AQA

Please write clearly in block capitals.

Centre number | | | | | | Candidate number | | | | |

Surname ______________________

Forename(s) ______________________

Candidate signature ______________________

AS
FURTHER MATHEMATICS

Paper 1

Model Answers

Monday 13 May 2019 Afternoon Time allowed: 1 hour 30 minutes

Materials
- You must have the AQA formulae and statistical tables booklet for A-level Mathematics and A-level Further Mathematics.
- You should have a scientific calculator that meets the requirements of the specification. (You may use a graphical calculator.)

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- You must answer each question in the space provided for that question. If you require extra space for your answer(s), use the lined pages at the end of this book. Write the question number against your answer(s).
- Do **not** write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 80.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

For Examiner's Use	
Question	Mark
1	
2	
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12	
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TOTAL	

J U N 1 9 7 3 6 6 1 0 1

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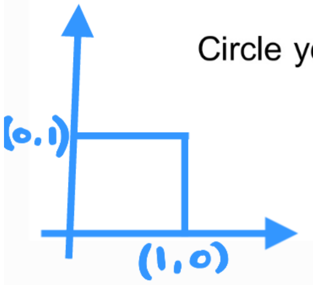
1 Which of the following matrices is an identity matrix?



Circle your answer.

[1 mark]

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Identity matrix must be square, and numbers on the diagonals must be ones, with the rest being zeros.

2 Which of the following expressions is the determinant of the matrix $\begin{bmatrix} a & 2 \\ b & 5 \end{bmatrix}$?

Circle your answer.

[1 mark]

$5a - 2b$ $\quad$ $2a - 5b$ $\quad$ $5b - 2a$ $\quad$ $2b - 5a$

If $\underline{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det \underline{M} = ad - bc$

$\therefore$ determinant of matrix $\begin{pmatrix} a & 2 \\ b & 5 \end{pmatrix} = 5a - 2b$

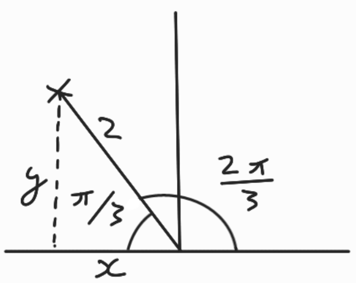
3 Point P has polar coordinates $\left(2, \frac{2\pi}{3}\right)$.

Which of the following are the Cartesian coordinates of P?

Circle your answer.

[1 mark]

$(1, -\sqrt{3})$ $\quad$ $(-\sqrt{3}, 1)$ $\quad$ $(\sqrt{3}, -1)$ $\quad$ $(-1, \sqrt{3})$



$x = 2\cos \pi/3 = 1$

$y = 2 \sin \pi/3 = \sqrt{3}$

$(-1, \sqrt{3})$

4 The line L has polar equation

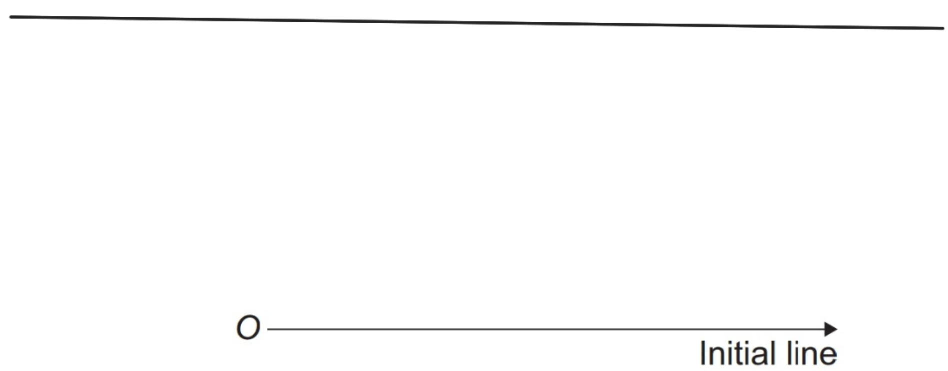
$$r = \frac{k}{\sin\theta}$$

$\Rightarrow$ straight line above initial line parallel to initial line

where k is a positive constant.

4 (a) Sketch L.

[1 mark]



4 (b) State the minimum distance between L and the point O.

