also divisible by 7	AO2.2a	R1	
	Completes a rigorous argument and explains how their argument proves the required result. **AG**	AO2.4	R1	Since $\mathrm{f}(0)$ is divisible by 7 *and* $\mathrm{f}(k)$ is divisible by 7 $\Rightarrow \mathrm{f}(k + 1)$ is divisible by 7 then, by induction, $\mathrm{f}(n) = 8^n - 7n + 6$ is divisible by 7 for all integers $n \geq 0$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
6 ALT				Let $f(n) = 8^n - 7n + 6$ $f(0) = 1 + 6 = 7$ $\Rightarrow f(n)$ is divisible by 7 when n=0 Consider $n = k$ Assume that f(k) is divisible by 7 $f(k+1) = 8^{k+1} - 7(k+1) + 6$ $= 8\left(8^k - 7k + 6\right) + 8 \times 7k - 7k - 1 - 48$ $= 8f(k) + 49k - 49$ $= 8f(k) + 7(7k - 7)$ $\therefore f(k+1)$ is divisible by 7 since $f(k)$ is divisible by 7 Therefore f(k) is divisible by 7 $\Rightarrow$ f(k + 1) is divisible by 7 Since $f(0)$ is divisible by 7 *and* f(k) is divisible by 7 $\Rightarrow$f(k + 1) is divisible by 7 then, by induction, $f(n) = 8^n - 7n + 6$ is divisible by 7 for all integers $n \geq 0$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
7(a)	Models the motion of the ball by forming an equation of motion	AO3.1b	M1	$m\frac{d^2x}{dt^2} = (-12.5mx) \times 2$ $\Rightarrow \frac{d^2x}{dt^2} = -25x$
	Uses SHM equations to form model for displacement	AO3.1a	M1	$\therefore x = A\sin(5t)$
	Uses initial condition to find the constant	AO1.1a	M1	$\Rightarrow \dot{x} = 5A\cos(5t)$
	Obtains correct value for constant	AO1.1b	A1	when $t = 0$, $\dot{x} = 0.75$ so $0.75 = 5A$ $A = 0.15$
	Interprets 'their' value to find minimum distance from *P*	AO3.2a	A1F	Hence $x = 0.15\sin(5t)$ Max displacement = 0.15 metres from O, when $\sin(5t) = \pm 1$, so minimum distance from *P* is 0.75 metres
(b)	Identifies a correct limitation of the model for example friction between ball and the surface or damping effect due to air	AO3.5b	B1	It is unlikely that the surface is perfectly smooth so friction will be acting. The ball will be likely to travel a smaller distance before coming to rest and the minimum distance of the ball from *P* may actually be greater than that calculated in part **(a)**.
	Correctly infers whether the distance is too big or too small based on the limitation they have identified. Accept any well-reasoned inference.	AO2.2b	R1	
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical Solution
8	Selects suitable split in choosing u and v using integration by parts	AO3.1a	M1	$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ $= \int_0^{\frac{\pi}{2}} (\sin^{n-1} x)(\sin x) \, dx$
	Uses integration by parts (allow one sign error)	AO1.1a	M1	$u = \sin^{n-1} x \quad \frac{du}{dx} = (n-1)\sin^{n-2} x \cos x$ $\frac{dv}{dx} = \sin x \quad v = -\cos x$
	Integrates fully correctly (no need to be simplified)	AO1.1b	A1	$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}}$ $+(n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$
	Uses the identity $\cos^2 x = 1 - \sin^2 x$ in $\int \cos^2 x \sin^{n-2} x \, dx$ (PI)	AO1.1b	B1	$I_n = [0] + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx$ $= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$ $= (n-1)(I_{n-2} - I_n)$
	Completes rigorous argument to show result with all steps in the argument clearly set out **AG**	AO2.1	R1	$\therefore I_n = (n-1) I_{n-2} - nI_n + I_n$ $\Rightarrow nI_n = (n-1) I_{n-2}$

