AQA MODEL ANSWERS

Please write clearly in block capitals.

Centre number		Candidate number	

Surname ____________________

Forename(s) ____________________

Candidate signature ____________________

A-level
FURTHER MATHEMATICS
Paper 2

Thursday 6 June 2019 Afternoon Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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TOTAL	

J U N 1 9 7 3 6 7 2 0 1

PB/Jun19/E2

7367/2

Answer **all** questions in the spaces provided.

1 Given that z is a complex number, and that z^* is the complex conjugate of z, which of the following statements is **not** always true?

Circle your answer.

[1 mark]

$(z^*)^* = z$ $\quad$ $zz^* = |z|^2$ $\quad$ $(-z)^* = -(z^*)$ $\quad$ $z - z^* = z^* - z$

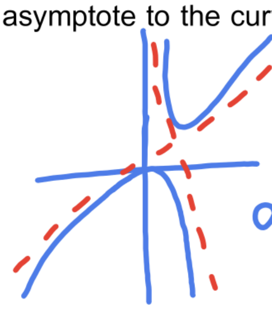
$(x+iy)-(x-iy) \overset{?}{=} (x-iy)-(x+iy)$

$2iy = -2iy \Rightarrow$ only true if $y=0$

2 Which of the straight lines given below is an asymptote to the curve

$$y = \frac{ax^2}{x-1}$$

where a is a non-zero constant?



$\frac{ax^2}{x-1} = ax+a+\frac{a}{x-1}$

Circle your answer.

[1 mark]

$y = ax + a$ $\quad$ $y = ax$ $\quad$ $y = ax - a$ $\quad$ $y = a$

$y=mx+c$ is an asymptote if $\lim_{x\to\infty}\left[f(x)-(mx+b)\right]=0$

$\lim_{x\to\infty} ax+a+\frac{a}{x-1}-(ax+a)=0$

3 The set $\mathcal{A}$ is defined by $\mathcal{A} = \{x : -\sqrt{2} < x < 0\} \cup \{x : 0 < x < \sqrt{2}\}$

Which of the inequalities given below has $\mathcal{A}$ as its solution?

Circle your answer.

[1 mark]

$|x^2 - 1| > 1$ $\quad$ $|x^2 - 1| \geq 1$ $\quad$ $|x^2 - 1| < 1$ $\quad$ $|x^2 - 1| \leq 1$

$x^2-1=1$ $\qquad$ $-x^2+1=1$

$x^2-2=0$ $\qquad$ $x^2=0$

$x=\pm\sqrt{2}$ $\qquad$ $x=0$

$\Rightarrow -\sqrt{2}<x<0$ & $0<x<\sqrt{2}$

Do not write outside the box

4 The positive integer k is such that

$$\sum_{r=1}^{k}(3r-k)=90$$

Find the value of k.

[3 marks]

$$\sum_{r=1}^{k}(3r-k)=3\sum_{r=1}^{k}r-\sum_{r=1}^{k}k$$

$$=3\left(\tfrac{1}{2}k(k+1)\right)-k(k)$$ using standard results

$$=\tfrac{3}{2}k(k+1)-k^2$$

$$\Rightarrow \tfrac{1}{2}k^2+\tfrac{3}{2}k=90 \rightarrow k^2+3k-180=0$$

$$(k+15)(k-12)=0$$

k must be +ve so $k=12$

Turn over for the next question

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5 A curve has equation $y = \cosh x$

