AQA MODEL ANSWERS

Please write clearly in block capitals.

Centre number						Candidate number				

Surname ____________________

Forename(s) ____________________

Candidate signature ____________________

A-level
FURTHER MATHEMATICS
Paper 1

Monday 3 June 2019 Morning Time allowed: 2 hours

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 100.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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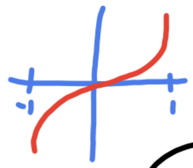
11	
12	
13	
14	
15	
TOTAL	

Answer **all** questions in the spaces provided.

1 Which one of these functions has the set $\{x : |x| < 1\}$ as its greatest possible domain?

values for x

Circle your answer.



[1 mark]

$\cosh x$ $\qquad$ $\cosh^{-1} x$ $\qquad$ $\tanh x$ $\qquad$ $\tanh^{-1} x$

2 The first two non-zero terms of the Maclaurin series expansion of $f(x)$ are x and $-\frac{1}{2}x^3$

$f(0) + f'(0)x + \ldots$

Which one of the following could be $f(x)$?

Circle your answer.

$\simeq 0 + \cos 0 \cdot x + \ldots$

[1 mark]

$xe^{\frac{1}{2}x^2}$ $\qquad$ $\frac{1}{2}\sin 2x$ $\qquad$ $x\cos x$ $\qquad$ $(1+x^3)^{-\frac{1}{2}}$

3 The function $f(x) = x^2 - 1$

Find the mean value of $f(x)$ from $x = -0.5$ to $x = 1.7$
Give your answer to three significant figures.

Circle your answer.

[1 mark]

-0.521 $\qquad$ -0.434 $\qquad$ -0.237 $\qquad$ 0.786

$$\int_{-0.5}^{1.7} (x^2 - 1)\,dx = \left[\frac{1}{3}x^3 - x\right]_{-0.5}^{1.7} = -0.52068$$

$$\frac{-0.52068}{2.2} = -0.237 \ (3 s.f.)$$

Do not write outside the box

4 Solve the equation $2z - 5\mathrm{i}z^* = 12$

[4 marks]

write $z = x+iy \Rightarrow z^* = x-iy$

so $2(x+iy) - 5i(x-iy) = 12$

equate the real & imaginary parts:

Re: $2x - 5y = 12$

Im: $2y - 5x = 0$

$y = \frac{5}{2}$

Sub in: $2x - 5(\frac{5}{2}x) = 12$

$\frac{4}{2}x - \frac{25}{2}x = 12 \Rightarrow -\frac{21}{2}x = \frac{24}{2}$

so $x = -\frac{8}{7}$ & $y = \frac{5}{2}\left(-\frac{8}{7}\right) = -\frac{20}{7}$

$z = -\frac{8}{7} - \frac{20}{7}i$

Turn over for the next question

Do not write outside the box

5 A plane has equation $\mathbf{r}.\begin{bmatrix}1\\1\\1\end{bmatrix} = 7$

A line has equation $\mathbf{r} = \begin{bmatrix}2\\0\\1\end{bmatrix} + \mu\begin{bmatrix}1\\0\\1\end{bmatrix}$

Calculate the acute angle between the line and the plane.

Give your answer to the nearest 0.1°

[3 marks]

