AQA

SPECIMEN MATERIAL

Please write clearly, in block capitals.

Centre number | | | | | |    Candidate number | | | | |

Surname ______________________________

Forename(s) ______________________________

Candidate signature ______________________________

# A-level
# FURTHER MATHEMATICS
## Paper 2

Exam Date    Morning    Time allowed: 2 hours

**Materials**
For this paper you must have:
- The AQA booklet of formulae and statistical tables.
- You may use a graphics calculator.

**Instructions**
- Use black ink or black ball-point pen. Pencil should be used for drawing.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

**Information**
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

**Advice**
Unless stated otherwise, you may quote formulae, without proof, from the booklet.
You do not necessarily need to use all the space provided.

Answer **all** questions in the spaces provided.

**1** Given that $z_1 = 4\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ and $z_2 = 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$

state the value of $\arg\left(\frac{z_1}{z_2}\right)$

Circle your answer.

**[1 mark]**

$\frac{\pi}{12}$ $\qquad$ $\frac{4}{3}$ $\qquad$ $\frac{7\pi}{12}$ $\qquad$ $2$

$$\arg\left(\frac{4e^{i\pi/3}}{2e^{i\pi/4}}\right) = \arg\left(4e^{i\pi/3}\right) - \arg\left(2e^{i\pi/4}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{4\pi - 3\pi}{12}$$

$$= \frac{\pi}{12}$$

**2** Given that $z$ is a complex number and that $z^*$ is the complex conjugate of $z$

prove that $zz^* - |z|^2 = 0$

**[3 marks]**

Let $z = a+bi$ which means $z^* = a-bi$

$$zz^* - |z|^2 = (a+bi)(a-bi) - \left(\sqrt{a^2+b^2}\right)^2$$
$$= a^2 + abi - abi + b^2 - (a^2+b^2)$$
$$= a^2+b^2 - (a^2+b^2)$$
$$= 0$$

3 The transformation T is defined by the matrix **M**. The transformation S is defined by the matrix $\mathbf{M}^{-1}$. Given that the point $(x, y)$ is invariant under transformation T, prove that $(x, y)$ is also an invariant point under transformation S.

**[3 marks]**

If a point is invariant under T this means that

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow M^{-1}M\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore $\begin{pmatrix} x \\ y \end{pmatrix}$ is invariant under S

**4** Solve the equation $z^3 = \mathrm{i}$, giving your answers in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$

**[4 marks]**

Let $z = \cos\theta + i\sin\theta$

$\Rightarrow z^3 = (\cos\theta + i\sin\theta)^3$

$= \cos 3\theta + i\sin 3\theta$ (De Moirve's Theorem)

so $i = \cos 3\theta + i\sin 3\theta$

Comparing real and imaginary parts:

$0 = \cos 3\theta$ | $1 = \sin 3\theta$

$\Rightarrow 3\theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}$ | $\Rightarrow 3\theta = \frac{\pi}{2}, \frac{5\pi}{2}, -\frac{3\pi}{2}$

$\Rightarrow \theta = \pm \frac{\pi}{6}, \pm \frac{\pi}{2}, \pm \frac{5\pi}{6}$ | $\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$

The values that satisfy both equations are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{\pi}{2}$

$z = e^{i\pi/6}, e^{-i\pi/2}, e^{5i\pi/6}$

**5** Find the smallest value $\theta$ of for which

$$(\cos\theta + \mathrm{i}\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-\mathrm{i})\ \{\theta \in \mathbb{R}\ : \theta > 0\}$$

**[4 marks]**

$(\cos\theta + i\sin\theta)^5 = \frac{1}{\sqrt{2}}(1-i)$

$\cos 5\theta + i\sin 5\theta = \frac{1}{\sqrt{2}}(1-i)$

Comparing real and imaginary parts:

$\cos 5\theta = \frac{1}{\sqrt{2}}$ $\qquad$ $\sin 5\theta = -\frac{1}{\sqrt{2}}$