Show that the arc length of the curve from $x = a$ to $x = b$, where $0 < a < b$, is equal to

$$\sinh b - \sinh a$$

[4 marks]

$$S = \int_a^b \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$ standard result

$$\frac{dy}{dx} = \sinh x$$

identity

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x \equiv \cosh^2 x$$

$$\therefore S = \int_a^b (\cosh^2 x)^{\frac{1}{2}} dx = \int_a^b \cosh x \, dx$$

$$= \left[\sinh x\right]_a^b$$

$$= \sinh b - \sinh a$$

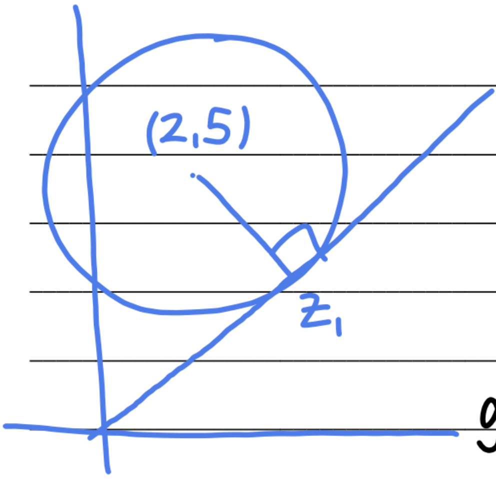
Do not write outside the box

6 A circle C in the complex plane has equation $|z - 2 - 5\text{i}| = a$

The point z_1 on C has the least argument of any point on C, and $\arg(z_1) = \frac{\pi}{4}$

Prove that $a = \frac{3\sqrt{2}}{2}$

[6 marks]



$\tan\left(\frac{\pi}{4}\right) = 1 = \frac{y}{x}$

$\arg(z_1) = \frac{\pi}{4} \Rightarrow z_1$ is on line $y = x$.

let $z_1 = k + ki$, as $y = x$.

gradient of tangent $= 1$ $\therefore$ gradient of radius $= -1$

gradients of perpendicular lines multiply to give -1

radius ⊥ tangent

$\therefore$ line between $(2,5)$ & (k,k) has gradient -1

$\Rightarrow \frac{5-k}{2-k} = -1 \Rightarrow 5 - k = k - 2$

$2k = 7$

$k = \frac{7}{2}$

equation of circle: $(x - x_1)^2 + (y - y_1) = a^2$

sub in values $(2,5)$ & $\left(\frac{7}{2}, \frac{7}{2}\right)$

$a^2 = \left(5 - \frac{7}{2}\right)^2 + \left(2 - \frac{7}{2}\right)^2 = \left(\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 = 2 \times \frac{9}{4} = \frac{9}{2} = 4.5$

$a^2 = 4.5$

$\Rightarrow a = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}$ (a is +ve root)

Do not write outside the box

7 The points A, B and C have coordinates $A(4, 5, 2)$, $B(-3, 2, -4)$ and $C(2, 6, 1)$

7 (a) Use a vector product to show that the area of triangle ABC is $\frac{5\sqrt{11}}{2}$

[4 marks]

$\vec{AB} = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \\ -6 \end{pmatrix}$ $\quad \vec{AC} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -7 & -3 & -6 \\ -2 & 1 & -1 \end{vmatrix} = \underline{i}(3+6) - \underline{j}(7-12) + \underline{k}(-7-6)$

$= \begin{pmatrix} 9 \\ 5 \\ -13 \end{pmatrix}$

$\text{area} = \frac{1}{2}|\vec{AB} \times \vec{AC}| = \frac{\sqrt{9^2+5^2+13^2}}{2} = \frac{5\sqrt{11}}{2}$

7 (b) The points A, B and C lie in a plane.

Find a vector equation of the plane in the form $\mathbf{r.n} = k$

[1 mark]

we need a vector that is ⊥ to the plane containing the points $A, B, \& C$. the cross product is ⊥ $\Rightarrow \underline{r} \cdot \begin{pmatrix} 9 \\ 5 \\ -13 \end{pmatrix} = k$

$\underline{n} = 9i + 5j - 13k$

pick point in plane: $\begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ -13 \end{pmatrix} = 36 + 25 - 26 = 35$

$\therefore \underline{r} \cdot \begin{pmatrix} 9 \\ 5 \\ -13 \end{pmatrix} = 35$

Do not write outside the box

7 (c) Hence find the exact distance of the plane from the origin.

