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AQA

SPECIMEN MATERIAL

Please write clearly, in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname ____________________

Forename(s) ____________________

Candidate signature ____________________

AS MODEL SOLUTIONS
FURTHER MATHEMATICS
Paper 1

Exam Date   Morning   Time allowed: 1 hour 30 minutes

Materials
For this paper you must have:
- The AQA booklet of formulae and statistical tables.
- You may use a graphics calculator.

Instructions
- Use black ink or black ball-point pen. Pencil should be used for drawing.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
Unless stated otherwise, you may quote formulae, without proof, from the booklet.
You do not necessarily need to use all the space provided.

Answer **all** questions in the spaces provided.

1 A reflection is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

State the equation of the line of invariant points.

Circle your answer.

[1 mark]

$x = 0$ $y = 0$ $y = x$ $y = -x$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

So for $\begin{pmatrix} x \\ y \end{pmatrix}$ to be invariant, $y=-y$ and therefore $y=0$

2 Find the mean value of $3x^2$ over the interval $1 \le x \le 3$

Circle your answer.

[1 mark]

$8\frac{2}{3}$ 10 13 26

Mean Value $= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3-1}\int_1^3 3x^2\,dx = \frac{1}{2}\left[x^3\right]_1^3 = \frac{1}{2}(27-1) = 13$

3 Find the equations of the asymptotes of the curve $x^2 - 3y^2 = 1$

Circle your answer.

[1 mark]

$y = \pm\, 3x$ $\qquad$ $y = \pm\frac{1}{3}x$ $\qquad$ $y = \pm\sqrt{3}\,x$ $\qquad$ $y = \pm\frac{1}{\sqrt{3}}x$

$x^2 - 3y^2 = 1$

So the curve ~~can~~ can never satisfy $x^2 - 3y^2 = 0$

$\Rightarrow x^2 = 3y^2$

$\Rightarrow x = \pm\sqrt{3}\,y$

$\Rightarrow y = \pm\frac{1}{\sqrt{3}}x$

Turn over for the next question

4 $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

4 (a) Find the value of k for which matrix **A** is singular.

[1 mark]

Singular $\Rightarrow$ det $A = 0$ so $ad - bc = 0 \Rightarrow k - 2 = 0$
and thus $k = 2$

4 (b) Describe the transformation represented by matrix **B**.

[1 mark]

Reflection in the y-axis (From the formula booklet)

4 (c) (i) Given that **A** and **B** are both non-singular, verify that $\mathbf{A}^{-1}\mathbf{B}^{-1} = (\mathbf{BA})^{-1}$.

[4 marks]

$BA = \begin{pmatrix} -1 & -2 \\ 1 & k \end{pmatrix} \Rightarrow (BA)^{-1} = \frac{1}{-k+2}\begin{pmatrix} k & 2 \\ -1 & -1 \end{pmatrix}$

$A^{-1} = \frac{1}{k-2}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix}$ and $B^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So $A^{-1}B^{-1} = \frac{1}{(k-2)(-1)}\begin{pmatrix} k & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{-k+2}\begin{pmatrix} k+0 & 0+(-2)(-1) \\ -1\times1+0 & 0+(-1)(1) \end{pmatrix}$

$= \frac{1}{-k+2}\begin{pmatrix} k & 2 \\ -1 & -1 \end{pmatrix}$

And thus $(BA)^{-1} = A^{-1}B^{-1}$

4 (c) (ii) Prove the result $\mathbf{M}^{-1}\mathbf{N}^{-1} = (\mathbf{NM})^{-1}$ for all non-singular square matrices **M** and **N** of the same size.

