Model Solutions

AQA

Please write clearly in block capitals.

Centre number

Candidate number

Surname

Forename(s)

Candidate signature

I declare this is my own work.

A-level
FURTHER MATHEMATICS
Paper 1

Friday 22 May 2020 Morning Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
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TOTAL	

J U N 2 0 7 3 6 7 1 0 1

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Answer **all** questions in the spaces provided.

1 Which of the integrals below is **not** an improper integral?

Circle your answer.

[1 mark]

$\int_0^\infty e^{-x}\,dx$ undefined values

$\int_0^2 \frac{1}{1-x^2}\,dx$ range 0-2 includes 1 $1-1^2=0$. cannot divide by 0.

$\int_0^1 \sqrt{x}\,dx$

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ cannot divide by 0.

2 Which one of the matrices below represents a rotation of 90° about the x-axis?

Circle your answer.

[1 mark]

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

Do not write outside the box

3 The quadratic equation $ax^2 + bx + c = 0$ $(a,\ b,\ c \in \mathbb{R})$ has real roots α and β.

One of the four statements below is incorrect.

Which statement is **incorrect**?

Tick (✓) **one** box.

[1 mark]

$c = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ ☐

$c = a \Rightarrow \alpha$ is the reciprocal of β ☐

$b < 0$ and $c < 0 \Rightarrow \alpha > 0$ and $\beta > 0$ ☑

$b = 0 \Rightarrow \alpha = -\beta$ ☐

Turn over for the next question

Do not write outside the box

4 It is given that $1 - 3i$ is one root of the quartic equation

$$z^4 - 2z^3 + pz^2 + rz + 80 = 0$$

where p and r are real numbers.

4 (a) Express $z^4 - 2z^3 + pz^2 + rz + 80$ as the product of two quadratic factors with real coefficients.

[4 marks]

As $1-3i$ is a root, then its conjugate pair, $1+3i$, is also a root.

$(z-(1-3i))(z-(1+3i))$

$= z^2 - z - 3iz - z + 1 + 3i + 3iz - 3i - 9i^2$

$= z^2 - 2z + 10$ is a factor

$(z^2 - 2z + 10)(z^2 + bz + 8) = z^4 - 2z^3 + pz^2 + rz + 80$

compare coefficients:

z^3 coeff: $b - 2 = -2$

$b = 0$

$\therefore$ quadratic is

$(z^2 - 2z + 10)(z^2 + 8)$

Do not write outside the box

4 (b) Find the value of p and the value of r.

[2 marks]

$(z^2-2z+10)(z^2+8)$

$=z^4+8z^2-2z^3-16z+10z^2+80$

$=z^4-2z^3+18z^2-16z+80$

$p=18$ $r=-16$

Turn over for the next question

Do not write outside the box

5 H_1 is the locus of points such that the distance from the point (5, 0) is twice the distance from the line $x = 2$

5 (a) Show that the equation of H_1 can be written in the form

$$(x-1)^2 - \frac{y^2}{q} = r$$

where q and r are integers.

[5 marks]

$\sqrt{(x-5)^2+y^2} = 2|x-2|$

$(x-5)^2 + y^2 = 4(x-2)^2$

$x^2 - 10x + 25 + y^2 = 4x^2 - 16x + 16$

$9 = 3x^2 - 6x - y^2$

$3 = x^2 - 2x - \frac{y^2}{3}$

complete the square

$3 = (x-1)^2 - 1 - \frac{y^2}{3}$

$4 = (x-1)^2 - \frac{y^2}{3}$

$r = 4 \quad q = 3$

Do not write outside the box

5 (b) H_2 is the hyperbola

$$x^2 - y^2 = 4$$

Describe fully a sequence of two transformations which maps the graph of H_2 onto the graph of H_1

[4 marks]

- Translation by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- There is a stretch parallel to the y-axis
- There is also a scale factor of $\sqrt{3}$.

Do not write outside the box

6 Let w be the root of the equation $z^7 = 1$ that has the smallest argument α in the interval $0 < \alpha < \pi$

6 (a) Prove that w^n is also a root of the equation $z^7 = 1$ for any integer n.