[1 mark]

$|\underline{n}| = 5\sqrt{11}$ magnitude of normal

$\underline{r} \cdot \underline{n}$ gives projection of $\underline{r}$ on $\underline{n}$ so distance is given by $\frac{\underline{r} \cdot \underline{n}}{|n|}$

$\frac{35}{5\sqrt{11}} = \frac{7}{\sqrt{11}} = \frac{7\sqrt{11}}{11}$

Turn over for the next question

Do not write outside the box

8 A parabola P_1 has equation $y^2 = 4ax$ where $a > 0$

P_1 is translated by the vector $\begin{bmatrix} b \\ 0 \end{bmatrix}$, where $b > 0$, to give the parabola P_2

8 (a) The line $y = mx$ is a tangent to P_2

Prove that $m = \pm\sqrt{\dfrac{a}{b}}$

Solutions using differentiation will be given no marks.

[4 marks]

translation $\begin{bmatrix} b \\ 0 \end{bmatrix}$: $x \to x - b$

so parabola becomes $y^2 = 4a(x-b)$

tangent to P_2 : equate y's so $m^2x^2 = 4a(x-b)$

$m^2x^2 - 4ax + 4ab = 0$

tangent $\Rightarrow$ only one solution to above equation $\Rightarrow$ repeated roots

$\therefore \Delta = 0$: $\underbrace{16a^2 - 4m^2(4ab)}_{b^2-4ac} = 0$

$16a^2 = 16m^2ab$

$m = \pm\sqrt{\dfrac{a}{b}}$

Do not write outside the box

8 (b) The line $y = \sqrt{\frac{a}{b}}\,x$ meets P_2 at the point D.

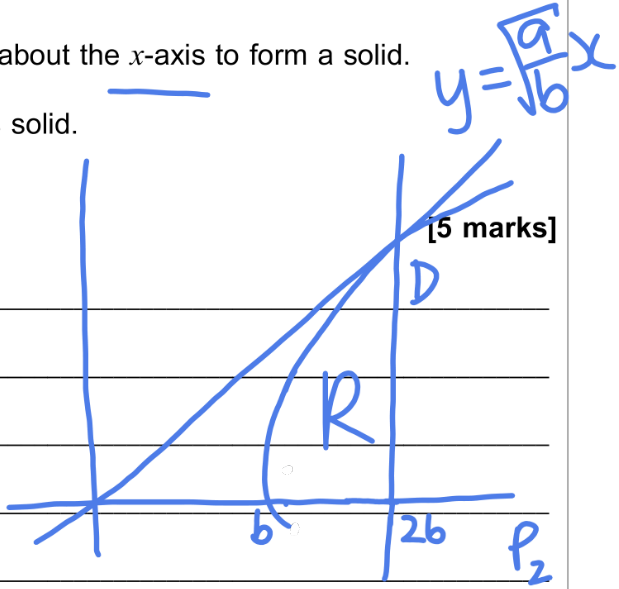
The finite region R is bounded by the x-axis, P_2 and a line through D perpendicular to the x-axis.

The region R is rotated through 2π radians about the x-axis to form a solid.

Find, in terms of a and b, the volume of this solid.

Fully justify your answer.

[5 marks]



first sketch problem

then find coord of D

tangent meets P_2:

$$\left(\sqrt{\frac{a}{b}}x\right)^2 = 4a(x-b)$$

$$ax^2 = 4ab(x-b)$$

$$x^2 - 4bx + 4b^2 = 0 \Rightarrow (x-2b)^2 = 0$$

$$\Rightarrow x = 2b$$

find where P_2 meets x-axis $\quad 0 = 4a(x-b) \therefore x = b$

So integration bounds for volume of revolution are $[b, 2b]$

$$V = \pi\int_b^{2b} 4a(x-b)\,dx = 4a\pi\left[\frac{x^2}{2} - bx\right]_b^{2b}$$

$$= 4a\pi\left\{\left(\frac{4b^2}{2} - 2b^2\right) - \left(\frac{b^2}{2} - b^2\right)\right\}$$

$$= 4a\pi\left\{0 - \left(-\frac{b^2}{2}\right)\right\}$$

$$= 2ab^2\pi$$

Do not write outside the box

9 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{-3}{5} & \frac{13}{10} \end{bmatrix}$$