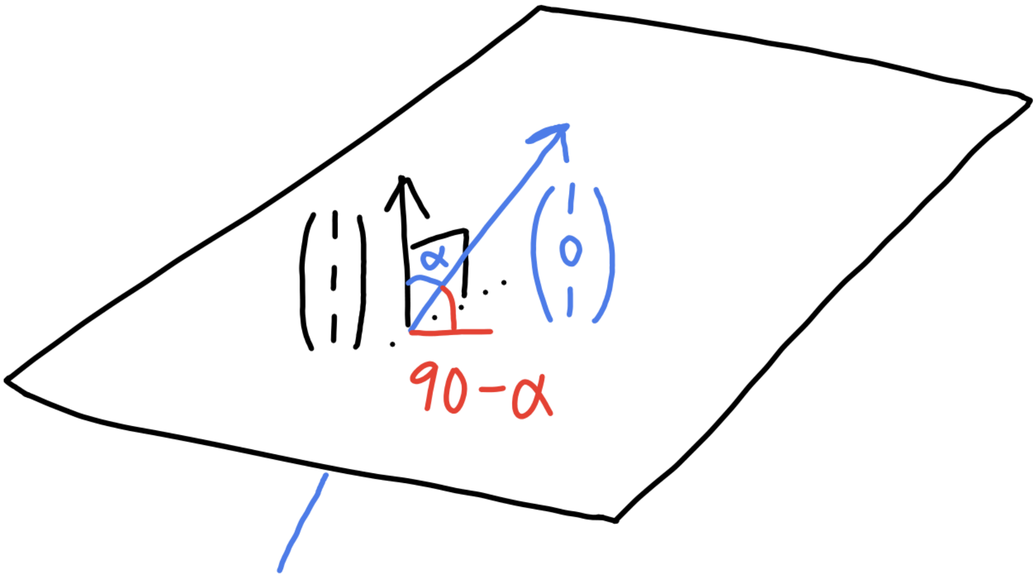
take scalar product of directional vectors:

$\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} = 2$ find length of vectors: $\sqrt{1^2+1^2+1^2} = \sqrt{3}$

$\sqrt{1^2+1^2} = \sqrt{2}$

$\therefore \cos\alpha = \frac{a\cdot b}{|a||b|} = \frac{2}{\sqrt{6}}$

but α is the angle between the normal & the line, not the plane & the line, so we need $\theta = 90 - \alpha = \boxed{54.7°}$



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6 (a) Show that

$$\cosh^3 x + \sinh^3 x = \frac{1}{4}e^{mx} + \frac{3}{4}e^{nx}$$

where m and n are integers.

[3 marks]

$\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\hookrightarrow \cosh^3 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^3 = \frac{1}{8}(e^{3x} + 3e^x + 3e^{-x} + e^{3x})$

$\sinh^3 x = \left[\frac{1}{2}(e^x - e^{-x})\right]^3 = \frac{1}{8}(e^{3x} - 3e^x + 3e^{-x} - e^{-3x})$

x	e^{2x}	$+e^{-2x}$	$+2$
e^x	e^{3x}	$+e^{-x}$	$+2e^x$
e^{-x}	$+e^x$	$+e^{-3x}$	$+2e^{-x}$

so $\sinh^3 x + \cosh^3 x = \frac{1}{8}(e^{3x} + e^{3x} + \cancel{3e^x - 3e^x} + 3e^{-x} + 3e^{-x} + \cancel{e^{-3x} - e^{-3x}})$

$= \boxed{\frac{1}{4}e^{3x} + \frac{3}{4}e^{-x}}$

$\therefore \boxed{m = 3}, \boxed{n = -1}$ Q.E.D.

6 (b) Hence find $\cosh^6 x - \sinh^6 x$ in the form

$$\frac{a\cosh(kx) + b}{8}$$

where a, b and k are integers.

[5 marks]

$\cosh^6 x - \sinh^6 x = (\cosh^3 x + \sinh^3 x)(\cosh^3 x - \sinh^3 x)$

$\hookrightarrow$ difference of two squares

$\cosh^3 x - \sinh^3 x = \frac{3}{4}e^x + \frac{1}{4}e^{-3x}$

$\therefore \cosh^6 x - \sinh^6 x = \left(\frac{1}{4}e^{3x} + \frac{3}{4}e^{-x}\right)\left(\frac{3}{4}e^x + \frac{1}{4}e^{-3x}\right)$

$= \frac{3}{16}e^{4x} + \frac{9}{16} + \frac{1}{16} + \frac{3}{16}e^{-4x}$

$= \frac{3}{8}\left(\frac{e^{4x} + e^{-4x}}{2}\right) + \frac{5}{8}$

$$= \frac{3\cosh 4x + 5}{8}$$

$a = 3, k = 4, b = 5$

Turn over for the next question

Do not write outside the box

7 Three non-singular square matrices, **A**, **B** and **R** are such that

$$\mathbf{AR} = \mathbf{B}$$

The matrix **R** represents a rotation about the z-axis through an angle θ and

$$\mathbf{B} = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7 (a) Show that **A** is independent of the value of θ.