[1 mark]

$(w^n)^7 = w^{7n} = (w^7)^n = 1^n = 1$

$\therefore$ w^n satisfies the equation $z^7 = 1$

6 (b) Prove that $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

[2 marks]

As $z^7 = 1$, $z^7 - 1 = 0$

Roots of $z^7 - 1 = 0$ are $1, w, w^2, w^3, w^4, w^5$ and w^6

z^6 term $= 0$ $\therefore$ sum of roots $= 0$

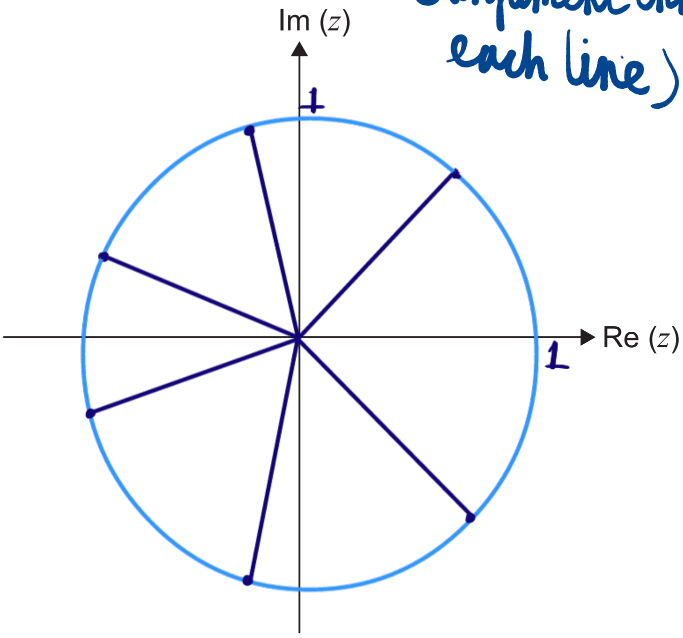
$\therefore$ $1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

(as required)

6 (c) Show the positions of w, w^2, w^3, w^4, w^5, and w^6 on the Argand diagram below.

[2 marks]

(argument changes by $\frac{\pi}{7}$ for each line)



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6 (d) Prove that

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$$

[4 marks]

$w = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$\Rightarrow w^6 = \cos\left(\frac{-2\pi}{7}\right) + i\sin\left(\frac{-2\pi}{7}\right)$

$= \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$

w^6 is the conjugate pair of w

$\therefore\ w + w^6 = 2\cos\frac{2\pi}{7}$

$\Rightarrow w^2 + w^5 = 2\cos\frac{4\pi}{7}$

$\Rightarrow w^3 + w^4 = 2\cos\frac{6\pi}{7}$

Using part b:

$1 + w + w^2 + w^3 + w^4 + w^5 + w^6 = 0$

$1 + (w + w^6) + (w^2 + w^5) + (w^3 + w^4) = 0$

$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$

$\therefore 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$

$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$ (as required)

Do not write outside the box

7 Three planes have equations

$$\begin{aligned}(4k+1)x - 3y + (k-5)z &= 3\\ (k-1)x + (3-k)y + 2z &= 1\\ 7x - 3y + 4z &= 2\end{aligned}$$

7 (a) The planes do **not** meet at a unique point.

Show that $k = 4.5$ is one possible value of k, and find the other possible value of k.

[3 marks]

make a matrice out of the equations given:

$$A = \begin{pmatrix} 4k+1 & -3 & k-5 \\ k-1 & 3-k & 2 \\ 7 & -3 & 4 \end{pmatrix}$$

$$\det A = (4k+1)[4(3-k)+6] + 3[4(k-1)-14] + (k-5)[-3(k-1)-7(3-k)]$$
$$= (4k+1)(18-4k) + 3(4k-18) + (k-5)(4k-18)$$
$$= 72k - 16k^2 + 18 - 4k + 12k - 54 + 4k^2 - 18k - 20k + 90$$
$$= -12k^2 + 42k + 54$$

$\det A = 0$

$\therefore \quad 12k^2 - 42k - 54 = 0$

$4k^2 - 14k - 18 = 0$

$(4k-18)(k+1) = 0$

$k = 4.5$ (as required)

and $k = -1$

Do not write outside the box

7 (b) For each value of k found in part **(a)**, identify the configuration of the given planes.