[5 marks]

eigenvalues: $\det(M - \lambda \mathbb{1}) = 0$

required

$\begin{vmatrix} \frac{1}{5}-\lambda & \frac{2}{5} \\ -\frac{3}{5} & \frac{13}{10}-\lambda \end{vmatrix} = \left(\frac{1}{5}-\lambda\right)\left(\frac{13}{10}-\lambda\right) + \frac{6}{25} \overset{!}{=} 0$

$\hookrightarrow \frac{13}{50} - \frac{1}{5}\lambda - \frac{13}{10}\lambda + \lambda^2 + \frac{6}{25} = 0$

$\frac{25}{50} - \frac{15}{10}\lambda + \lambda^2 \overset{!}{=} 0$

$\Rightarrow 0.5 - 1.5\lambda + \lambda^2 = 0$

$2\lambda^2 - 3\lambda + 1 = 0$

$(2\lambda - 1)(\lambda - 1) = 0 \therefore \lambda = 1 \;\&\; \lambda = 0.5$

eigenvectors: $M\underline{v} = \lambda \underline{v} \Rightarrow (M - \lambda)\underline{v} = 0$

$\lambda = 1: \begin{pmatrix} -\frac{4}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{3}{10} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow 2y - 4x = 0 \; (\times 5)$

$y = 2x$

$\lambda = \frac{1}{2}: \begin{pmatrix} -\frac{3}{10} & \frac{2}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow 8y - 6x = 0 \; (\times 10)$

$4y = 3x$

So $\lambda = 1, \underline{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\lambda = 0.5, \underline{v} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

9 (b) Find matrices **U** and **D** such that **D** is a diagonal matrix and $\mathbf{M} = \mathbf{U}\mathbf{D}\mathbf{U}^{-1}$

[2 marks]

U is made of eigenvectors, $D = \mathrm{diag}(\lambda_1, \lambda_2)$

$\lambda=1 \quad \lambda=0.5$

Do not write outside the box

So $U=\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix} \to D=\begin{pmatrix}1 & 0\\0 & \frac{1}{2}\end{pmatrix}$

eigenvalues in order corresponding to their eigenvectors given in $\underline{U}$

9 (c) Given that $\mathbf{M}^n \to \mathbf{L}$ as $n \to \infty$, find the matrix $\mathbf{L}$.

[4 marks]

find U^{-1} $\det U = 3-8=-5 \Rightarrow U^{-1}=\frac{-1}{5}\begin{pmatrix}3 & -4\\-2 & 1\end{pmatrix}$

$M=UDU^{-1} \Rightarrow M^n = UD^nU^{-1}$

$= -\frac{1}{5}\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}1^n & 0\\0 & (\frac{1}{2})^n\end{pmatrix}\begin{pmatrix}3 & -4\\-2 & 1\end{pmatrix}$

$\therefore L = \lim_{n\to\infty} M^n = -\frac{1}{5}\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}3 & -4\\-2 & 1\end{pmatrix}$

$= -\frac{1}{5}\begin{pmatrix}1 & 4\\2 & 3\end{pmatrix}\begin{pmatrix}3 & -4\\0 & 0\end{pmatrix}$

$= -\frac{1}{5}\begin{pmatrix}3 & -4\\6 & -8\end{pmatrix} = \begin{pmatrix}-0.6 & 0.8\\-1.2 & 1.6\end{pmatrix}$

9 (d) The transformation represented by $\mathbf{L}$ maps all points onto a line.

Find the equation of this line.