[1 mark]

Distance $= k$

5 A hyperbola H has the equation

$$\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$$

where a is a positive constant.

5 (a) Write down the equations of the asymptotes of H.

[1 mark]

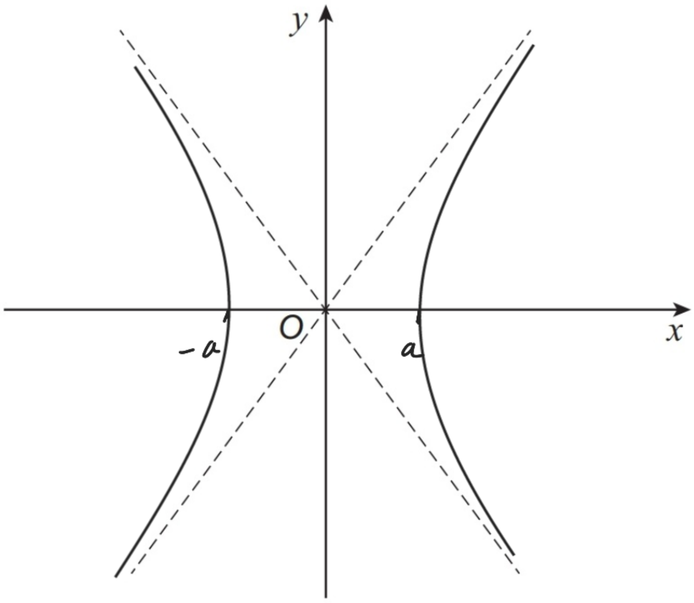
Equation of asymptotes: $y = 0 + \frac{\sqrt{4a^2}}{\sqrt{a^2}}(x - 0)$

$y = \pm 2x$

5 (b) Sketch the hyperbola H on the axes below, indicating the coordinates of any points of intersection with the coordinate axes.

The asymptotes have already been drawn.

[2 marks]



5 (c) The finite region bounded by H, the positive x-axis, the positive y-axis and the line $y = a$ is rotated through 360° about the y-axis.

Show that the volume of the solid generated is ma^3, where $m = 3.40$ correct to three significant figures.

[5 marks]

$$\text{Volume} = \pi \int x^2 \, dy$$

$$\frac{x^2}{a^2} - \frac{y^2}{4a^2} = 1$$

$$x^2 = a^2 + \frac{y^2 a^2}{4a^2}$$

$$x^2 = \frac{y^2}{4} + a^2$$

$$\Rightarrow \text{volume} = \pi \int_0^a \left(\frac{y^2}{4} + a^2 \right) dy$$