In each case fully justify your answer, stating whether or not the equations of the planes form a consistent system.

[4 marks]

When $K = 4.5$:

$$A = \begin{pmatrix} 19 & -3 & -0.5 \\ 3.5 & -1.5 & 2 \\ 7 & -3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 19 & -3 & -0.5 \\ 3.5 & -1.5 & 2 \\ 7 & -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$19x - 3y - 0.5z = 3 \quad - \text{ 1}$$
$$3.5x - 1.5y + 2z = 1 \quad - \text{ 2}$$
$$7x - 3y + 4z = 2 \quad - \text{ 3}$$

The system of equations is consistent. Two planes are the same and intersect the third plane.

when $K = -1$:

$$A = \begin{pmatrix} -3 & -3 & -6 \\ -2 & 4 & 2 \\ 7 & -3 & 4 \end{pmatrix}$$

7 b continued).

$$\begin{pmatrix} -3 & -3 & -6 \\ -2 & 4 & 2 \\ 7 & -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$-3x - 3y - 6z = 3$ — 1
$-2x + 4y + 2z = 1$ — 2
$7x - 3y + 4z = 2$ — 3

1 − 3
$(-3x - 3y - 6z) - (7x - 3y + 4z) = 3 - 2$
$-10x - 10z = 1$ — 4

$4 \times$ 1 $- 3 \times$ 2
$(-12x - 12y - 24z) - (-6x + 12y + 6z) = 12 - 3$
$-6x - 30z = 9$ — 5

5 $- 3 \times$ 4
$(-6x - 30z) - (-30x - 30z) = 9 - 3$
$24x = 6$
$x = \frac{1}{4}$

Insert 'x' into 4
$\frac{7}{2} = -10z \qquad z = -\frac{7}{20}$

Insert x and z into 1
$-\frac{3}{4} - 6\left(-\frac{7}{20}\right) - 3 = 3y$
$y = -\frac{11}{20}$

Insert x, y and z into 2
$-2\left(\frac{1}{4}\right) + 4\left(\frac{-11}{20}\right) + 4\left(\frac{-7}{20}\right) = -\frac{41}{10} \neq 1$

∴ The system of equations is inconsistent.

The 3 planes form a prism

Do not write outside the box

8 The three roots of the equation

$$4x^3 - 12x^2 - 13x + k = 0$$

where k is a constant, form an arithmetic sequence.

Find the roots of the equation.

[6 marks]

Let the roots be $\alpha - G$, α, $\alpha + G$

$\Sigma: \alpha = (\alpha - G) + \alpha + (\alpha + G) = 3\alpha$

$3\alpha = -\frac{b}{a} = \frac{12}{4} = 3$

$\alpha = 1$

$\Sigma: \alpha\beta = (\alpha - G)\alpha + (\alpha + G)\alpha + (\alpha + G)(\alpha - G)$

Knowing $\alpha = 1$

$(1 - G) + (1 + G) + (1 + G)(1 - G) = \frac{c}{a} = \frac{-13}{4}$

$2 + (1 - G^2) = \frac{-13}{4}$

$3 - G^2 = \frac{-13}{4}$

$\frac{25}{4} = G^2$

$G = \pm \frac{5}{2}$

$\therefore$ the roots are : -1.5, 1, 3.5

Turn over for the next question

Do not write outside the box

9 The function f is defined by

$$f(x) = \frac{x(x+3)}{x+4} \qquad (x \in \mathbb{R},\ x \neq -4)$$

9 (a) Find the interval (a, b) in which $f(x)$ does not take any values.

Fully justify your answer.