[2 marks]

$\begin{pmatrix}x'\\y'\end{pmatrix} = L\begin{pmatrix}x\\y\end{pmatrix}$

$\Rightarrow x' = -\frac{1}{5}(3x-4y),\ y' = -\frac{1}{5}(6x-8y)$

$\times 2$

$\therefore y = 2x$

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10 Prove by induction that $f(n) = n^3 + 3n^2 + 8n$ is divisible by 6 for all integers $n \geq 1$

[7 marks]

for $n=1$, $f(1) = 1^3 + 3(1)^2 + 8(1) = 12 = 2 \times 6$

$f(1)$ is divisible by 6, so statement is true for $n=1$.

assume the statement is true for $n=k$, so $f(k) = 6m$ for some integer m.

for $n=k+1$: $f(k+1) = (k+1)^3 + 3(k+1)^2 + 8(k+1)$

$= k^3 + 3k^2 + 3k + 1 + 3(k^2 + 2k + 1) + 8(k+1)$

$= k^3 + 6k^2 + 17k + 12$

$f(k+1) - f(k) = 3k^2 + 9k + 12$

$\therefore f(k+1) = 6m + 3k^2 + 9k + 12$

$3k^2 + 9k + 12 = 3(k^2 + 3k + 4)$ $\therefore$ is a multiple of 3.

$3k^2 + 9k + 12 = 3(k^2 + 3k) + 12$

$k^2 + 3k = k(k+3)$ $\Rightarrow$ either k or $k+3$ is even (difference is 3 so can't both be even).

even $\times$ odd = even, so $k^2 + 3k$ is even, 4 is even $\therefore 3(k^2 + 3k + 4)$ is an even multiple of 3 (divisible by 3 & 2) so is divisible by 6. thus, if $f(k)$ is divisible by 6, $f(k+1)$ is a sum of multiples of 6, hence is also a multiple of 6.

So, if we assume the statement is true for $f(k)$, it's true for $f(k+1)$. as it is true for $f(1)$, so by induction, we conclude it is true for all $k \in \mathbb{Z}^+$

Turn over for the next question

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ANSWER IN THE SPACES PROVIDED

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11 The line L_1 has equation

$$\frac{x-2}{3} = \frac{y+4}{8} = \frac{4z-5}{5}$$

The line L_2 has equation

$$\left(\mathbf{r} - \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) \times \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \mathbf{0}$$

Find the shortest distance between the two lines, giving your answer to three significant figures.

[8 marks]

$\lambda = \frac{x-2}{3} = \frac{y+4}{8} = \frac{4z-5}{5}$ ⇒ Vector for L_1

given by $\underline{r}_1 = \begin{pmatrix} 2 \\ -4 \\ 5/4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ 5/4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 5/4 \end{pmatrix} + \lambda' \begin{pmatrix} 12 \\ 32 \\ 5 \end{pmatrix}$

×4

L_2 vector: $\underline{r}_2 = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

now we have vector equations for both lines, we find the vector between them & the vector ⊥ to both

$L_1 \rightarrow L_2$: $\begin{pmatrix} 2 \\ -4 \\ 5/4 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -7/4 \end{pmatrix}$

$\underline{n} = \begin{pmatrix} 12 \\ 32 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 12 & 32 & 5 \\ 2 & 1 & 3 \end{vmatrix} = \underline{i}(96-5) - \underline{j}(36-10) + \underline{k}(12-64)$

$= \begin{pmatrix} 91 \\ -26 \\ -52 \end{pmatrix}$

Do not write outside the box

$|\underline{n}| = \sqrt{91^2 + 26^2 + 52^2} = \sqrt{11661} = 13\sqrt{69}$

project $(L_1 \to L_2)$ on $|\underline{n}|$ & divide by length of $\underline{n}$

$$\frac{\begin{pmatrix} 4 \\ -4 \\ -\frac{7}{4} \end{pmatrix} \cdot \begin{pmatrix} 91 \\ -26 \\ -52 \end{pmatrix}}{13\sqrt{69}} = \frac{43}{\sqrt{69}} = 5.17659\ldots = 5.18 \ (3 s.f.)$$

Turn over for the next question

Do not write outside the box

12 Abel and Bonnie are trying to solve this mathematical problem:

> $z = 2 - 3\text{i}$ is a root of the equation
> $2z^3 + mz^2 + pz + 91 = 0$
>
> Find the value of m and the value of p.