$$= \pi \left[\frac{y^3}{12} + a^3 y \right]_0^a$$

$= \pi\left(\frac{a^3}{12} + a^3 - 0\right)$

$= \frac{13\pi}{12} a^3$

$\approx 3.403 a^3$

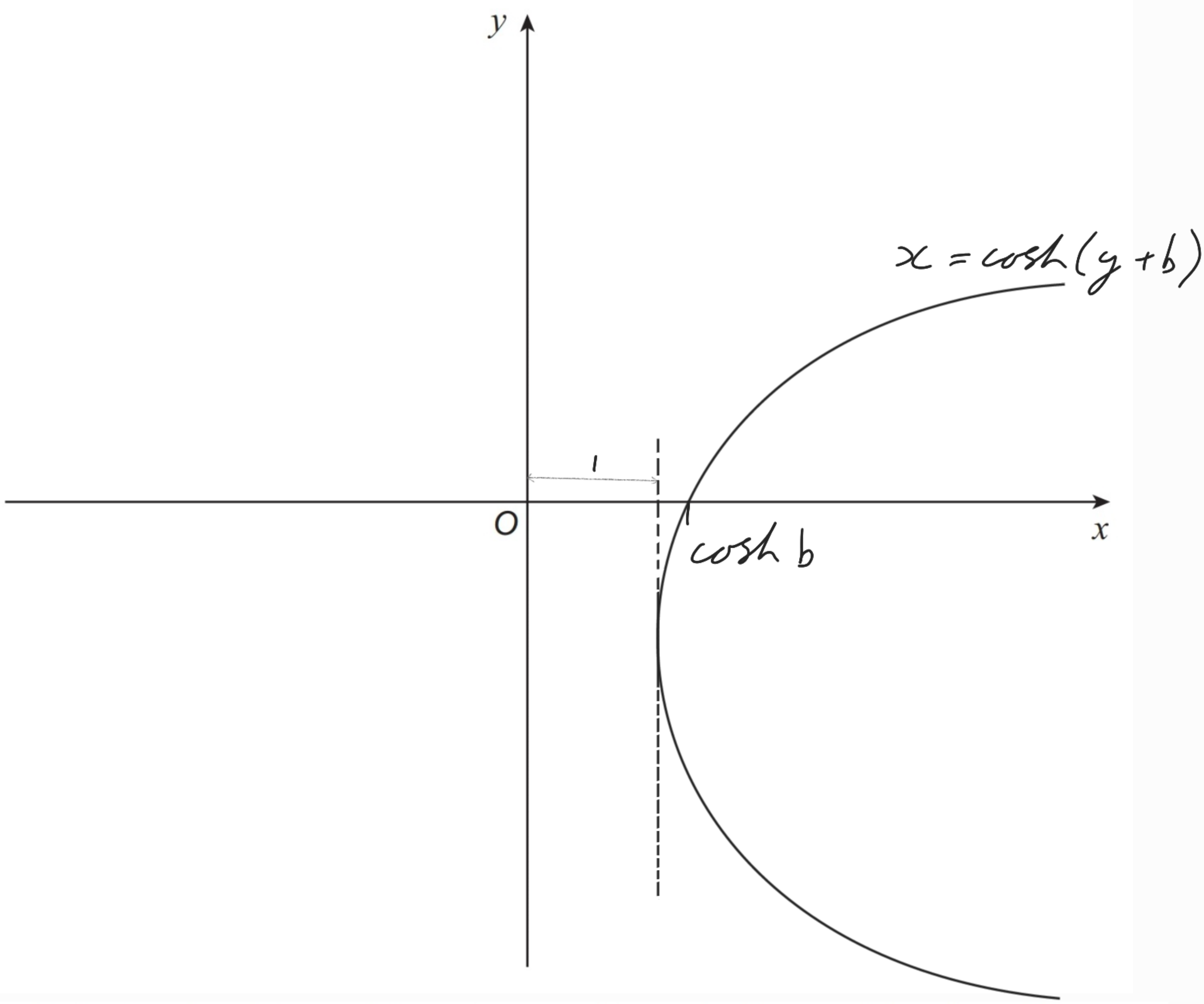
$= 3.40 a^3$ (3 sig figs)

6 (a) On the axes provided, sketch the graph of

$$x = \cosh(y+b)$$ $\equiv y = \cosh x$, translated left by b then reflected in $y = x$

where b is a positive constant.

[4 marks]



6 (b) Determine the minimum distance between the graph of $x = \cosh(y+b)$ and the y-axis.

[1 mark]

(See diagram). Minimum distance = 1

7 (a) Show that

$$\frac{1}{r-1} - \frac{1}{r+1} \equiv \frac{A}{r^2-1}$$

where A is a constant to be found.

[1 mark]

Multiplied by $\frac{r+1}{r+1}$

Multiplied by $\frac{r-1}{r-1}$

This gives: $\frac{r+1}{(r+1)(r-1)} - \frac{r-1}{(r-1)(r+1)}$

$= \frac{r-r+1+1}{(r+1)(r-1)}$

$= \frac{2}{r^2-1}$

$A = 2$

7 (b) Hence use the method of differences to show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} \equiv \frac{an^2+bn+c}{4n(n+1)}$$

where a, b and c are integers to be found.

[4 marks]

$$\sum_{r=2}^{n}\left(\frac{2}{r^2-1}\right) = \sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{1}{r+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \dots$$

$$+ \left(\cancel{\frac{1}{n-3}} - \cancel{\frac{1}{n-1}}\right) + \left(\cancel{\frac{1}{n-2}} - \frac{1}{n}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore \sum_{r=2}^{n}\left(\frac{2}{r^2-1}\right) = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{3}{2} - \frac{(n+1)+n}{n(n+1)}$$

$$\Rightarrow \sum_{r=2}^{n}\left(\frac{1}{r^2-1}\right) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$$

$$= \frac{3(n^2+n)-2(2n+1)}{4n(n+1)}$$

$$= \frac{3n^2-n-2}{4n(n+1)}$$

8 Given that $z_1 = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$ and $z_2 = 2\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$