$5\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \ldots$ $\qquad$ $\Rightarrow 5\theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \ldots$

$\theta = \frac{\pi}{20}, \frac{7\pi}{20}, \frac{9\pi}{20}, \frac{3\pi}{4}, \ldots$ $\qquad$ $\Rightarrow \theta = \frac{\pi}{20}, \frac{7\pi}{20}, \ldots$

The smallest value that satisfies both conditions is $\theta = \frac{7\pi}{20}$

**6** Prove that $8^n - 7n + 6$ is divisible by 7 for all integers $n \geq 0$

**[5 marks]**

Let $p(n) = 8^n - 7n + 6$

Let $n = 0$

$p(0) = 8^0 - 7(0) + 6$

$= 1 - 0 + 6$

$= 7$ which is divisible by 7

So the statement is true for $n = 0$

Assume that for $n = k$, the statement is true such that

$p(k) = 8^k - 7^k + 6 = 7M$ where $M \in \mathbb{N}$,

Let $n = k+1$

$p(k+1) = 8^{k+1} - 7(k+1) + 6$

$= 8^k + 7(8^k) - 7k - 7 + 6$

$= (8^k - 7^k + 6) + 7(8^k - 1)$

$= 7(M + 8^k - 1)$

So the statement is true for $n=k+1$ when assumed true for $n=k$.

As it is also true for $n=0$, the statement is true for all $n$ $n \in \mathbb{N}$,

**7** A small, hollow, plastic ball, of mass $m$ kg is at rest at a point $O$ on a polished horizontal surface. The ball is attached to two identical springs. The other ends of the springs are attached to the points $P$ and $Q$ which are 1.8 metres apart on a straight line through $O$.

The ball is struck so that it moves away from $O$, towards $P$ with a speed of 0.75 m s$^{-1}$.

As the ball moves, its displacement from $O$ is $x$ metres at time $t$ seconds after the motion starts.

The force that each of the springs applies to the ball is $12.5mx$ newtons towards $O$.

The ball is to be modelled as a particle. The surface is assumed to be smooth and it is assumed that the forces applied to the ball by the springs are the only horizontal forces acting on the ball.

**7 (a)** Find the minimum distance of the ball from $P$, in the subsequent motion.

**[5 marks]**

Using $F=ma$, $-12.5mx - 12.5mx = ma$

$-25x = a$

$\Rightarrow \frac{d^2x}{dt^2} = -25x$ so $\frac{d^2x}{dt^2} + 25x = 0$

Auxillary equation: $\lambda^2 + 25 = 0$ so $\lambda = \pm 5i$

We now have $x = e^0(A\sin 5t + B\cos t)$

When $t=0$, $x=0$: $0 = B$ $\Rightarrow x = A\sin 5t$ and $\dot{x} = 5A\cos 5t$

When $t=0$, $\dot{x} = 0.75$: $0.75 = 5A$ and $A = 0.15$

So $x = 0.15\sin 5t$

$x$ is largest when $\sin 5t = \pm 1$ so the particle is 0.15m from O. Therefore, the minimum distance from P is 0.75m

**7 (b)** In practice the minimum distance predicted by the model is incorrect.

Is the minimum distance predicted by the model likely to be too big or too small?

Explain your answer with reference to the model.

**[2 marks]**

It is likely that there is some friction between the surface and the particle so it'll travel a shorter distance. This will mean the minimum distance of the ball from P will be larger than 0.75m

**8** Given that $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx \quad n \geq 0$

show that $n I_n = (n-1)I_{n-2} \quad n \geq 2$

**[5 marks]**

$I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin x \sin^{n-1} x \, dx$

$u = \sin^{n-1} x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)\sin^{n-2} x \cos x \qquad v = \cos x$

$I_n = \left[-\sin^{n-1} x \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} (n-1)\cos x \sin^{n-2} x \cos x \, dx$