[4 marks]

Consider $(NM) \times (M^{-1}N^{-1})$

Let $X = (NM) \times (M^{-1}N^{-1})$

$= N \times M \times M^{-1} \times N^{-1}$

$= N \times N^{-1}$

$= I$

So $X = (NM) \times (M^{-1}N^{-1}) = I \Rightarrow M^{-1}N^{-1} = (NM)^{-1}$

Therefore, the result $M^{-1}N^{-1} = (NM)^{-1}$ for all square matricies N and M that are the same size

Turn over for the next question

5 The region bounded by the curve with equation $y = 3 + \sqrt{x}$, the x-axis and the lines $x = 1$ and $x = 4$ is rotated through 2π radians about the x-axis.

Use integration to show that the volume generated is $\frac{125\pi}{2}$

[5 marks]

$y = 3 + \sqrt{x} \implies y^2 = 9 + 6\sqrt{x} + x$

Solid of revolution around x-axis: $V = \int_a^b \pi y^2 \, dx$

$V = \pi \int_1^4 (9 + 6\sqrt{x} + x) \, dx = \pi \int_1^4 (9 + 6x^{\frac{1}{2}} + x) \, dx = \pi \left[9x + 4x^{3/2} + \frac{1}{2}x^2 \right]_1^4$

$V = \pi \left[\left(9(4) + 4(4)^{3/2} + \frac{1}{2}(4)^2\right) - \left(9(1) + 4(1)^{3/2} + \frac{1}{2}(1)^2\right) \right]$

$= \pi \left[76 - \frac{27}{2} \right] = \pi \left[\frac{1}{2}(152 - 27) \right] = \frac{\pi}{2}(125)$

The volume of the region is $\frac{125\pi}{2}$

6 (a) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that

$x = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$ where $t = \tanh x$

[4 marks]

$\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \Rightarrow t = \frac{e^{2x} - 1}{e^{2x} + 1}$

To find the inverse function, we need to rearrange for x

$t(e^{2x}+1) = e^{2x} - 1 \Rightarrow te^{2x} - e^{x} = -1 - t$

$\Rightarrow e^{2x}(t-1) = -1(1+t)$

$\Rightarrow e^{2x} = \frac{1+t}{1-t}$

$\Rightarrow 2x = \ln\left(\frac{1+t}{1-t}\right)$

$\Rightarrow x = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right)$

Question 6 continues on the next page

6 (b) (i) Prove $\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

[4 marks]

$\cosh^3 x = \left[\frac{1}{2}(e^x+e^{-x})\right]^3 = \frac{1}{8}\left(e^{3x} + 3e^{2x}e^{-x} + 3e^{x}e^{-2x} + e^{-3x}\right)$

$= \frac{1}{8}\left(e^{3x}+e^{-3x} + 3(e^{x}+e^{-x})\right)$

$= \frac{1}{4}\left(\frac{1}{2}(e^{3x}+e^{-3x})\right) + \frac{3}{4}\left(\frac{1}{2}(e^{x}+e^{-x})\right)$

$= \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x = RHS$

So $\cosh^3 x = \frac{1}{4}\cosh 3x + \frac{3}{4}\cosh x$

6 (b) (ii) Show that the equation $\cosh 3x = 13\cosh x$ has only one positive solution.

Find this solution in exact logarithmic form.

[4 marks]

$\cosh^3 x = \frac{1}{4}(13\cosh x) + \frac{3}{4}\cosh x$

$= \frac{13+3}{4}(\cosh x) = 4\cosh x$

$\cosh^3 x - 4\cosh x = 0$

$\cosh x(\cosh^2 x - 4) = 0$

$\cosh x(\cosh x - 2)(\cosh x + 2) = 0$

Range of $\cosh x$: $\cosh x \geqslant 1 \ \forall x$ so $\cosh x \neq 0$ and $\cosh x \neq -2$

So $\cosh x = 2 \Rightarrow x = \ln(2 + \sqrt{2^2 - 1}) = \ln(2+\sqrt{3})$

$x = \ln(2+\sqrt{3})$ (Only one possible solution)