[3 marks]

rotation θ about θ: $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as needed for a rotation

$\det R = \cos\theta(\cos\theta) + \sin\theta(\sin\theta) = \sin^2\theta + \cos^2\theta = 1$

$\Rightarrow R^{-1} = \begin{bmatrix} [X] & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} [X^{-1}] & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

sub. Matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

↪ rotation $(-\theta)$ about z

rearrange $AR = B$: $A = BR^{-1}$

$$= \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\cos^2\theta - \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta & 0 \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ A is independent of θ

7 (b) Give a full description of the single transformation represented by the matrix **A**.
[1 mark]

reflection in $x=0$ plane

Turn over for the next question

Do not write outside the box

8 (a) If $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to prove that

$$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$

[3 marks]

de Moivre's gives: $z^n = (\cos\theta + i\sin\theta)^n$
$= \cos n\theta + i\sin n\theta$
& $\frac{1}{z^n} = z^{-n}$
$= \cos(-n\theta) + i\sin(-n\theta)$
$\cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta$ $= \cos n\theta - i\sin n\theta$

$\Rightarrow z^n - \frac{1}{z^n} = \cancel{\cos n\theta} + i\sin n\theta - (\cancel{\cos n\theta} - i\sin n\theta)$
$= 2i\sin n\theta$

8 (b) Express $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$

[4 marks]

expand using binomial theorem:
$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$
use result in (a) with $n=1$: $\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5$
so $32i\sin^5\theta = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z^1 - z^{-1})$
↪ group into pairs of form $\left(z^n - \frac{1}{z^n}\right)$
$\Rightarrow 32i\sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$
by de Moivre's

$\Rightarrow \sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$

Do not write outside the box

8 (c) Hence show that

$$\int_0^{\frac{\pi}{3}} \sin^5 \theta \, d\theta = \frac{53}{480}$$

[3 marks]

$$\int_0^{\pi/3} \sin^5 \theta \, d\theta = \frac{1}{16}\int_0^{\pi/3} (\sin 5\theta - 5\sin 3\theta + 10\sin\theta)\, d\theta$$

$$= \frac{1}{16}\left[\frac{-1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right]_0^{\pi/3}$$

$$= \left(-\frac{1}{80}\cos\frac{5\pi}{3} + \frac{5}{48}\cos\frac{3\pi}{3} - \frac{5}{8}\cos\frac{\pi}{3}\right) -$$

$$\left(-\frac{1}{80}\cos 0 + \frac{5}{48}\cos 0 - \frac{5}{8}\cos 0\right)$$

$$= \frac{-1}{80}\times\frac{1}{2} + \frac{5}{48}\times(-1) - \frac{5}{8}\times\frac{1}{2} - \left(-\frac{1}{80} + \frac{5}{48} - \frac{5}{8}\right)$$

$$= \frac{53}{480}$$

Turn over for the next question

Do not write outside the box

9 (a) Solve the equation $z^3 = \sqrt{2} - \sqrt{6}\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $0 \leq \theta < 2\pi$

[5 marks]

write in Eulerian form: $z^3 = \sqrt{2+6}\ \exp\left(\tan^{-1}\left(-\frac{\sqrt{6}}{\sqrt{2}}\right)\right)$

$= 2\sqrt{2}\, e^{-\pi i/3}$

$\left(\overbrace{re^{i\theta}}^{z}\right)^3 = 2\sqrt{2}\, e^{-i\pi/3} \Rightarrow r = \sqrt{2},\ 3\theta = -\frac{\pi}{3} + 2k\pi$

$\therefore\ k=0,\ \theta = -\pi/9 \rightarrow$ not in range

$k=1,\ \theta = 5\pi/9$

$k=2,\ \theta = 11\pi/9$

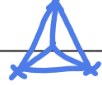
$k=3,\ \theta = 17\pi/9$

so $z = \sqrt{2}\, e^{\frac{5\pi i}{9}},\ \sqrt{2}\, e^{\frac{11\pi i}{9}},\ \sqrt{2}\, e^{\frac{17\pi i}{9}}$

9 (b) The transformation represented by the matrix $\mathbf{M} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ acts on the points on an Argand Diagram which represent the roots of the equation in part **(a)**.