$= 0 + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x \, dx = (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x \, dx$

$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int_0^{\pi/2} \sin^n x \, dx$

$I_n = (n-1)I_{n-2} - (n-1)I_n$

$I_n = (n-1)I_{n-2} - nI_n + I_n$

$0 = (n-1)I_{n-2} - nI_n$

$nI_n = (n-1)I_{n-2}$

**9** A student claims:

"Given any two non-zero square matrices, **A** and **B**, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$"

**9 (a)** Explain why the student's claim is incorrect giving a counter example.

**[2 marks]**

If one of the matricies is singular then it has no inverse so the statement will be incorrect

Eg $A = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$ is singular so $A^{-1}$ doesn't exist

**9 (b)** Refine the student's claim to make it fully correct.

**[1 mark]**

Given any two non-singular square matrices, A and B, then $(AB)^{-1} = B^{-1}A^{-1}$

**9 (c)** Prove that your answer to part **(b)** is correct.

**[3 marks]**

Let A and B be non-singular so invertible

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ (Matrix multiplication is associative)

$= AIA^{-1}$

$= AA^{-1}$

$= I$

so $(AB)(B^{-1}A^{-1}) = I$

$\Rightarrow B^{-1}A^{-1} = (AB)^{-1}$

Therefore $B^{-1}A^{-1} = (AB)^{-1}$

**10** Evaluate the improper integral $\int_0^{\infty} \frac{4x - 30}{(x^2 + 5)(3x + 2)} \mathrm{d}x$, showing the limiting process used.

Give your answer as a single term.

**[8 marks]**

$$\frac{4x-30}{(x^2+5)(3x+2)} = \frac{Ax+B}{x^2+5} + \frac{C}{3x+2}$$

$$4x-30 = (Ax+B)(3x+2) + C(x^2+5)$$

$$= 3Ax^2 + 3Bx + 2A + 3Bx + Cx^2 + 5C$$

$$= x^2(3A+C) + x(3B+2A) + 2B+5C$$

$\Rightarrow 0 = 3A + C \qquad 3B + 2A = 4 \qquad 2B + 5C = -30$

$C = -3A$

$\downarrow$

$2B + 5(-3A) = -30$

$2B - 15A = -30$ ①

$3B + 2A = 4$ ②

Putting ① and ② into a scientific calculator yields: $A = 2$, $B = 0$

$C = -3A = -3(2) = -6$

$$\int_0^{\infty} \frac{4x-30}{(x^2+5)(3x+2)} dx = \int_0^{\infty} \frac{2x}{x^2+5} dx - \int_0^{\infty} \frac{6}{3x+2} dx$$

$$= \Big[\ln(x^2+5) - 2\ln(3x+2)\Big]_0^{\infty}$$

$$= \lim_{a \to \infty} \Big[\ln(x^2+5) - 2\ln(3x+2)\Big]_0^{a}$$

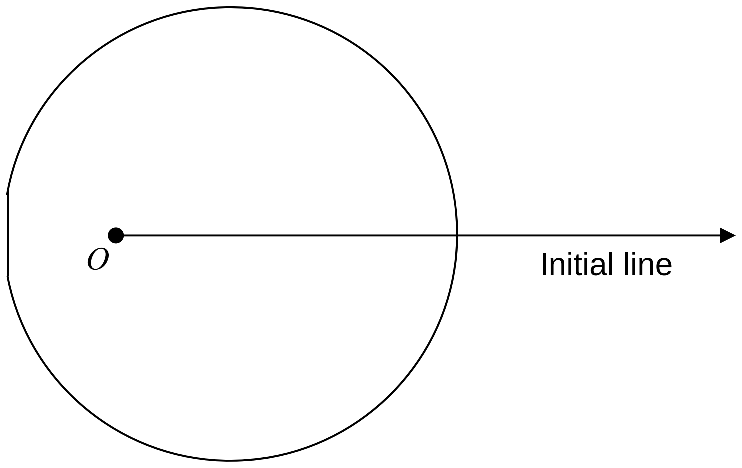
$$= \lim_{a \to \infty} \left[ \ln\left( \frac{x^2+5}{(3x+2)^2} \right) \right]_0^a$$