Q	Marking Instructions	AO	Marks	Typical Solution
				ALT $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-2} x \sin^2 x \ dx$ $= \int_0^{\frac{\pi}{2}} \sin^{n-2} x \left(1 - \cos^2 x\right) \ dx$ $= I_{n-2} - \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \ dx$ $= I_{n-2} - \int_0^{\frac{\pi}{2}} (\sin^{n-2} x \cos x) \cos x \ dx$ $u = \cos x \qquad \frac{du}{dx} = -\sin x$ $\frac{dv}{dx} = \sin^{n-2} x \cos x \qquad v = \frac{1}{(n-1)} \sin^{n-1} x$ $I_n = I_{n-2} - \left[\frac{1}{(n-1)} \sin^{n-1} x \cos x \right]_0^{\frac{\pi}{2}}$ $- \int_0^{\frac{\pi}{2}} \frac{1}{(n-1)} \sin x \sin^{n-2} x \ dx$ $= I_{n-2} - [0] - \frac{1}{(n-1)} \int_0^{\frac{\pi}{2}} \sin^n x \ dx$ $= I_{n-2} - \frac{1}{(n-1)} I_n$ $\therefore (n-1) I_n = (n-1) I_{n-2} - I_n$ $\Rightarrow n I_n = (n-1) I_{n-2}$
	Total		**5**	

Q	Marking Instructions	AO	Marks	Typical Solution
9(a)	Explains that the claim is incorrect as **singular** square matrices do not have inverses.	AO2.3	E1	Statement is incorrect if either matrix is singular/has determinant equal to zero as the inverse will not exist
	Correctly gives an example of a singular matrix.	AO1.1b	B1	Eg $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ is singular
(b)	Correctly refines the statement using 'non-singular' or equivalent wording	AO2.3	B1	Given any two **non-singular** square matrices, **A** and **B**, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
(c)	Correctly recalls the inverse property for matrices **A** and **B** (seen at least once)	AO1.2	B1	**A** and **B** are non-singular so inverses exist hence
	Correctly uses associativity by regrouping (seen at least once)	AO2.5	B1	**A** and **B** are non-singular so inverses exist hence $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1}$ $= \mathbf{A}\boldsymbol{I}\mathbf{A}^{-1}$ $= \mathbf{AA}^{-1}$ $= \boldsymbol{I}$
	Correctly applies the identity property throughout and concludes their rigorous mathematical argument with no errors or omissions	AO2.1	R1	Since $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \boldsymbol{I}$ Then $(\mathbf{AB})^{-1} = (\mathbf{B}^{-1}\mathbf{A}^{-1})$
	Total		**6**	

Q	Marking Instructions	AO	Marks	Typical Solution
10	Splits integrand into partial fractions of the form $\frac{Ax+B}{x^2+5}+\frac{C}{3x+2}$	AO3.1a	M1	$\frac{4x-30}{(x^2+5)(3x+2)} \equiv \frac{Ax+B}{x^2+5}+\frac{C}{3x+2}$ $\Rightarrow 4x-30 \equiv (Ax+B)(3x+2)+C(x^2+5)$ Compare coefficients of x: $4=2A+3B$
	Sets up an identity from which to solve for A, B and C	AO1.1a	M1	Compare coeffcients of x^2: $0=3A+C$ Compare constant terms
	Obtains correct values of A, B and C CAO	AO1.1b	A1	$-30=2B+5C$ $\therefore -30=2B-15A$ $-30=2B-15(\frac{4-3B}{2})$
	Integrates 'their' two terms correctly FT provided both M1 marks awarded	AO1.1b	A1F	$B=0$ $A=2$ $C=-6$ $\int \frac{4x-30}{(x^2+5)(3x+2)}dx = \int \frac{2x}{x^2+5}-\frac{6}{3x+2}dx$ $= \ln(x^2+5)-2\ln(3x+2)+c$
	Applies the laws of logs to 'their' integral correctly	AO1.1a	M1	$\int_0^\infty \frac{4x-30}{(x^2+5)(3x+2)}dx$
	Applies limits (a and 0) to 'their' integral correctly	AO1.1a	M1	$=\lim_{a\to\infty}\int_0^a \frac{2x}{x^2+5}-\frac{6}{3x+2}dx$
	Shows the limiting process used with clear detailed working	AO2.1	R1	$=\lim_{a\to\infty}\left[\ln(x^2+5)-2\ln(3x+2)\right]_0^a$
	Obtains correct single term solution CAO	AO1.1b	A1	$=\lim_{a\to\infty}\left[\ln\left(\frac{x^2+5}{(3x+2)^2}\right)\right]_0^a$ $=\lim_{a\to\infty}\left[\ln\left(\frac{a^2+5}{9a^2+12x+4}\right)-\ln\left(\frac{5}{4}\right)\right]$ $=\ln\left(\frac{1}{9}\right)-\ln\left(\frac{5}{4}\right)$ $=\ln\left(\frac{4}{45}\right)$
	Total		**8**	