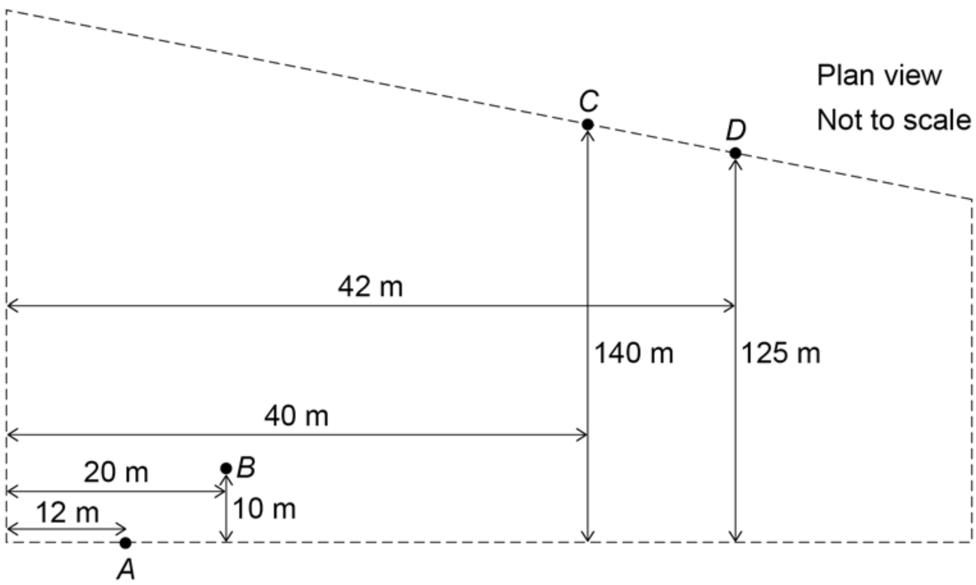
Turn over for the next question

7 A lighting engineer is setting up part of a display inside a large building. The diagram shows a plan view of the area in which he is working.

He has two lights, which project narrow beams of light.

One is set up at a point 3 metres above the point *A* and the beam from this light hits the wall 23 metres above the point *D*.

The other is set up 1 metre above the point *B* and the beam from this light hits the wall 29 metres above the point *C*.



7 (a) By creating a suitable model, show that the beams of light intersect.

[6 marks]

$r_A = \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\left(\begin{pmatrix}30\\125\\23\end{pmatrix} - \begin{pmatrix}0\\0\\3\end{pmatrix}\right)$ [This is the light starting 3m above A (which we take to be the origin) and ending up 23m above D)]

$r_B = \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\left(\begin{pmatrix}28\\140\\29\end{pmatrix} - \begin{pmatrix}8\\10\\1\end{pmatrix}\right)$ [Vector equation of a line representing the light starting at 1m above B and ending 29m above C]

$r_A = \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}30\\125\\20\end{pmatrix}$ and $r_B = \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\begin{pmatrix}20\\130\\28\end{pmatrix}$

$\Rightarrow 30\lambda = 8 + 20\mu$ and $125\lambda = 10 + 130\mu$

Solving these two equations simultaneously yields: $\lambda = \frac{3}{5}$ and $\mu = \frac{1}{2}$

Substituting these values into the third component:

$3 + \frac{3}{5}(20) = 15$ and $1 + \frac{1}{2}(28) = 15$ so the lines intersect

7 (b) Find the angle between the two beams of light.

[3 marks]

$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta \Rightarrow \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$

$\begin{pmatrix}30\\125\\20\end{pmatrix} \cdot \begin{pmatrix}20\\130\\28\end{pmatrix} = 17,410$

$\cos\theta = \frac{17,410}{\sqrt{30^2+125^2+20^2}\sqrt{20^2+130^2+28^2}} = 0.9951$

$\Rightarrow \theta = 5.6^\circ$

7 (c) State one way in which the model you created in part **(a)** could be refined.

[1 mark]

Take the width of the beams into account

8 A curve has polar equation $r = 3 + 2\cos\theta$, where $0 \le \theta < 2\pi$

8 (a) (i) State the maximum and minimum values of r.