$$= \lim_{a \to \infty} \left[ \ln\left( \frac{a^2+5}{9a^2+12a+4} \right) - \ln\left( \frac{5}{4} \right) \right]$$

$$= \ln\left( \frac{1}{9} \right) - \ln\left( \frac{5}{4} \right) = \ln\left( \frac{4}{45} \right)$$

So $\displaystyle\int_0^\infty \frac{4x-30}{(x^2+5)(3x+2)}\, dx = \ln\left(\frac{4}{45}\right)$

**11** The diagram shows a sketch of a curve *C*, the pole $O$ and the initial line.




The polar equation of *C* is $r = 4 + 2\cos\theta$ , $-\pi \le \theta \le \pi$

**11 (a)** Show that the area of the region bounded by the curve *C* is 18π

**[4 marks]**

$A = \int_{-\pi}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} (4+2\cos\theta)^2 \, d\theta = 2\int_{-\pi}^{\pi} (2+\cos\theta)^2 \, d\theta$

$A = 2\int_{-\pi}^{\pi} (4 + 4\cos\theta + \cos^2\theta) \, d\theta = \int_{-\pi}^{\pi} 8 + 8\cos\theta + 2\cos^2\theta \, d\theta$ $(\cos 2\theta = 2\cos^2\theta - 1)$

$A = \int_{-\pi}^{\pi} (8 + 8\cos\theta + (1+\cos 2\theta)) \, d\theta = \left[8\theta + 8\sin\theta + \theta + \frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$

$A = \left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi}^{\pi}$

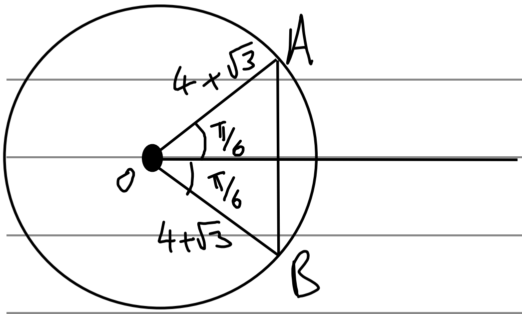
$= (9\pi + 0 + 0) - (-9\pi + 0 + 0)$

$= 18\pi$

**11 (b)** Points $A$ and $B$ lie on the curve $C$ such that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $AOB$ is an equilateral triangle.

Find the polar equation of the line segment $AB$

**[4 marks]**




$r = 4 + 2\cos\theta$

$= 4 + 2\cos(\pi/6)$

$= 4 + 2\sqrt{3}$

$y = r\cos\theta$

$r\cos\theta = (4+\sqrt{3})\cos\theta$

$= (4+\sqrt{3})\cos\frac{\pi}{3}$

$r\cos\theta = (4+\sqrt{3})\frac{\sqrt{3}}{2}$ for $-\pi/6 \le \theta \le \pi/6$

**12** $\mathbf{M} = \begin{bmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{bmatrix}$

**12 (a)** Given that 4 is an eigenvalue of **M**, find a corresponding eigenvector.

**[3 marks]**

$$\begin{pmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$-x+2y-z = 4x \Rightarrow -5x+2y-z=0$ ①

$2x+2y-2z=4y \Rightarrow 2x-2y-2z=0$ ②

$-x-2y-z=4z \Rightarrow -x-2y-5z=0$ ③

① + ②: $-3x-3z=0$

$\Rightarrow x = z$

② + 2×③: $-6y-12z=0$

$2z=-y$

Let $x=1$ so $z=-1$ and $y=2$

The corresponding eigenvector is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