Q	Marking Instructions	AO	Marks	Typical Solution
11(a)	Uses $\frac{1}{2}\int r^2 \, \mathrm{d}\theta$ or $\int_0^{\pi} r^2 \, \mathrm{d}\theta$ OE	AO1.1a	M1	$\frac{1}{2}\int (4+2\cos\theta)^2 \, \mathrm{d}\theta$ $\frac{1}{2}\int_{-\pi}^{\pi}(16+16\cos\theta+4\cos^2\theta) \, \mathrm{d}\theta$ $\int_{-\pi}^{\pi}(8+8\cos\theta+(1+\cos 2\theta)) \, \mathrm{d}\theta$ $=\left[8\theta+8\sin\theta+\theta+\frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$ $=\left[9\theta+8\sin\theta+\frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$ $=(9\pi+0+0)-(-9\pi+0+0)$ $= 18\pi$
	Rewrites $\cos^2\theta$ in terms of $\cos 2\theta$	AO1.1a	M1	
	Correctly integrates 'their' expression, ft non-zero coefficients.	AO1.1b	A1F	
	Obtains required answer from fully correct mathematical argument	AO2.1	R1	
(b)	Selects appropriate method to determine polar equation by equating *OA* and *OB* to find θ	AO3.1a	M1	Let $A(r_1, \theta_1)$ and $B(r_2, \theta_2)$ $OA = OB \Rightarrow r_1 = r_2$ $\Rightarrow 4+2\cos\theta_1 = 4+2\cos\theta_2$ $\Rightarrow \theta_1 = -\theta_2$ Angle $AOB = \frac{\pi}{3} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{3}$ $\Rightarrow \theta_1 = \frac{\pi}{6}$ and $\theta_2 = -\frac{\pi}{6}$ $OA = OB = 4+\sqrt{3}$ *AB* is perpendicular to the initial line Polar equation of *AB* is $r\cos\theta = \left(4+\sqrt{3}\right)\frac{\sqrt{3}}{2}$ for $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$
	Uses the above to find two values of θ and hence deduce the lengths of *OA* and *OB* Award this mark for correct deduction using 'their' values of θ	AO2.2a	R1	
	Uses the correct polar equation for a perpendicular line $r = d\sec\theta$	AO3.1a	M1	
	Obtains a correct equation for *AB* (including correct specified range) CAO	AO1.1b	A1	
	Total		**8**	

Q	Marking Instructions	AO	Marks	Typical Solution
12(a)	Forms appropriate equation using **M**λ =λ**v**	AO1.1a	M1	$\begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 4\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
	Eliminates one variable	AO1.1a	M1	$-5a + 2b - 2c = 0$ $2a - 2b - 2c = 0$ $-a - 2b - 5c = 0$ $3a + 3c = 0$
	Deduces a correct eigenvector	AO2.2a	A1	eigenvector is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Q	Marking Instructions	AO	Marks	Typical Solution
(b)	Forms the characteristic equation of **M**	AO3.1a	M1	$\begin{vmatrix} -1-\lambda & 2 & -1 \\ 2 & 2-\lambda & -2 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$
	Obtains the correct characteristic equation - unsimplified	AO1.1b	A1	$(-1-\lambda)\left[(2-\lambda)(-1-\lambda)-4\right]$ $-2(-2-2\lambda-2)-1(-4+2-\lambda)=0$
	Obtains roots and identifies them as eigenvalues for 'their' characteristic equation	AO1.1b	A1F	$-\lambda^3+12\lambda+16=0$ $(4-\lambda)(\lambda^2+4\lambda+4)=0$ $-(\lambda+2)(\lambda-4)(\lambda+2)=0$ Eigenvalues are 4, –2, –2
	Forms an appropriate matrix equation using the eigenvalue –2 FT 'their' eigenvalue	AO3.1a	M1	$\begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
	Expands and simplifies to obtain a single equation in x, y and z FT 'their' matrix equation provided both M1 marks have been awarded	AO1.1b	A1F	$x+2y-z=0$
	Correctly deduces two linearly independent eigenvectors CAO	AO2.2a	A1	$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
	Correctly identifies that the matrix **D** must include 4 and 'their' other eigenvalue(s)	AO1.2	B1F	$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$
	Correctly identifies the corresponding **U** matrix from 'their' eigenvectors	AO1.1b	A1F	$\mathbf{U} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$
	Total		**11**	