Abel says he has solved the problem.

Bonnie says there is not enough information to solve the problem.

12 (a) Abel's solution begins as follows:

> Since $z = 2 - 3\text{i}$ is a root of the equation,
> $z = 2 + 3\text{i}$ is another root.

State **one extra** piece of information about m and p which could be added to the problem to make the beginning of Abel's solution correct.

[1 mark]

m & p are real numbers

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12 (b) Prove that Bonnie is right.

[4 marks]

$z^3 + \frac{mz^2}{2} + \frac{p}{2}z + \frac{91}{2} = 0$

So product of roots $= -\frac{91}{2}$

as $(2-3i)$ is one root: $(2-3i)\beta\gamma = -\frac{91}{2}$

$-\frac{91}{2} \div (2-3i) = -\frac{91}{2(2-3i)} \times \frac{2+3i}{2+3i} = -\frac{91(2+3i)}{2(4+9)}$

$= -\frac{91(2+3i)}{26}$

$= -\frac{7(2+3i)}{2}$

so roots could be $(2+3i)$ & $-\frac{7}{2}$,

giving $m = -2 \times (2+3i+2-3i-\frac{7}{2})^2 = -1$ &

$p = +2 \times (4+9+(2+3i)\times-\frac{7}{2}+(2-3i)\times-\frac{7}{2}) = -2$

but roots could also be $(4+6i)$ & $-\frac{7}{4}$,

giving $m = -2 \times (2-3i+4+6i-\frac{7}{4}) \neq -1$

$\hookrightarrow$ different values for m & p.

there is more than one set of possible values for m & p, so we don't have enough info to solve the problem.

$\therefore$ Bonnie is right.

Do not write outside the box

13 (a) Explain why $\int_3^{\infty} x^2 e^{-2x}\, dx$ is an improper integral.

[1 mark]

the upper limit is infinity, making it improper.

13 (b) Evaluate $\int_3^{\infty} x^2 e^{-2x}\, dx$

Show the limiting process.

[9 marks]

Start by calculating the integral without the limits.

use integration by parts: $u = x^2 \quad v' = e^{-2x}$

$u' = 2x \quad v = -\frac{e^{-2x}}{2}$

$$\rightarrow \int x^2 e^{-2x}\, dx = -\frac{x^2 e^{-2x}}{2} + \int x e^{-2x}\, dx$$

if we differentiate this, we get 1, leaving us with an integral we can compute

integrate by parts again:

$u = x \quad v' = e^{-2x}$

$u' = 1 \quad v = -\frac{e^{-2x}}{2}$

$$\int x e^{-2x}\, dx = \left(-\frac{xe^{-2x}}{2}\right) + \int \frac{e^{-2x}}{2}\, dx = -\frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}$$

$$\therefore \int x^2 e^{-2x}\, dx = -\frac{x^2 e^{-2x}}{2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}$$

$$\int_3^{\infty} x^2 e^{-2x}\, dx = \lim_{n \to \infty} \int_3^{n} x^2 e^{-2x}\, dx$$

Change to finite limit for proper integral

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$$= \lim_{n \to \infty} \left\{ \left[-\frac{x^2 e^{-2x}}{2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4} \right]_3^n \right\}$$

$n \to \infty : e^{-2n} \to 0$, this dominates over the increase of $x \to \infty$.

$$\hookrightarrow \int_3^\infty x^2 e^{-2x}\,dx = \frac{9e^{-6}}{2} + \frac{3e^{-6}}{2} + \frac{e^{-6}}{4} = \frac{25e^{-6}}{4}$$

Turn over for the next question

Do not write outside the box

14 Let

$$S_n = \sum_{r=1}^{n} \frac{1}{(r+1)(r+3)}$$

where $n \geq 1$

14 (a) Use the method of differences to show that

$$S_n = \frac{5n^2 + an}{12(n+b)(n+c)}$$

where a, b and c are integers.