8 (a) Find the value of $|z_1 z_2|$

[1 mark]

$|z_1 z_2| = 2 \times 2 = 4$

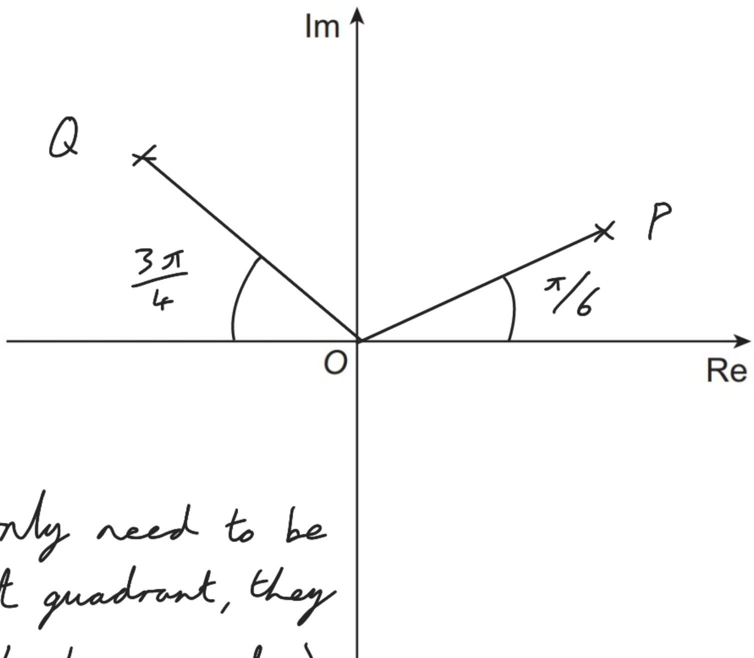
8 (b) Find the value of $\arg\left(\frac{z_1}{z_2}\right)$

[1 mark]

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6} - \frac{3\pi}{4} = -\frac{7\pi}{12}$

8 (c) Sketch z_1 and z_2 on the Argand diagram below, labelling the points as P and Q respectively.

[2 marks]



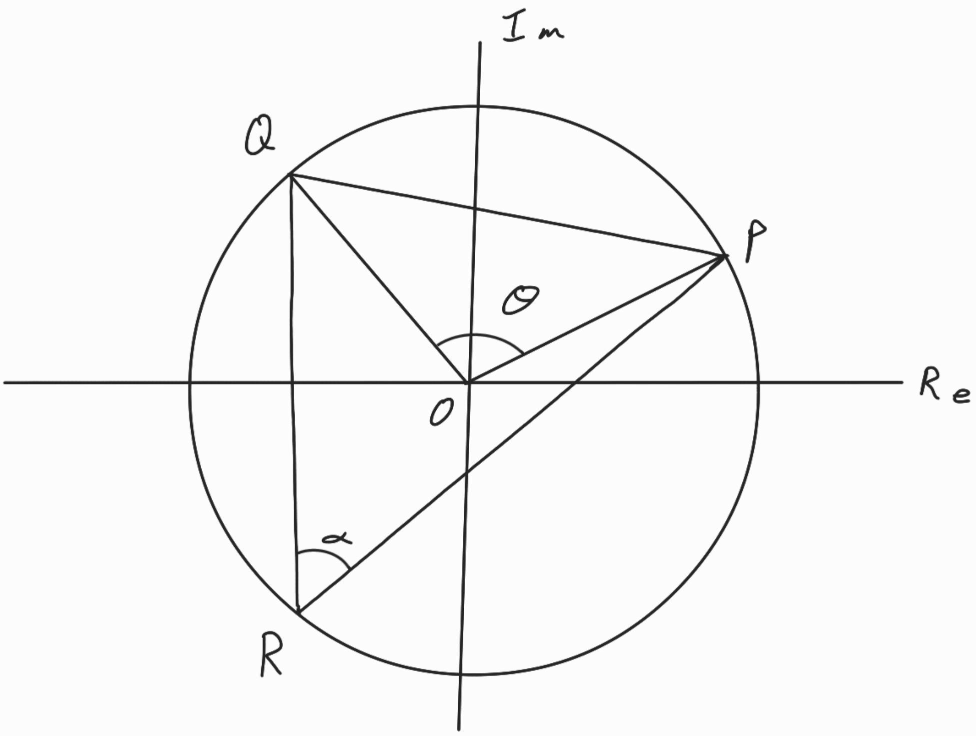
(These points only need to be in the correct quadrant, they do not have to be accurate).