**12 (b)** Given that **MU** = **UD**, where **D** is a diagonal matrix, find possible matrices for **D** and **U**.

**[8 marks]**

$$\det(M - I\lambda) = \begin{vmatrix} -1-\lambda & 2 & -1 \\ 2 & 2-\lambda & -2 \\ -1 & -2 & -1-\lambda \end{vmatrix} = 0$$

$= (-1-\lambda)\left[(2-\lambda)(-1-\lambda)-4\right] - 2(-2-2\lambda-2) - 1(-4+2-\lambda)$

$= (-1-\lambda)(\lambda^2-\lambda-6) - 2(-4-2\lambda) - (-2-\lambda)$

$= -\lambda^2+\lambda+6-\lambda^3+\lambda^2+6\lambda+8-4\lambda+2+\lambda$

$= -\lambda^3+12\lambda+16$

$\Rightarrow 0 = \lambda^3-12\lambda-16$

$\lambda = 4: (4)^3-12(4)-16 = 64-48-16 = 0$ so $(\lambda-4)$ is a factor of $\lambda^3-12\lambda+16=0$

| | $\lambda^2$ | $4\lambda$ | $4$ |
|---|---|---|---|
| $\lambda$ | $\lambda^3$ | $+4\lambda^2$ | $4\lambda$ |
| $-4$ | $-4\lambda^2$ | $-16\lambda$ | $-16$ |

$\Rightarrow \lambda^3-12\lambda-16 = (\lambda-4)(\lambda^2+4\lambda+4)$

$= (\lambda-4)(\lambda+2)^2$

$(\lambda-4)(\lambda+2)^2 = 0 \Rightarrow$ Eigenvalues are $4, -2, -2$

$$\lambda = 2: \begin{pmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Using a scientific calculator yields $x = -2y+z$

$y = y$

$z = z$

so $x+2y-z=0$

As $x+2y-z=0$, we can get two eigenvectors $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\\1\end{pmatrix}$.

$D=\begin{pmatrix}4&0&0\\0&-2&0\\0&0&-2\end{pmatrix}$ and $U=\begin{pmatrix}1&1&-1\\2&0&1\\-1&1&1\end{pmatrix}$

**13** **S** is a singular matrix such that

$$\det \mathbf{S} = \begin{vmatrix} a & a & x \\ x-b & a-b & x+1 \\ x^2 & a^2 & ax \end{vmatrix}$$

Express the possible values of $x$ in terms of $a$ and $b$.

**[7 marks]**

S is singular => det S = 0

$$\begin{vmatrix} a & a & x \\ x-b & a-b & x+1 \\ x^2 & a^2 & ax \end{vmatrix} = \begin{vmatrix} 0 & a & x \\ x-a & a-b & x+1 \\ x^2-a^2 & a^2 & ax \end{vmatrix}$$

$(C_1' = C_1 - C_2)$

$$= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a+b & x+1 \\ x+a & a^2 & ax \end{vmatrix}$$

$$= (x-a)\begin{vmatrix} 0 & a & x \\ 1 & a-b & x+1 \\ x+a & 0 & 0 \end{vmatrix}$$

$(R_3' = R_3 - aR_1)$

$$= (x-a)(x+a)\begin{vmatrix} a & x \\ a-b & x+1 \end{vmatrix}$$

$$= (x-a)(x+a)(a+bx) = 0$$

$x = a, -a, -\frac{a}{b}$

**14** Given that the vectors **a** and **b** are perpendicular, prove that

$|(\mathbf{a} + 5\mathbf{b}) \times (\mathbf{a} - 4\mathbf{b})| = k|\mathbf{a}||\mathbf{b}|$ , where $k$ is an integer to be found.

Explicitly state any properties of the vector product that you use within your proof.