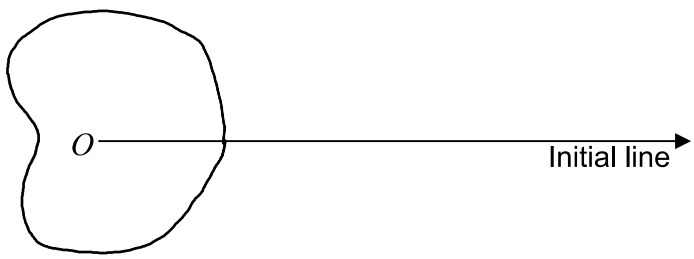
[2 marks]

Maximum of $\cos\theta = 1$ so $r = 3 + 2(1) = 5$ $r = 5$

Minimum of $\cos\theta = -1$ so $r = 3 + 2(-1) = 1$ $r = 1$

8 (a) (ii) Sketch the curve.

[2 marks]



8 (b) The curve $r = 3 + 2\cos\theta$ intersects the curve with polar equation $r = 8\cos^2\theta$, where $0 \le \theta < 2\pi$

Find all of the points of intersection of the two curves in the form $[r, \theta]$.

[6 marks]

Let $c = \cos\theta \Rightarrow r = 3 + 2c \quad r = 8c^2$

Equating the two equations: $3 + 2c = 8c^2$

$8c^2 - 2c - 3 = 0$

$8c^2 - 6c + 4c - 3 = 0$

$2c(4c - 3) + (4c - 3) = 0$

$(2c + 1)(4c - 3) = 0$

$\cos\theta = -\frac{1}{2}, \ \cos\theta = \frac{3}{4}$

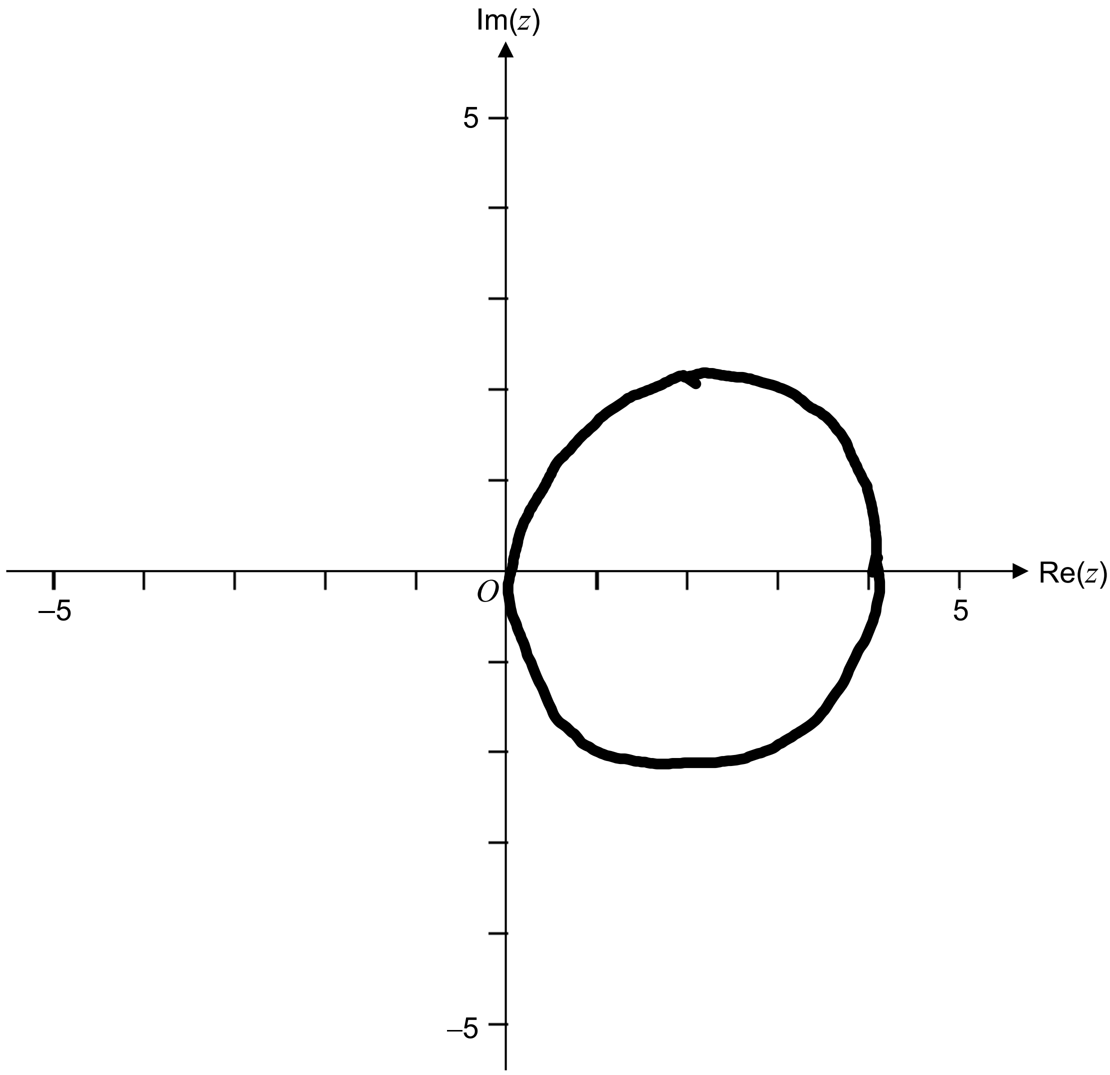
$\Rightarrow \theta = 0.723, \ 5.56, \ \frac{4\pi}{3}, \ \frac{2\pi}{3}$