Find the exact area of the shape formed by joining the transformed points.

[4 marks]

area of triangle $= \frac{1}{2}ab\sin c$. shape formed by roots is a triangle made of 3 smaller triangles: 

angle between roots: $\frac{11\pi}{9} - \frac{5\pi}{9} = \frac{6}{9}\pi = \frac{2}{3}\pi$

total area $= 3 \times \frac{1}{2} \times \sqrt{2} \times \sqrt{2} \times \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

Do not write outside the box

scale factor = det M = (5×3) − (1×1) = 14

transformed area = $14 \times \frac{3\sqrt{3}}{2} = 21\sqrt{3}$

Turn over for the next question

Do not write outside the box

10 The points $A(5, -4, 6)$ and $B(6, -6, 8)$ lie on the line L. The point C is $(15, -5, 9)$.

10 (a) D is the point on L that is closest to C.

Find the coordinates of D.

[6 marks]

let L be represented by vector $\underline{r}$.

direction of $\underline{r} = \begin{pmatrix} 6 \\ -6 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\therefore \underline{r} = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

any line between L & C : $\underline{r} - \underline{c} = \underline{v} = \begin{pmatrix} 5-15+\mu \\ -4+5-2\mu \\ 6-9+2\mu \end{pmatrix}$

$\underline{v} = \begin{pmatrix} -10+\mu \\ 1-2\mu \\ -3+2\mu \end{pmatrix}$

for shortest distance, $\underline{v}$ needs to be $\perp$ to L $\Rightarrow$

required ?

$0 \overset{!}{=} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -10+\mu \\ 1-2\mu \\ -3+2\mu \end{pmatrix} \Rightarrow -10+\mu-2+4\mu-6+4\mu=0 \Rightarrow \mu=2$

$D = \begin{pmatrix} 5 \\ -4 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -8 \\ 10 \end{pmatrix}$

Do not write outside the box

10 (b) Hence find, in exact form, the shortest distance from C to L.

[2 marks]

$$\overrightarrow{CD} = \begin{pmatrix} 7 \\ -8 \\ 10 \end{pmatrix} - \begin{pmatrix} 15 \\ -5 \\ 9 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \\ 1 \end{pmatrix}$$

$$|\underline{CD}| = \sqrt{8^2 + 3^2 + 1^2}$$

$$= \sqrt{74} \quad \text{exact form}$$

Turn over for the next question

Do not write outside the box

11 Find the general solution of the differential equation

this can guide your solution

$$x\frac{dy}{dx} - 2y = \frac{x^3}{\sqrt{4 - 2x - x^2}}$$

where $0 < x < \sqrt{5} - 1$

[7 marks]

divide by x to isolate $\frac{dy}{dx}$: $\frac{dy}{dx} - \frac{2y}{x} = \frac{x^2}{\sqrt{4-2x-x^2}}$

this is in the form $\frac{dy}{dx} + f(x)y = g(x)$ so we can use an integrating factor $\exp\left(\int^x f(x')dx'\right)$

$e^{\int_0^x -\frac{2}{x'}dx'} = e^{-2\ln x} = x^{-2}$

$\times x^{-2}$: $x^{-2}\frac{dy}{dx} - 2yx^{-3} = \frac{1}{\sqrt{4-2x-x^2}}$

$\frac{d}{dy}(?) = \frac{1}{x^2} \Rightarrow ? = \frac{y}{x^2}$ $\quad \frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{\sqrt{4-2x-x^2}}$

RHS has form similar to $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$

$\Rightarrow$ complete the square: $4-2x-x^2 = 4-(x+1)^2+1$
$= 5-(x+1)^2$

$\Rightarrow \int^x \frac{1}{\sqrt{5-(x'+1)^2}}dx' = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$

general solution

$\frac{y}{x^2} = \sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$

Do not write outside the box

$\therefore\ y = x^2\left\{\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + c\right\}$

$\sin^{-1}u$ has domain $0 \leqslant u \leqslant 1 \Rightarrow 0 \leqslant \frac{x+1}{\sqrt{5}} \leqslant 1$

Turn over for the next question

Do not write outside the box

12 Three planes have equations

$$4x - 5y + \ z = 8$$
$$3x + 2y - kz = 6$$
$$(k-2)x + ky - 8z = 6$$

where k is a real constant.