Q	Marking Instructions	AO	Marks	Typical Solution
13	Explains that det**M**=0 when **M** is singular (Seen anywhere)	AO2.4	R1	**S** is singular $\Rightarrow \begin{vmatrix} a & a & x \\ x-b & a-b & x+1 \\ x^2 & a^2 & ax \end{vmatrix} = 0$
	Seeks factor by combining rows or columns to find a first linear factor for example $C_1' = C_1 - C_2$	AO3.1a	M1	det **S** $= \begin{vmatrix} 0 & a & x \\ x-a & a-b & x+1 \\ x^2-a^2 & a^2 & ax \end{vmatrix}$
	Extracts first factor correctly	AO1.1b	A1	$= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a-b & x+1 \\ x+a & a^2 & ax \end{vmatrix}$
	Combines rows or columns to find a second linear factor $R_3' = R_3 - aR_1$	AO1.1a	M1	det **S** $= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a-b & x+1 \\ x+a & 0 & 0 \end{vmatrix}$
	Extracts second factor correctly	AO1.1b	A1	$= (x-a)(x+a)\begin{vmatrix} a & x \\ a-b & x+1 \end{vmatrix}$
	Completes expansion and obtains final factor	AO1.1b	A1	$= (x-a)(x+a)(a+bx)$
	Deduces correct values of x FT 'their' factors	AO2.2a	A1F	$(x-a)(x+a)(a+bx) = 0$ $x = a, -a, -\frac{a}{b}$
	Total		**7**	

Q	Marking Instructions	AO	Marks	Typical Solution
14	Uses vector product and expands brackets correctly	AO1.1a	M1	$\lvert(\mathbf{a}+5\mathbf{b})\times(\mathbf{a}-4\mathbf{b})\rvert$
	Uses the correct notation and correct order with the vector product.	AO2.5	B1	$=\lvert\mathbf{a}\times\mathbf{a}-4\mathbf{a}\times\mathbf{b}+5\mathbf{b}\times\mathbf{a}-20\mathbf{b}\times\mathbf{b}\rvert$ $=\lvert\mathbf{0}-4\mathbf{a}\times\mathbf{b}+5\mathbf{b}\times\mathbf{a}-\mathbf{0}\rvert$
	Reduces the number of terms in 'their' expression by using $\mathbf{a}\times\mathbf{a}=\mathbf{b}\times\mathbf{b}=\mathbf{0}$	AO1.1a	M1	since **a** is parallel to **a** and **b** is parallel to **b** then $\mathbf{a}\times\mathbf{a}=\mathbf{0}$ and $\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $=\lvert-4\mathbf{a}\times\mathbf{b}-5\mathbf{a}\times\mathbf{b}\rvert$ since $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
	and explains their reasoning (must have clear statement that $\mathbf{a}\times\mathbf{a}=\mathbf{0}$)	AO2.4	E1	$=\lvert-9\mathbf{a}\times\mathbf{b}\rvert$ $=9\lvert\mathbf{a}\times\mathbf{b}\rvert$
	Uses $-\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ to collect 'their' terms together	AO1.1a	M1	$=9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90$
	and explains their reasoning (must have clear statement that $-\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{a}$ **OE)**	AO2.4	E1	$=9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$
	Recalls correctly the formula for the modulus of the vector product (may see $\lvert\mathbf{a}\rvert\times\lvert\mathbf{b}\rvert\sin\theta$ or may see $\lvert\mathbf{a}\rvert\times\lvert\mathbf{b}\rvert\sin 90^\circ$)	AO1.2	B1	
	Obtains $\lvert\mathbf{a}\times\mathbf{b}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ since vectors **a** and **b** are perpendicular	AO1.1b	A1	
	Completes a fully correct proof giving an answer of $9\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$ CAO	AO2.2a	R1	
	Total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
15(a)	Commences an argument by correctly expanding brackets and simplifying final term to 1/16	AO1.1a	M1	$(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})$ $=1-\frac{1}{4}e^{2i\theta}-\frac{1}{4}e^{-2i\theta}+\frac{1}{16}$ $=\frac{17}{16}-\frac{1}{4}(\cos 2\theta+i\sin 2\theta)-\frac{1}{4}(\cos 2\theta-i\sin 2\theta)$ $=\frac{17}{16}-\frac{1}{2}\cos 2\theta$ $=\frac{1}{16}(17-8\cos 2\theta)$
	Substitutes correctly for both $e^{2i\theta}$ and $e^{-2i\theta}$ in terms of $\cos 2\theta$ and $\sin 2\theta$ (seen anywhere in solution)	AO1.1b	B1	
	Completes argument and reaches stated result by collecting terms and simplifying correctly, no errors in working seen **AG**	AO2.1	R1	
(b)	Identifies series as a geometric series and states first term and common ratio correctly	AO1.1b	B1	Geometric series with first term $r=e^{2i\theta}$ and common ratio $a=\frac{1}{4}e^{2i\theta}$
	States and uses sum to infinity formula correctly FT incorrect values for first term and common ratio	AO1.1b	B1F	$S_\infty=\frac{a}{1-r}=\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}$