$r = 3 + 2c$ so at $c = \frac{3}{4}, r = \frac{9}{2}$ and at $c = -\frac{1}{2}, r = 2$

Points of intersection: $(\frac{9}{2}, 0.723), (\frac{9}{2}, 5.56), (2, \frac{2\pi}{3})$ and $(2, \frac{4\pi}{3})$

9 (a) Sketch on the Argand diagram below, the locus of points satisfying the equation $|z - 2| = 2$

[2 marks]



9 (b) Given that $|z - 2| = 2$ and $\arg(z - 2) = -\dfrac{\pi}{3}$, express z in the form $a + b\mathrm{i}$, where a and b are real numbers.

[3 marks]

$z - 2 = re^{i\theta}$

$= 2e^{-\frac{i\pi}{3}} = 2(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3})$

$z = 2\cos -\frac{\pi}{3} + 2i\sin -\frac{\pi}{3} + 2$

$a = 2\cos -\frac{\pi}{3} + 2 = 3$ $\quad b = 2\sin -\frac{\pi}{3} = -\sqrt{3}$

$z = 3 - \sqrt{3}i$

Turn over for the next question

10 (a) Prove that

$$6 + 3\sum_{r=1}^{n}(r+1)(r+2) = (n+1)(n+2)(n+3)$$

[6 marks]

$6 + 3\sum_{r=1}^{n}(r+1)(r+2) = 6 + 3\sum_{r=1}^{n} r^2 + 3r + 2$

$= 6 + 3\sum_{r=1}^{n} r^2 + 9\sum_{r=1}^{n} r + 6\sum_{r=1}^{n} 1$

$= 6 + 3\left(\frac{n(n+1)(2n+1)}{6}\right) + 9\left(\frac{n(n+1)}{2}\right) + 6n$

$= \frac{1}{2}n(n+1)(2n+1) + \frac{9}{2}n(n+1) + 6(n+1)$

$= (n+1)\left[\frac{1}{2}n(2n+1) + \frac{9}{2}n + 6\right]$

$= (n+1)\left[n^2 + \frac{1}{2}n + \frac{9}{2}n + 6\right]$

$= (n+1)\left[n^2 + 5n + 6\right]$

$= (n+1)(n+2)(n+3)$

10 (b) Alex substituted a few values of n into the expression $(n + 1)(n + 2)(n + 3)$ and made the statement:

"For all positive integers n,

$$6 + 3\sum_{r=1}^{n}(r + 1)(r + 2)$$

is divisible by 12."