8 (d) A third complex number w satisfies both $|w| = 2$ and $-\pi < \arg w < 0$

Given that w is represented on the Argand diagram as the point R, find the angle $P\hat{R}Q$.

Fully justify your answer.

[3 marks]



$|w| = 2$ is a circle

$\theta = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$

$P\hat{O}Q$

angle at circumference $= \frac{1}{2} \times$ angle at centre

$\therefore$ α is half of θ, for any position of R

$\therefore \quad \alpha = \frac{7\pi}{12} \div 2 = \frac{7\pi}{24}$

$P\hat{R}Q = \boxed{\frac{7\pi}{24}}$

9 (a) Saul is solving the equation

$$2\cosh x + \sinh^2 x = 1$$

He writes his steps as follows:

$$2\cosh x + \sinh^2 x = 1$$

$$2\cosh x + 1 - \cosh^2 x = 1$$

$$2\cosh x - \cosh^2 x = 0$$

$$\cosh x \neq 0 \ \therefore\ 2 - \cosh x = 0$$

$$\cosh x = 2$$

$$x = \pm\cosh^{-1}(2)$$

Identify and explain the error in Saul's method.

[2 marks]

In line 2, $\sinh^2 x$ is replaced with $1 - \cosh^2 x$.
This is wrong, $\sinh^2 x \neq 1 - \cosh^2 x$
$\sinh^2 x = \cosh^2 x - 1$

9 (b) Anna is solving the different equation

$$\sinh^2(2x) - 2\cosh(2x) = 1$$

and finds the correct answers in the form $x = \frac{1}{p}\cosh^{-1}(q + \sqrt{r})$, where p, q and r are integers.

Find the possible values of p, q and r.

Fully justify your answer.

[5 marks]

$\sinh^2(2x) = \cosh^2(2x) - 1$

(sub this identity into equation in question)

$\therefore \cosh^2(2x) - 1 - 2\cosh(2x) = 1$

$\cosh^2(2x) - 2\cosh(2x) - 2 = 0$

$\therefore \cosh(2x) = \dfrac{2 \pm \sqrt{(-2)^2 - 4\times 1 \times -2}}{2} = 1 \pm \sqrt{3}$

However $\cosh(2x) \geq 1 \therefore \cosh(2x) = 1 + \sqrt{3}$

$\therefore x = \pm \frac{1}{2} \cosh^{-1}(1 + \sqrt{3})$

$p = \pm 2$, $q = 1$, $r = 3$

10 (a) Using the definition of $\cosh x$ and the Maclaurin series expansion of e^x, find the first three non-zero terms in the Maclaurin series expansion of $\cosh x$.

[3 marks]

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

Using Maclaurin series expansion of e^x and e^{-x}:

$\cosh x = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$

$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24}$ (first 3 non-zero terms)

10 (b) Hence find a trigonometric function for which the first three terms of its Maclaurin series are the same as the first three terms of the Maclaurin series for $\cosh(ix)$.

[3 marks]

Sub ix into expression for $\cosh x$ from (a):

$\cosh(ix) = 1 + \frac{(ix)^2}{2} + \frac{(ix)^4}{24}$

$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$

= Maclaurin's expansion of $\cos x$

11 (a) Curve C has equation

$$y = \frac{x^2 + px - q}{x^2 - r}$$

where p, q and r are positive constants.

Write down the equations of its asymptotes.

[2 marks]

$x^2 - r \Rightarrow 0$ at asymptote

$x^2 - r = 0$

$x^2 = r$

$x = \pm\sqrt{r}$

as $x \to \infty$, $y \to 1$ $\therefore$ asymptote at $y = 1$

11 (b) Find the set of possible y-coordinates for the graph of

$$y = \frac{x^2 + x - 6}{x^2 - 1}, \qquad x \neq \pm 1$$

giving your answer in exact form.

No credit will be given for solutions based on differentiation.