[5 marks]

let $f(x) = y$

$y = \frac{x(x+3)}{x+4}$

$y(x+4) = x(x+3)$

$yx + 4y = x^2 + 3x$

$x^2 + x(3-y) - 4y = 0$

Taking the discriminant:

$(3-y)^2 - (4 \times 1 \times -4y) < 0$

$9 - 6y + y^2 + 16y < 0$

$y^2 + 10y + 9 < 0$

$(y+9)(y+1) < 0$

$-9 < y < -1$

$-9 < f(x) < -1$

So $f(x)$ does not take any values in the interval $(-9, -1)$

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9 (b) Find the coordinates of the two stationary points of the graph of $y = f(x)$

[2 marks]

Method 1

Differentiate using the quotient rule

$$\Rightarrow \frac{g(x)h'(x) - h(x)g'(x)}{(g(x))^2} = f'(x)$$

where $g(x) = x + 4$

$g'(x) = 1$

$h(x) = x(x+3) = x^2 + 3x$

$h'(x) = 2x + 3$

$$f'(x) = \frac{(x+4)(2x+3) - (x^2+3x)(1)}{(x+4)^2}$$

$0 = (x+4)(2x+3) - (x^2+3x)$

$= 2x^2 + 3x + 8x + 12 - x^2 - 3x$

$= x^2 + 8x + 12$

$= (x+6)(x+2)$

$x = -6 \quad x = -2$

$y = -9 \quad y = -1$

stationary points are $(-6, -9)$ and $(-2, -1)$

Method 2 (much quicker way)

Substitute -9 and -1 into $f(x)$

$$-9 = \frac{x(x+3)}{x+4}$$

$-9x - 36 = x^2 + 3x$

$x^2 + 12x + 36 = 0$

$(x+6)^2 = 0$

$x = -6$

$\therefore$ stationary point is $(-6, -9)$

$$-1 = \frac{x(x+3)}{x+4}$$

$-x - 4 = x^2 + 3x$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

Stationary point is $(-2, -1)$

9 (c) Show that the graph of $y = f(x)$ has an oblique asymptote and fin

[2 marks]

$$f(x) = \frac{x^2+3x}{x+4} = \frac{\cancel{(x+4)}(x-1)}{\cancel{(x+4)}} - \frac{4}{x+4}$$