**[9 marks]**

$|(\underline{a}+5\underline{b}) \times (\underline{a}-4\underline{b})| = |\underline{a}\times\underline{a} - 4\underline{a}\times\underline{b} + 5\underline{b}\times\underline{a} - 20\underline{b}\times\underline{b}|$ (Expanding brackets)

$= |\underline{0} - 4\underline{a}\times\underline{b} + 5\underline{b}\times\underline{a} - \underline{0}|$ (Since $\underline{a}$ is parallel to itself and $\underline{b}$ is parallel to itself $\underline{a}\times\underline{a} = \underline{0}$ and $\underline{b}\times\underline{b} = 0$

$= |-4\underline{a}\times\underline{b} - 5\underline{a}\times\underline{b}|$ (Since $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$)

$= |-9\underline{a}\times\underline{b}|$

$= 9|\underline{a}\times\underline{b}|$

$= 9|\underline{a}||\underline{b}| \sin 90$ $(\underline{a}\times\underline{b} = |\underline{a}||\underline{b}|\hat{\underline{n}} \sin\theta \Rightarrow |\underline{a}\times\underline{b}| = |\underline{a}||\underline{b}|\sin\theta)$

$= 9|\underline{a}||\underline{b}|$

**15 (a)** Show that $(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta}) = \frac{1}{16}(17-8\cos 2\theta)$

**[3 marks]**

$(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta}) = 1-\frac{1}{4}e^{2i\theta}-\frac{1}{4}e^{-2i\theta}+\frac{1}{16}$

$= \frac{17}{16}-\frac{1}{4}(\cos 2\theta+i\sin 2\theta)-\frac{1}{4}(\cos 2\theta-i\sin 2\theta)$

$= \frac{17}{16}-\frac{1}{4}\cos 2\theta-\frac{i}{4}\sin 2\theta-\frac{1}{4}\cos 2\theta+\frac{i}{4}\sin 2\theta$

$= \frac{17}{16}-\frac{1}{2}\cos 2\theta$

$= \frac{1}{16}(17-8\cos 2\theta)$

**15 (b)** Given that the series $e^{2i\theta}+\frac{1}{4}e^{4i\theta}+\frac{1}{16}e^{6i\theta}+\frac{1}{64}e^{8i\theta}+\ldots$ has a sum to infinity, express this sum to infinity in terms of $e^{2i\theta}$

**[2 marks]**

Geometric series with $a=e^{2i\theta}$, $r=\frac{1}{4}e^{2i\theta}$

$S_\infty = \frac{a}{1-r} = \frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}} = \frac{4e^{2i\theta}}{4-e^{2i\theta}}$

**15 (c)** Hence show that $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \cos 2n\theta = \frac{16\cos 2\theta - 4}{17 - 8\cos 2\theta}$

**[4 marks]**

$S = \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \cos 2n\theta \quad = e^{2i\theta} + \frac{1}{4}e^{4i\theta} + \frac{1}{16}e^{6i\theta} + \cdots$

So $S = \mathrm{Re}\left(e^{2i\theta} + \frac{1}{4}e^{4i\theta} + \frac{1}{16}e^{6i\theta} + \cdots\right)$

$= \mathrm{Re}\left(\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}}\right) = \mathrm{Re}\left(\frac{e^{2i\theta}}{1-\frac{1}{4}e^{2i\theta}} \times \frac{1-\frac{1}{4}e^{-2i\theta}}{1-\frac{1}{4}e^{-2i\theta}}\right) = \mathrm{Re}\left(\frac{e^{2i\theta} - \frac{1}{4}}{(1-\frac{1}{4}e^{2i\theta})(1-\frac{1}{4}e^{-2i\theta})}\right)$

$= \mathrm{Re}\left(\frac{\cos 2\theta - \frac{1}{4} + i\sin 2\theta}{\frac{1}{16}(17 - 8\cos 2\theta)}\right) = \frac{\cos 2\theta - \frac{1}{4}}{\frac{1}{16}(17 - 8\cos 2\theta)} = \frac{16\cos 2\theta - 4}{17 - 8\cos 2\theta}$