[6 marks]

use partial fractions: $\frac{1}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$

$$1 = A(r+3) + B(r+1)$$

$$\left.\begin{array}{l} r=-3:\ 1=-2B \Rightarrow B = -\frac{1}{2} \\ r=-1:\ 1=2A \Rightarrow A = \frac{1}{2} \end{array}\right\} \frac{1}{2(r+1)} - \frac{1}{2(r+3)} = \frac{1}{(r+1)(r+3)}$$

$$\therefore 2S_n = \sum_{r=1}^{n} \frac{1}{r+1} - \frac{1}{r+3} = \frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}} + \dots$$

$$+ \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n}} - \frac{1}{n+2} + \cancel{\frac{1}{n+1}}$$

$$+ \frac{1}{n+3}$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

$$= \frac{3(n+2)(n+3)}{2\times3\times(n+2)(n+3)} + \frac{2(n+2)(n+3)}{2\times3\times(n+2)(n+3)} - \frac{3\times2(n+3)}{2\times3\times(n+2)(n+3)} - \frac{2\times3(n+2)}{2\times3\times(n+2)(n+3)}$$

$$= \frac{5(n+2)(n+3) - 6(n+3+n+2)}{6(n+2)(n+3)}$$

Do not write outside the box

$$\Rightarrow \qquad S_n = \frac{5(n^2+5n+6)-6(2n+5)}{12(n+3)(n+2)}$$

$$= \frac{5n^2+13n}{12(n+2)(n+3)}$$

Question 14 continues on the next page

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14 (b) Show that, for any number k greater than $\frac{12}{5}$, if the difference between $\frac{5}{12}$ and S_n is less than $\frac{1}{k}$, then

$$n > \frac{k - 5 + \sqrt{k^2 + 1}}{2}$$

[6 marks]

difference between $\frac{5}{12}$ & $S_n < \frac{1}{k}$: $\frac{5}{12} - \frac{(5n^2+13n)}{12(n+2)(n+3)} < \frac{1}{k}$

$$\Rightarrow \frac{5(n+2)(n+3)-(5n^2+13n)}{12(n+2)(n+3)} < \frac{1}{k}$$

$$\frac{12n+30}{12(n+2)(n+3)} < \frac{1}{k}$$

both denominators >0 so $k(12n+30) < 12(n+2)(n+3)$

$$\Rightarrow 12n^2 + (60-12k)n + (72-30k) > 0$$

$\div 6$: $2n^2 + (10-2k)n + (12-5k) > 0$

$k > \frac{12}{5}$ so $(12-5k) < 0 \Rightarrow$ above equation has a +ve & −ve root

+ve root: $\frac{(2k-10)+\sqrt{(10-2k)^2-8(12-5k)}}{4} =$

$$\frac{(2k-10)+\sqrt{(100+4k^2-40)-96+40k}}{4} =$$

$$\frac{(2k-10)+\sqrt{4k^2+4}}{4} = \frac{2(k-5)+2\sqrt{k^2+1}}{4}$$

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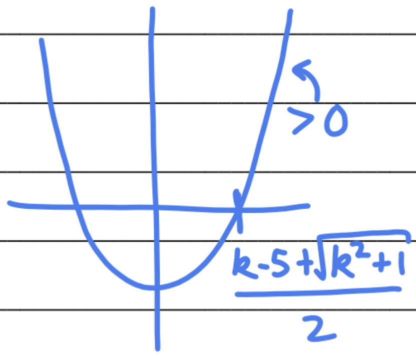
$$= \frac{(k-5)+\sqrt{k^2+1}}{2}$$

as the root is +ve, if $12n^2+(60-12k)n+(72-30k)$ is >0, then $n > \frac{(k-5)+\sqrt{k^2+1}}{2}$