$$= x - 1 - \frac{4}{x+4}$$

Asymptote is $y = x - 1$

Question 9 continues on the next page

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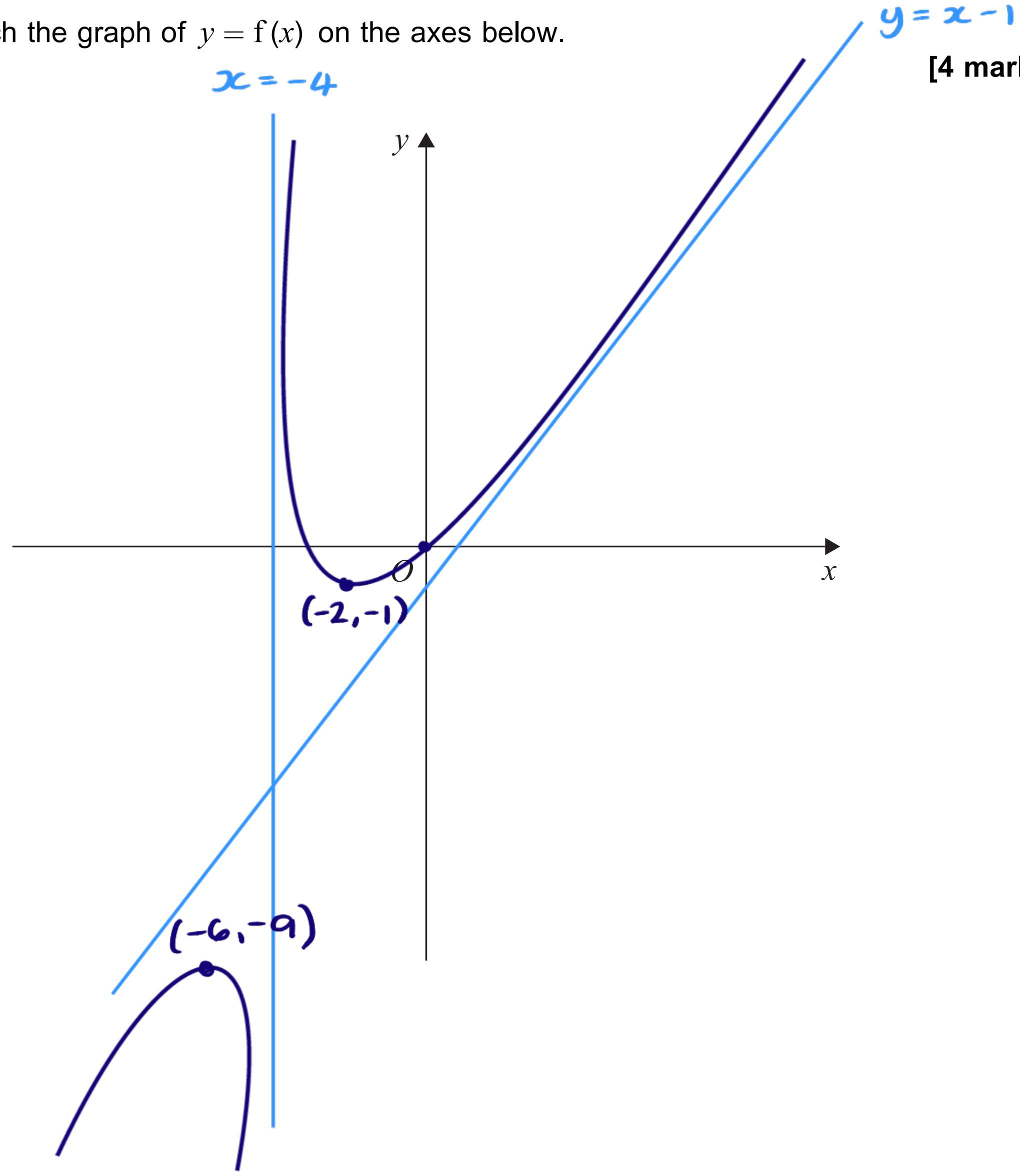
9 (d) Sketch the graph of $y = f(x)$ on the axes below.

[4 marks]



Turn over for the next question

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10 (a) Find the general solution of the differential equation

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{x+3}{x(x-1)(x^2+3)} \qquad (x > 1)$$

[8 marks]

Integrating factor :

$e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

Multiple both sides of the equation by x^2

$$\frac{dy}{dx}x^2 + 2yx = \frac{x(x+3)}{(x-1)(x^2+3)}$$

$$\frac{d}{dx}(x^2 y) = \frac{x^2+3x}{(x-1)(x^2+3)}$$

$$x^2 y = \int \frac{x^2+3x}{(x-1)(x^2+3)} dx$$

$\Downarrow$

Using partial fractions :

$$\frac{x^2+3x}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

$$x^2+3x = A(x^2+3) + (Bx+C)(x-1)$$

Compare coefficients :

x^2 coeff : $1 = A + B$ — 1

x coeff : $3 = -B + C$ — 2

constant : $0 = 3A - C$ — 3

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① + ②

$4 = A + C$ — ④

④ + ③ $\qquad A = 1, \quad C = 3, \quad B = 0$

$4 = 4A$

$$\therefore \frac{x^2+3x}{(x-1)(x^2+3)} = \frac{1}{x-1} + \frac{3}{x^2+3}$$

$$\Rightarrow x^2y = \int (x-1)^{-1} + \frac{3}{x^2+3}\,dx$$

$$x^2y = \ln(x-1) + \frac{3}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$\therefore \text{GS: } y = \frac{1}{x^2}\ln(x-1) + \frac{3}{x^2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{c}{x^2}$$