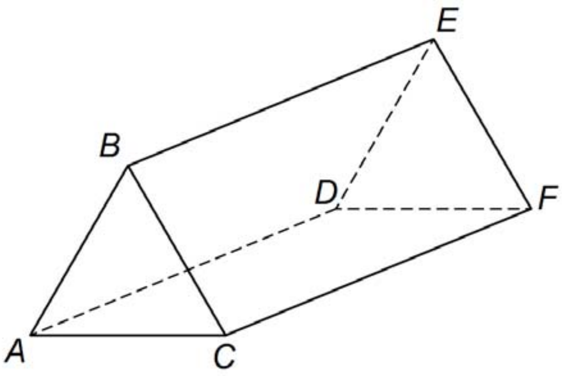
**15 (d)** Deduce a similar expression for $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \sin 2n\theta$

**[1 mark]**

$S = \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \sin 2n\theta =$ Imaginary parts of the series in part b

$= \frac{\sin 2\theta}{\frac{1}{16}(17 - 8\cos 2\theta)} = \frac{16\sin 2\theta}{17 - 8\cos 2\theta}$

**16** A designer is using a computer aided design system to design part of a building. He models part of a roof as a triangular prism *ABCDEF* with parallel triangular ends *ABC* and *DEF*, and a rectangular base *ACFD*. He uses the metre as the unit of length.




The coordinates of *B*, *C* and *D* are (3, 1, 11), (9, 3, 4) and (–4, 12, 4) respectively.

He uses the equation $x - 3y = 0$ for the plane *ABC*.

He uses $\left[\mathbf{r} - \begin{pmatrix} -4 \\ 12 \\ 4 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for the equation of the line *AD*.

Find the volume of the space enclosed inside this section of the roof.

**[9 marks]**

We first need to find the coordinates of A let $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the coordinates

They satisfy the equations of the line and plane : $a - 3b = 0$

$a = 3b$

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} -4 \\ 12 \\ 4 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ -12 \\ 4 \end{pmatrix} = \underset{\sim}{0}$$

$$\begin{pmatrix} a+4 \\ b-12 \\ c-4 \end{pmatrix} \times \begin{pmatrix} 4 \\ -12 \\ 0 \end{pmatrix} = \underset{\sim}{0} \Rightarrow \begin{pmatrix} 12(c-4) \\ 4(c-4) \\ -12(a+4) - 4(b-12) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$12(c-4)=0 \Rightarrow c=4 \qquad -12(a+4)-4(b-12)=0$

$3(a+4)+(b-12)=0$

$b=3a \qquad 3a+12+b-12=0$

So $b=3a=-3a \qquad b=-3a$

$\Rightarrow a=b=0$

$\therefore$ A has coordinates $\begin{pmatrix}0\\0\\4\end{pmatrix}$

$\Rightarrow \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}3\\1\\11\end{pmatrix}-\begin{pmatrix}0\\0\\4\end{pmatrix}=\begin{pmatrix}3\\1\\7\end{pmatrix}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$=\begin{pmatrix}9\\3\\4\end{pmatrix}-\begin{pmatrix}0\\0\\4\end{pmatrix}=\begin{pmatrix}9\\3\\0\end{pmatrix}$

Area of triangle $=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\left|\begin{pmatrix}3\\1\\7\end{pmatrix}\times\begin{pmatrix}9\\3\\0\end{pmatrix}\right|=\frac{21\sqrt{10}}{2}$

$|\overrightarrow{AD}|=\left|\begin{pmatrix}-4\\12\\4\end{pmatrix}-\begin{pmatrix}0\\0\\0\end{pmatrix}\right|=\left|\begin{pmatrix}-4\\12\\0\end{pmatrix}\right|=4\sqrt{10}$

Volume $=\frac{21\sqrt{10}}{2}\times 4\sqrt{10}=420\text{m}^3$

**END OF QUESTIONS**