Disprove Alex's statement.

[2 marks]

Consider the case when $n=4$,

$6+3\sum_{r=1}^{n}(r+1)(r+2) = (n+1)(n+2)(n+3) = (5)(6)(7) = 35\times 6 = 17.5\times 12$

This is not a multiple of 12 and therefore Alex's statement is false

Turn over for the next question

11 The equation $x^3 - 8x^2 + cx + d = 0$ where c and d are real numbers, has roots α, β, γ.

When plotted on an Argand diagram, the triangle with vertices at α, β, γ has an area of 8.

Given $\alpha = 2$, find the values of c and d.

Fully justify your solution.

[5 marks]

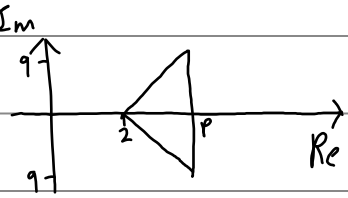
As the coefficients of the equation are real, $\beta = p+qi$ and $\gamma = p-qi$ (complex conjugate pair)

Sum of roots $(\Sigma\alpha) = 8$

$\alpha+\beta+\gamma=8$

$\alpha + p+qi+p-qi = 8$

$2+2p=8 \Rightarrow p=3$



Area $= 8$ so $(p-2)q=8$

$\Rightarrow q=8$

$\beta = 3+8i$ and $\gamma = 3-8i$

$d = -\alpha\beta\gamma = -146$

$c = \Sigma\alpha\beta = 2(3+8i)+2(3-8i)+(3+8i)(3-8i) = 85$

12 A curve, C_1 has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{5x^2 - 12x + 12}{x^2 + 4x - 4}$

The line $y = k$ intersects the curve, C_1

12 (a) (i) Show that $(k + 3)(k - 1) \geq 0$

[5 marks]

$k = \dfrac{5x^2 - 12x + 12}{x^2 + 4x + 4} \Rightarrow k(x^2+4x+4) = 5x^2 - 12x + 12$

$kx^2 + 4kx + 4k - 5x^2 + 12x - 12 = 0$

$x^2(k-5) + 4(k+3)x - 4(k+3) = 0$

$y = k$ intersects C_1 so the roots of the above equation are real

$\Rightarrow b^2 - 4ac \geqslant 0$

$\Rightarrow (4(k+3))^2 - 4(k-5)(-4(k+3)) \geqslant 0$

$\Rightarrow 16(k+3)^2 + 16(k-5)(k+3) \geqslant 0$

$\Rightarrow 16(k+3)[k+3+k-5] \geqslant 0$

$\Rightarrow 16(k+3)(2k-2) \geqslant 0$

$\Rightarrow 32(k+3)(k-1) \geqslant 0$

$\Rightarrow (k+3)(k-1) \geqslant 0$

Question 12 continues on the next page

12 (a) (ii) Hence find the coordinates of the stationary point of C_1 that is a maximum point.

[4 marks]

Maximum at $k=-3$ or $k=1$

Substituting k into $(k-5)x^2+4(k+3)x-4(k+3)=0$

$k=-3$

$=7 \; -8x^2=0$ so $x=0$

Maximum point of C_1 is $(0,-3)$

12 (b) Show that the curve C_2 whose equation is $y = \dfrac{1}{f(x)}$, has no vertical asymptotes.

[2 marks]

$b^2 - 4ac = (-12)^2 - 4(5)(12) < 0$

So $5x^2 - 12x + 12 \neq 0 \ \forall x$ so C_1 has no vertical asymptotes

12 (c) State the equation of the line that is a tangent to both C_1 and C_2.

[1 mark]

$y = 1$

END OF QUESTIONS

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