10 (b) Find the particular solution for which $y = 0$ when $x = 3$

Give your answer in the form $y = \mathrm{f}(x)$

[2 marks]

when $y = 0, x = 3$

$$0 = \frac{1}{9}\ln 2 + \frac{1}{3\sqrt{3}}\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) + \frac{c}{9}$$

$$0 = \frac{1}{9}\left(\ln 2 + \frac{\pi\sqrt{3}}{3} + c\right)$$

$$c = -\ln 2 - \frac{\pi\sqrt{3}}{3}$$

Particular Solution:

$$y = \frac{1}{x^2}\left(\ln(x-1) + \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \ln 2 - \frac{\pi\sqrt{3}}{3}\right)$$

Do not write outside the box

11 The lines l_1, l_2 and l_3 are defined as follows.

$$l_1 : \left(\mathbf{r} - \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}\right) \times \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \mathbf{0}$$

$$l_2 : \left(\mathbf{r} - \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}\right) \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \mathbf{0}$$

$$l_3 : \left(\mathbf{r} - \begin{bmatrix} -5 \\ 12 \\ -4 \end{bmatrix}\right) \times \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix} = \mathbf{0}$$

11 (a) (i) Explain how you know that two of the lines are parallel.

[1 mark]

l_1 and l_2 are parallel as the direction vectors are multiples of eachother.

$$\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = -1 \times \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Do not write outside the box

11 (a) (ii) Show that the perpendicular distance between these two parallel lines is 7.95 units, correct to three significant figures.

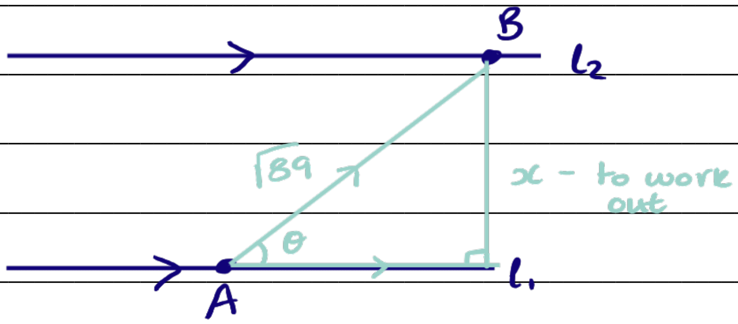
[5 marks]

Let point A be $(1,5,-1)$ on l_1.

Let point B be $(-3,2,7)$ on l_2.

$$\vec{AB} = \begin{pmatrix}-3\\2\\7\end{pmatrix} - \begin{pmatrix}1\\5\\-1\end{pmatrix} = \begin{pmatrix}-4\\-3\\8\end{pmatrix}$$

$$|\vec{AB}| = \sqrt{(-4)^2 + (-3)^2 + 8^2} = \sqrt{89}$$



Work out θ $\Rightarrow$ $\theta = \cos^{-1}\left(\frac{|a.b|}{|a||b|}\right)$

$$a \cdot b = \begin{pmatrix}-4\\-3\\8\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\-3\end{pmatrix} = 8 - 3 - 24 = -19$$

$|a| = \sqrt{89}$ $\quad |b| = \sqrt{(-2)^2 + 1 + (-3)^2} = \sqrt{14}$

$$\theta = \cos^{-1}\frac{|a.b|}{|a||b|} = \cos^{-1}\frac{|-19|}{\sqrt{89}\cdot\sqrt{14}} = 57.4\ldots^\circ$$

$x = \sqrt{89} \times \sin 57.4\ldots$

$= 7.950741205$

$= 7.95$ units (3sf)

Turn over ▶

Do not write outside the box

11 (b) Show that the lines l_1 and l_3 meet, and find the coordinates of their point of intersection.