[6 marks]

let $k = \dfrac{x^2 + x - 6}{x^2 - 1}$

$k(x^2 - 1) = x^2 + x - 6$

$kx^2 - x^2 - x - k + 6 = 0$

$(k - 1)x^2 - x + 6 - k = 0$

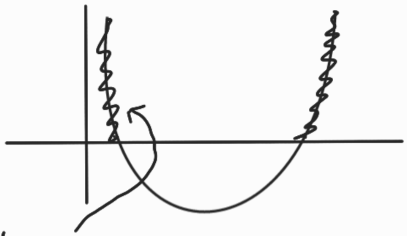
$\Rightarrow b^2 - 4ac \geq 0$

$1 - 4(k-1)(6-k) \geq 0$

$1 - 4(6k - k^2 - 6 + k) \geq 0$

$4k^2 - 28k + 25 \geq 0$

by calculator: $y = \frac{7 \pm 2\sqrt{6}}{2}$



values ≥ 0 are needed

$\Rightarrow y \geq \frac{7+2\sqrt{6}}{2}$

$y \leq \frac{7-2\sqrt{6}}{2}$

12 The matrix **A** is given by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

12 (a) Prove by induction that, for all integers $n \geq 1$,

$$\mathbf{A}^n = \begin{bmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{bmatrix}$$

[4 marks]

$n = 1$:

LHS: $A^1 = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

RHS: $\begin{pmatrix} 1 & 3^1 - 1 \\ 0 & 3^1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$\therefore$ true for $n = 1$

assume true for $n = k$:

$\therefore A^k = \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix}$

$n = k+1$:

$$A^k \times A = \begin{pmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

$$A^{k+1} = \begin{pmatrix} 1 & 2 + 3(3^k - 1) \\ 0 & 3 \times 3^k \end{pmatrix}$$

$$A^{k+1} = \begin{pmatrix} 1 & 3^{k+1} - 1 \\ 0 & 3^{k+1} \end{pmatrix}$$ $\therefore$ also true for $n = k+1$

$\therefore$ Statement is true for $n=1$, and true for $n = k+1$ when assumed true for $n = k$ $\Rightarrow$ True for all integers $n \geq 1$

12 (b) Find all invariant lines under the transformation matrix **A**.

Fully justify your answer.

[6 marks]

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$x' = x + 2y$ and $y' = 3y$

let $y = mx + c$ and $y' = mx' + c$, sub into equations for images:

$x' = x + 2(mx + c)$ ① and $mx' + c = 3(mx + c)$ ②

Sub ① into ②:

$$m(x + 2mx + 2c) + c \equiv 3mx + 3c$$

Equating coefficients :

$x : m + 2m^2 = 3m$, constants : $2mc + c = 3c$

$m(m-1) = 0$ $c(m-1) = 0$

$m = 0$ or $m = 1$ $c = 0$ or $m = 1$

$y = 0x + 0$ $y = 1x + c$

$\therefore$ Invariant lines are : $y = 0$

$y = x$

12 (c) Find a line of invariant points under the transformation matrix **A**.

[2 marks]

$x = x + 2y$ and $y = 3y$

Sub in $y = x$: $x = x + 2x$ $x \neq 3x$

Not consistent

$x = 3x$ $x \neq 3x$

Not consistent

Sub in $y = 0$: $x = x + 0$

Consistent

$0 = 3 \times 0$

Consistent

$\therefore$ $y = 0$ is the only line of invariant points

13 Line l_1 has Cartesian equation

$$x - 3 = \frac{2y + 2}{3} = 2 - z$$

13 (a) Write the equation of line l_1 in the form

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a parameter and **a** and **b** are vectors to be found.

[2 marks]

$$\frac{x-3}{1} = \frac{y-(-1)}{1.5} = \frac{z-2}{-1}$$

$$\therefore \underline{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1.5 \\ -1 \end{pmatrix}$$

13 (b) Line l_2 passes through the points $P(3, 2, 0)$ and $Q(n, 5, n)$, where n is a constant.

13 (b) (i) Show that the lines l_1 and l_2 are **not** perpendicular.