The planes do **not** meet at a unique point.

12 (a) Find the possible values of k.

[3 marks]

don't meet @ unique point $\Rightarrow$ determinant $= 0$

as matrix: $\underbrace{\begin{pmatrix} 4 & -5 & 1 \\ 3 & 2 & -k \\ (k-2) & k & -8 \end{pmatrix}}_{M}$ $\rightarrow \det M = 9k^2 - 9k - 180$

$\doteq 0$

dividing both sides by 9.

$\hookrightarrow k^2 - k - 20 = 0$

$\rightarrow$ $k = 5$ or $k = -4$

Added notes:

$\det M = 4(-16 + k^2) + 5(-24 + k^2 - 2k) + 1(3k - 2k + 4)$

$= -64 + 4k^2 - 120 + 5k^2 - 10k + k + 4$

$= 9k^2 - 9k - 180$

$= 0$

Do not write outside the box

12 (b) For each value of k found in part **(a)**, identify the configuration of the given planes.

Fully justify your answer, stating in each case whether or not the equations of the planes form a consistent system.

[5 marks]

$\boxed{k=5}$ system of equations: $\begin{bmatrix} 4 & -5 & 1 & 8 \\ 3 & 2 & -5 & 6 \\ 3 & 5 & -8 & 6 \end{bmatrix}$

$3x+2y-5z=6$
$3x+5y-8z=6$
$-3y+3z=0 \Rightarrow y=z$

sub in: $3x-3y=6 \Rightarrow x=y+2$
① ② $4x-4y=8 \Rightarrow x=y+2$ } consistent

$k=5$: consistent, line of intersection/sheaf

$\boxed{k=-4}$ $\begin{bmatrix} 4 & -5 & 1 & 8 \\ 3 & 2 & 4 & 6 \\ -6 & -4 & -8 & 6 \end{bmatrix} \Rightarrow \begin{matrix} 3x+2y+4z=6 \\ -6x-4y-8z=6 \end{matrix}$ } inconsistent

$k=-4$: inconsistent, 2 parallel, distinct planes with 3rd crossing both

Turn over for the next question

Do not write outside the box

13 The equation $z^3 + kz^2 + 9 = 0$ has roots α, β and γ.

13 (a) (i) Show that

$$\alpha^2 + \beta^2 + \gamma^2 = k^2$$

[3 marks]

know $\alpha+\beta+\gamma = -k$ & $\alpha\beta+\beta\gamma+\gamma\alpha = 0$

so can find $\alpha^2+\beta^2+\gamma^2$ from $(\alpha+\beta+\gamma)^2$

$$(\underbrace{\alpha+\beta+\gamma}_{\text{known}})^2 = \alpha^2+\beta^2+\gamma^2+2(\underbrace{\alpha\beta+\beta\gamma+\gamma\alpha}_{\text{known}})$$

$\hookrightarrow (-k)^2 = \alpha^2+\beta^2+\gamma^2+2(0)$

$\therefore k^2 = \alpha^2+\beta^2+\gamma^2$ (as required)

13 (a) (ii) Show that

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = -18k$$

[4 marks]

Similarly, expand $(\alpha\beta+\beta\gamma+\gamma\alpha)^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta^2\gamma+2\alpha^2\beta\gamma+2\alpha\beta\gamma^2$

$= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$\alpha\beta\gamma = -9$

$\alpha+\beta+\gamma = -k$

$\alpha\beta+\beta\gamma+\gamma\alpha = 0$

$\therefore 0 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+18k$

$\Rightarrow \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = -18k$ (as required)

Do not write outside the box

13 (b) The equation $9z^3 - 40z^2 + rz + s = 0$ has roots $\alpha\beta + \gamma$, $\beta\gamma + \alpha$ and $\gamma\alpha + \beta$.