[5 marks]

$$l_1 : \underline{r}_1 = \begin{pmatrix} 1-2\lambda \\ 5+\lambda \\ -1-3\lambda \end{pmatrix}$$

$$l_3 : \underline{r}_3 = \begin{pmatrix} -5+4\mu \\ 12 \\ -4+9\mu \end{pmatrix}$$

If they meet, $\underline{r}_1 = \underline{r}_3$

$$\begin{pmatrix} 1-2\lambda \\ 5+\lambda \\ -1-3\lambda \end{pmatrix} = \begin{pmatrix} -5+4\mu \\ 12 \\ -4+9\mu \end{pmatrix}$$

- $1-2\lambda = -5+4\mu$

$6 = 2\lambda + 4\mu \Rightarrow 3 = \lambda + 2\mu$ — 1

- $5+\lambda = 12 \Rightarrow \lambda = 7$
- $-1-3\lambda = -4+9\mu$

$3 = 3\lambda + 9\mu \Rightarrow 1 = \lambda + 3\mu$ — 2

when $\lambda = 7$

1 $\rightarrow$ $3 = (7) + 2\mu$

$-4 = 2\mu$

$\mu = -2$

Show equations are consistent by plugging $\mu = -2$ and $\lambda = 7$ into 2

$1 = (7) + 3(-2)$

$1 = 7 - 6$ $\therefore$ consistent

$\Rightarrow l_1$ and l_3 do meet.

Insert $\lambda = 7$ back into l_1 equation:

$$\begin{pmatrix} 1-2(7) \\ 5+(7) \\ -1-3(7) \end{pmatrix} = \begin{pmatrix} -13 \\ 12 \\ -22 \end{pmatrix}$$

l_1 and l_3 meet at $(-13, 12, -22)$

Turn over for the next question

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12 (a) Use the definition of the cosh function to prove that

$$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) \qquad \text{for } a > 0$$

[6 marks]

Let $y = \cosh^{-1}\left(\frac{x}{a}\right)$

$x = a\cosh y$

As $\cosh y = \frac{e^y + e^{-y}}{2}$

$x = \frac{a}{2}\left(e^y + e^{-y}\right)$

$\frac{2x}{a} = e^y + e^{-y}$

Multiply each side by e^y to make a quadratic:

$\frac{2x}{a} \times e^y = e^{2y} + 1$

$0 = e^{2y} - \frac{2x}{a}e^y + 1$

$e^y = \frac{\frac{2x}{a} \pm \sqrt{\frac{4x^2}{a^2} - 4}}{2}$

$e^y = \frac{x \pm \sqrt{x^2 - a^2}}{a}$ ⇐ Product of roots = 1, so one root is greater than 1 and other is less than 1. As $\cosh^{-1}\left(\frac{x}{a}\right)$ needs to be ≥ 0 then $e^y \geq 1$. So larger root is chosen.

Do not write outside the box

$$e^y = \frac{x + \sqrt{x^2 - a^2}}{a}$$

$$y = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

As $y = \cosh^{-1}\left(\frac{x}{a}\right)$

$$\cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

(as required)

12 (b) The formulae booklet gives the integral of $\dfrac{1}{\sqrt{x^2 - a^2}}$ as

$$\cosh^{-1}\left(\frac{x}{a}\right) \quad \text{or} \quad \ln\left(x + \sqrt{x^2 - a^2}\right) + c$$

Ronald says that this contradicts the result given in part **(a)**.

Explain why Ronald is wrong.

[2 marks]

$$\ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right) = \ln\left(x + \sqrt{x^2 + a^2}\right) - \ln a$$

$\therefore\ c = -\ln a$

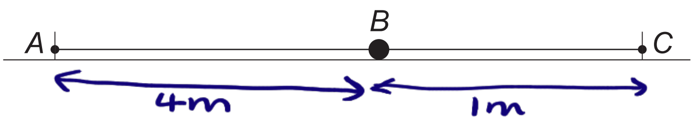
This does not contradict the result given in part (a).

Do not write outside the box

13 Two light elastic strings each have one end attached to a particle B of mass $3c$ kg, which rests on a smooth horizontal table.

The other ends of the strings are attached to the fixed points A and C, which are 8 metres apart.

ABC is a horizontal line.



String AB has a natural length of 4 metres and a stiffness of $5c$ newtons per metre.