[3 marks]

Perpendicular $\Rightarrow$ scalar product of direction vectors $= 0$

Direction vector of $l_2 = \begin{pmatrix} n-3 \\ 5-2 \\ n-0 \end{pmatrix} = \begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix}$

scalar product : $\begin{pmatrix} 1 \\ 1.5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix} = n - 3 + 4.5 - n = 1.5$

scalar product $\neq 0$ $\therefore$ l_1 and l_2 are not perpendicular

13 (b) (ii) Explain briefly why lines l_1 and l_2 cannot be parallel.

[2 marks]

Parallel $\Rightarrow \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix}$, $n - 3 = 2$

$n = 5$ and $n = -2$

n cannot $= 5$ and -2, $\therefore$ l_1 and l_2 are not parallel

13 (b) (iii) Given that θ is the acute angle between lines l_1 and l_2, show that

$$\cos\theta = \frac{p}{\sqrt{34n^2 + qn + 306}}$$

where p and q are constants to be found.

[3 marks]

$$l_1 \cdot l_2 = |l_1||l_2| \times \cos\theta$$

$$\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} n-3 \\ 3 \\ n \end{pmatrix} = \sqrt{2^2 + 3^2 + 2^2} \times \sqrt{(n-3)^2 + 3^2 + n^2} \times \cos\theta$$

$$2n - 6 + 9 - 2n = \sqrt{17} \times \sqrt{2n^2 - 6n + 18} \times \cos\theta$$

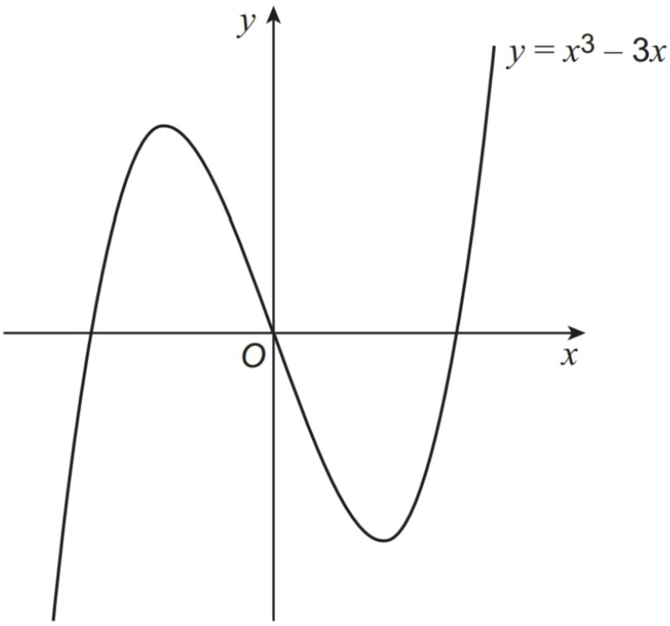
$$3 = \sqrt{34n^2 - 102n + 306} \times \cos\theta$$

$$\cos\theta = \frac{3}{\sqrt{34n^2 - 102n + 306}}$$

$p = 3$

$q = -102$

14 The graph of $y = x^3 - 3x$ is shown below.



The two stationary points have x-coordinates of -1 and 1

The cubic equation

$$x^3 - 3x + p = 0$$

where p is a real constant, has the roots α, β and γ.

The roots α and β are **not** real.

14 (a) Explain why $\alpha + \beta = -\gamma$

[1 mark]

Coefficient of $x^2 = 0$

$\therefore \alpha + \beta + \gamma = 0$

$\alpha + \beta = -\gamma$

14 (b) Find the set of possible values for the real constant p.

[2 marks]

Maximum point : $x = -1$, $y = (-1)^3 + 3 = 2$ $(-1, 2)$

Minimum point : $x = 1$, $y = 1^3 - 3 = -2$ $(1, -2)$

$\therefore p > 2$, $p < -2$

14 (c) $f(x) = 0$ is a cubic equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$

14 (c) (i) Show that the constant term of $f(x)$ is $p + 2$

[3 marks]

Let $w = x + 1$

$x = w - 1$

Sub into cubic equation:

$(w-1)^3 - 3(w-1) + p = 0$

$w^3 - 3w^2 + 3w - 1 - 3w + 3 + p = 0$

$w^3 - 3w^2 + p + 2 = 0$

constant term $= p + 2$ (as required)

14 (c) (ii) Write down the x-coordinates of the stationary points of $y = f(x)$

[1 mark]

$x = 0$, $x = 2$