13 (b) (i) Show that

divide equation by 9

$$k = -\frac{40}{9}$$

[1 mark]

sum of roots $= +\frac{40}{9} \Rightarrow \alpha\beta+\gamma+\beta\gamma+\alpha+\gamma\alpha+\beta = \frac{40}{9}$

$\alpha+\beta+\gamma = -k$ & $\alpha\beta+\beta\gamma+\gamma\alpha = 0$ from (a)

$\therefore -k = \frac{40}{9}$

$k = -\frac{40}{9}$ (as required)

Question 13 continues on the next page

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13 (b) (ii) Without calculating the values of α, β and γ, find the value of s.

Show working to justify your answer.

[6 marks]

product of roots $= -\frac{s}{9} = (\alpha\beta+\gamma)(\beta\gamma+\alpha)(\gamma\alpha+\beta)$

$= \alpha^2\beta^2\gamma^2 + \alpha\beta^3\gamma + \alpha^3\beta\gamma + \alpha^2\beta^2 + \alpha\beta\gamma^3 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha\beta\gamma$

$= (\alpha\beta\gamma)^2 + \alpha\beta\gamma + \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha\beta\gamma(\alpha^2+\beta^2+\gamma^2)$

$= (-9)^2 - 9 - 18k - 9k^2$

$= -\frac{232}{9}$

$\Rightarrow \boxed{s = 232}$

Turn over for the next question

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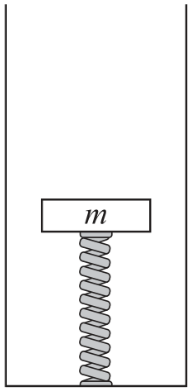
14 **In this question use $g = 10\,\text{m}\,\text{s}^{-2}$**

A light spring is attached to the base of a long tube and has a mass m attached to the other end, as shown in the diagram.

The tube is filled with oil.

When the compression of the spring is ε metres, the thrust in the spring is $9m\varepsilon$ newtons.



The mass is held at rest in a position where the compression of the spring is $\dfrac{20}{9}$ metres.

The mass is then released from rest. During the subsequent motion the oil causes a resistive force of $6mv$ newtons to act on the mass, where $v\,\text{m}\,\text{s}^{-1}$ is the speed of the mass.

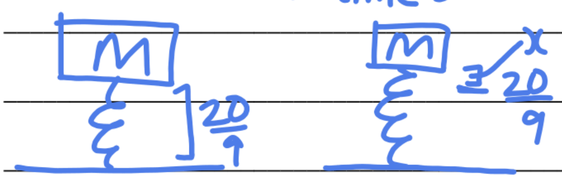
At time t seconds after the mass is released, the displacement of the mass above its starting position is x metres.

14 (a) Find x in terms of t.

[10 marks]

this set up produces simple harmonic motion

start → time t



starting compression $\varepsilon_0 = \dfrac{20}{9}$

then $\varepsilon(x) = \dfrac{20}{9} - x$

Do not write outside the box

equation of motion: $F = ma$

$\hookrightarrow 9m\varepsilon - W - 6m\dot{x} = m\ddot{x}$

$9m\left(\frac{20}{9} - x\right) - mg - 6m\dot{x} = m\ddot{x}$

$\Rightarrow \ddot{x} + 6\dot{x} + 9x = 10$

$\hookrightarrow \lambda^2 + 6\lambda + 9 = 0$ auxiliary equation

$(\lambda + 3)^2 = 0 \quad \therefore \lambda = -3$ (repeated root)

repeat root

$\Rightarrow x = (At + B)e^{-3t}$ complementary function

let $x' = \gamma$: $\dot{x}' = 0, \ddot{x}' = 0$

sub in: $9\gamma = 10 \therefore \gamma = \frac{10}{9}$ particular integral

general solution: $x = Ae^{-3t} + Bte^{-3t} + \frac{10}{9}$

initial conditions: $x = 0$ @ $t = 0 \Rightarrow A = \frac{-10}{9}$

$\dot{x} = -3Ae^{-3t} + Be^{-3t} - 3Bte^{-3t}$

$\dot{x} = 0$ @ $t = 0 \Rightarrow 0 = -3A + B = -3\left(\frac{10}{9}\right) + B \Rightarrow B = -\frac{30}{9}$

$\therefore x = -\frac{10}{9}e^{-3t} - \frac{10}{3}te^{-3t} + \frac{10}{9}$

14 (b) State, giving a reason, the type of damping which occurs.