Q	Marking Instructions	AO	Marks	Typical Solution
(c)	Deduces that the series in part **(c)** is related to the real part of the series in part **(b)**	AO2.2a	R1	Series stated = real part of the series $e^{2i\theta}+\frac{1}{4}e^{4i\theta}+\frac{1}{16}e^{6i\theta}+\frac{1}{64}e^{8i\theta}+....$
	Selects an appropriate method by using the result in part **(b)** and multiplying appropriately to realise the denominator	AO3.1a	M1	Using result from previous part $\dfrac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}=\dfrac{e^{2i\theta}}{(1-\frac{1}{4}e^{2i\theta})}\times\dfrac{(1-\frac{1}{4}e^{-2i\theta})}{(1-\frac{1}{4}e^{-2i\theta})}$
	Substitutes to obtain an expression with cosines and sines only – using part **(a)** FT incorrect sum to infinity provided M1 has been awarded	AO1.1b	A1F	$=\dfrac{e^{2i\theta}-\frac{1}{4}}{(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})}$ $\dfrac{\cos 2\theta-\frac{1}{4}+i\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}$
	Identifies the real part and correctly completes the argument to reach the stated result. Only award for an error-free fully correct solution	AO2.1	R1	Real part = $\dfrac{\cos 2\theta-\frac{1}{4}}{\frac{1}{16}(17-8\cos 2\theta)}=\dfrac{16\cos 2\theta-4}{17-8\cos 2\theta}$
(d)	Identifies the imaginary part and states the correct expression	AO2.2a	R1	Required series = imaginary part of the given series hence $\dfrac{\sin 2\theta}{\frac{1}{16}(17-8\cos 2\theta)}=\dfrac{16\sin 2\theta}{17-8\cos 2\theta}$
	Total		**10**	

Q	Marking Instructions	AO	Marks	Typical Solution
16	Uses the mathematical model to find the volume by first finding the coordinate of *A*. To award this mark must see an attempt to find coords of *A*, **and** an attempt at volume of prism	AO3.4	M1	$x = 4t - 4$ $y = 12 - 12t$ $z = 4$ $4t - 4 - 3(12 - 12t) = 0$
	Selects method involving both equation of plane and equation of line to find coords of *A* **Either** using parametric form **or** using cross product Ignore sign errors	AO3.1a	M1	$40t$-40 = 0 $t = 1$ $\begin{pmatrix} 0 & 0 & 4 \end{pmatrix}$ OR
	Either collects terms together and solves to find value of parameter for 'their' equation **Or** correctly calculates cross product for 'their' vectors	AO1.1b	A1F	$\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -4 \\ 12 \\ 4 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 3y+4 \\ y-12 \\ z-4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
	Deduces the correct coordinates of *A*	AO2.2a	A1	$12(z-4) = 0 \Rightarrow z = 4$ $-12(3y+4) - 4(y-12) = 0$
	Selects a correct approach to calculate the volume of the prism.	AO3.1a	M1	$\Rightarrow y = 0, x = 0$ A has coordinates $(0, 0, 4)$
	Finds two sides of the triangle *ABC* in vector form FT 'their' *A*	AO1.2	A1F	$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$
	Finds area of *ABC* FT 'their' *A*	AO1.1b	A1F	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -21 \\ 63 \\ 0 \end{pmatrix}$
	Finds length of prism FT 'their' *A*	AO1.1b	A1F	Area $ABC = \dfrac{21\sqrt{10}}{2}$
	Gives their answer in context by correctly finding the volume of the roof with correct units. FT 'their' prism	AO1.1b	A1F	$d = 4\sqrt{10}$ Volume = $V = \dfrac{21\sqrt{10}}{2} \times 4\sqrt{10} = 420\ \text{m}^3$
	Total		**9**	
	Total		**100**	