Turn over for the next question

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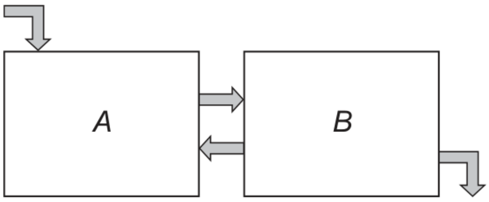

Two tanks, *A* and *B*, each have a capacity of 800 litres.

At time $t = 0$ both tanks are full of pure water.

When $t > 0$, water flows in the following ways:

- Water with a salt concentration of μ grams per litre flows into tank *A* at a constant rate
- Water flows from tank *A* to tank *B* at a rate of 16 litres per minute
- Water flows from tank *B* to tank *A* at a rate of r litres per minute (b)
- Water flows out of tank *B* through a waste pipe
- The amount of water in each tank remains at 800 litres.

At time t minutes $(t \geq 0)$ there are x grams of salt in tank *A* and y grams of salt in tank *B*.

This system is represented by the coupled differential equations

(b)

$$\frac{dx}{dt} = 36 - 0.02x + 0.005y \qquad (1)$$

$$\frac{dy}{dt} = 0.02x - 0.02y \qquad (2)$$

15 (a) Find the value of r.

[2 marks]

from B to A $\Rightarrow$ +ve y coefficient in $\frac{dx}{dt}$

$r = 0.005 \times 800 = 4$

capacity of tanks

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15 (b) Show that $\mu = 3$

[3 marks]

constant rate : flow into A = flow out of A

$16 - 4 = 12$ L/min $\Rightarrow$ carries 12μg/min

36 is constant value in $\frac{dx}{dt}$ so $12\mu = 36$

$\therefore \mu = 3$

15 (c) Solve the coupled differential equations to find both x and y in terms of t.

[9 marks]

differentiate (2) : $\frac{d^2y}{dt^2} = 0.02\dot{x} - 0.02\dot{y}$

$0.02\dot{x} = \ddot{y} + 0.02\dot{y}$

$x = 50\dot{y} + y$ & $\dot{x} = 50\ddot{y} + \dot{y}$

sub in (1) : $50\ddot{y} + \dot{y} = 36 - 0.02(50\dot{y} + y) + 0.005y$

$50\ddot{y} + 2\dot{y} + 0.015y = 36$

CF : $50m^2 + 2m + 0.015 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{2^2 - 4(50)(0.015)}}{2(50)}$

$= \frac{-2 \pm \sqrt{1}}{100}$

$\Rightarrow m = \frac{-1}{100}$ & $m = \frac{-3}{100}$

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PI: let $y=\lambda \Rightarrow \dot{y}=\ddot{y}=0$. sub in: $0.015\lambda=36$

same form as RHS $\lambda=2400$

$\therefore y=Ae^{-0.03t}+Be^{-0.01t}+2400$

$x=50\dot{y}+y$ so we need $\dot{y}$:

$\dot{y}=-0.03Ae^{-0.03t}-0.01Be^{-0.01t}$

$\Rightarrow x=-0.5Ae^{-0.03t}+0.5Be^{-0.01t}+2400$

initial conditions: @ $t=0$, $x=0$ & $y=0$

$\Rightarrow A+B+2400=0$ (y) ①

$-0.5A+0.5B+2400=0$ (x) ②

① + 2 × ②: $2B+7200=0$

$B=-3600$

$A-3600+2400=0$

$A=1200$

so $x=-600e^{-0.03t}-1800e^{-0.01t}+2400$

$y=1200e^{-0.03t}-3600e^{-0.01t}+2400$

END OF QUESTIONS

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