[1 mark]

the auxiliary equation has equal roots, so we conclude the system undergoes critical damping

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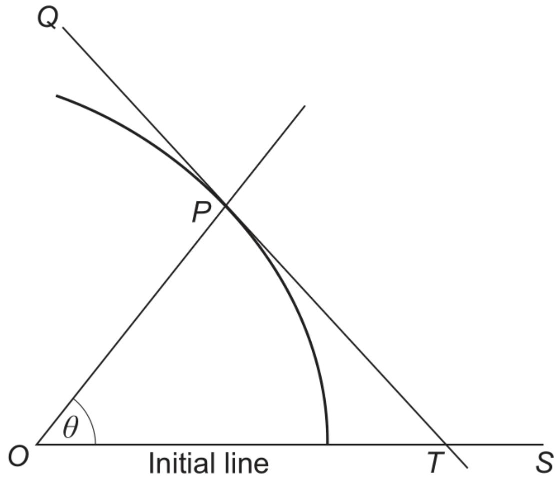
15 The diagram shows part of a spiral curve.

The point P has polar coordinates (r, θ) where $0 \le \theta \le \frac{\pi}{2}$

The points T and S lie on the initial line and O is the pole.

TPQ is the tangent to the curve at P.



15 (a) Show that the gradient of TPQ is equal to

$$\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}$$

[4 marks]

polar → Cartesian: $x = r\cos\theta$ & $y = r\sin\theta$

← r may be a function of θ

$\therefore \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$

gradient $= \frac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$ (as required)

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15 (b) The curve has polar equation

$$r = e^{(\cot b)\theta}$$

where b is a constant such that $0 < b < \frac{\pi}{2}$

Use the result of part **(a)** to show that the angle between the line *OP* and the tangent *TPQ* does not depend on θ.

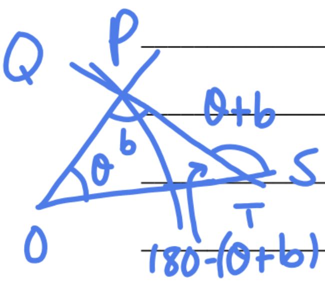
[7 marks]

$r = e^{(\cot b)\theta}$ so $\frac{dr}{d\theta} = (\cot b)e^{(\cot b)\theta}$

sub in answer for (a): $\frac{dy}{dx} = \frac{(\cot b)\cancel{e^{(\cot b)\theta}}\sin\theta + \cancel{e^{(\cot b)\theta}}\cos\theta}{(\cot b)\cancel{e^{(\cot b)\theta}}\cos\theta - \cancel{e^{(\cot b)\theta}}\sin\theta}$

$$= \frac{\cot b\sin\theta + \cos\theta}{\cot b\cos\theta - \sin\theta}$$

$$= \frac{\frac{\cos b}{\sin b}\sin\theta + \cos\theta}{\frac{\cos b}{\sin b}\cos\theta - \sin\theta}$$



$\times \frac{\sin b}{\sin b}$ $= \frac{\cos b\sin\theta + \sin b\cos\theta}{\cos b\cos\theta - \sin b\sin\theta}$

by compound angle formulae $= \frac{\sin(\theta+b)}{\cos(\theta+b)} = \tan(\theta+b)$

gradient $= \tan(\theta+b)$ $\therefore$ $\angle STP = \theta + b$

$\therefore$ by exterior angle of triangle, $\angle OPT = b$

so the angle between OP & TPQ is independent of θ

END OF QUESTIONS

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