String BC has a natural length of 1 metre and a stiffness of c newtons per metre.

The particle is pulled a distance of $\frac{1}{3}$ metre from its equilibrium position towards A, and released from rest.

13 (a) Show that the particle moves with simple harmonic motion.

[8 marks]

Total extension of both strings is 3.

$3 = y_{AB} + y_{BC}$ — 1

Tension:

$5\cancel{c}y_{AB} = \cancel{c}y_{BC}$

$5y_{AB} = y_{BC}$ — 2

Replace y_{BC} to $5y_{AB}$ into — 1

$3 = y_{AB} + 5y_{AB}$

$3 = 6y_{AB}$

$\frac{1}{2} = y_{AB}$

$\Rightarrow \frac{5}{2} = y_{BC}$

$5c\left(\frac{1}{2} - x\right) - c\left(\frac{5}{2} + x\right) = 3c\ddot{x}$ ← acceleration

$\cancel{\frac{5}{2}c} - 5cx - \cancel{\frac{5}{2}c} - xc = 3c\ddot{x}$

$-6\cancel{c}x = 3\cancel{c}\ddot{x}$

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$-2x = \ddot{x}$

↳ of the form $\ddot{x} = -\omega^2 x$

$\therefore$ SHM

13 (b) Find the speed of the particle when it is at a point P, a distance $\frac{1}{4}$ metre from the equilibrium position. Give your answer to two significant figures.

[4 marks]

Using part (a)

$\omega^2 = 2 \quad \therefore \quad \omega = \sqrt{2}$

$\Rightarrow A = \frac{1}{3}$

Using equation $v^2 = \omega^2(A^2 - x^2)$

$v^2 = 2\left(\frac{1}{3^2} - \frac{1}{4^2}\right)$

$v^2 = 2\left(\frac{1}{9} - \frac{1}{16}\right)$

$v^2 = \frac{7}{72}$

$v = \frac{\sqrt{14}}{12} = 0.31 \text{ ms}^{-1}$ (2sf)

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14 (a) Given that

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$

express $\sinh(m+1)x$ and $\sinh(m-1)x$ in terms of $\sinh mx$, $\cosh mx$, $\sinh x$ and $\cosh x$

[1 mark]

$\sinh(m+1)x = \sinh mx \cosh x + \cosh mx \sinh x$

$\sinh(mx+x)$

$\sinh(m-1)x = \sinh mx \cosh x - \cosh mx \sinh x$

$\sinh(mx-x)$

14 (b) Hence find the sum of the series

$$C_n = \cosh x + \cosh 2x + \cdots + \cosh nx$$

in terms of $\sinh x$, $\sinh nx$ and $\sinh(n+1)x$

[5 marks]

$\sinh(m+1)x - \sinh(m-1)x$

$= 2\cosh mx \sinh x$

Using different values of m in $\sinh(m+1)x - \sinh(m-1)x$ equation

$f(1)$: $\cancel{\sinh 2x} - \sinh 0 = 2\cosh x \sinh x$

$f(2)$: $\cancel{\sinh 3x} - \sinh x = 2\cosh 2x \sinh x$

$f(3)$: $\sinh 4x - \sinh 2x = 2\cosh 3x \sinh x$

$f(n-2)$: $\cancel{\sinh(n-1)x} - \cancel{\sinh(n-3)x} = 2\cosh(n-2)x \sinh x$

$f(n-1)$: $\sinh nx - \cancel{\sinh(n-2)x} = 2\cosh(n-1)x \sinh x$

$f(n)$: $\sinh(n+1)x - \cancel{\sinh(n-1)x} = 2\cosh nx \sinh x$

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$\sinh(n+1)x + \sinh nx - \sinh x = 2\sinh x(\cosh x + \cosh 2x + \dots + \cosh nx)$

$= 2\sinh x C_n$

$\therefore C_n = \dfrac{\sinh(n+1)x + \sinh nx - \sinh x}{2\sinh x}$

END OF QUESTIONS

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