Further Kinematics

In this chapter, you will learn to use vectors to model the motion of a particle moving in 2-dimensions. We will go through problems involving constant acceleration, projectile motion and variable acceleration.

Vectors in kinematics

You can use the vector equivalent of the SUVAT formulae to solve problems involving constant acceleration. Of course, any velocities, positions and accelerations should be given as vectors. The vector SUVAT equations are:

- $v = u + at$
- $r = ut + \frac{1}{2}at^2$

r represents the position vector, u and v represent the initial and final velocities respectively

If a particle starts at position vector r_0 and moves with constant velocity v, then its position vector r at a time t can be expressed as:

- $r = r_0 + vt$

Note that the displacement from the initial position will therefore be given by vt. We will now go through an example where we will need to apply the above formulae:

Example 1: A particle A starts at the point with position vector $(12i + 12j)$. The initial velocity of A is $(-i + j)$ ms^{-1}, and it has constant acceleration $(2i - 4j)ms^{-2}$. Another particle, B, has initial velocity i ms^{-1} and constant acceleration $2j$ ms^{-2}. After 3 seconds the two particles collide. Find: a) the speeds of the two particles when they collide. b) the position vector of the point where the two particles collide.

a) Using $v = u + at$ for A: You can use i-j notation too but column notation is slightly easier to work with.	$v = \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}2\\-4\end{pmatrix}(3)$ $\therefore v = \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}6\\-12\end{pmatrix} = \begin{pmatrix}5\\-11\end{pmatrix}$ Speed $= \sqrt{(5)^2 + (-11)^2} = \sqrt{146}\ ms^{-1}$
Using $v = u + at$ for B:	$v = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}(3) = \begin{pmatrix}1\\6\end{pmatrix}$ So speed is $\sqrt{(1)^2 + (6)^2} = \sqrt{37}\ ms^{-1}$
b) Using $r = ut + \frac{1}{2}at^2$ for A:	$r = \begin{pmatrix}-1\\1\end{pmatrix}(3) + \frac{1}{2}\begin{pmatrix}2\\-4\end{pmatrix}(3^2)$ $r = \begin{pmatrix}-3\\3\end{pmatrix} + \frac{9}{2}\begin{pmatrix}2\\-4\end{pmatrix} = \begin{pmatrix}6\\-15\end{pmatrix}$

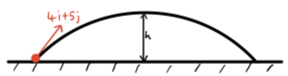
These problems are of the same nature as those you learnt to solve in Chapter 8 of Stats/Mechanics Year 1 and should be treated in the same way. The only difference here is that we use vectors, instead of treating each dimension of motion separately.

Vector methods with projectiles

You also need to be able to solve projectile motion problems using vectors. Recall that projectile motion is motion with constant acceleration and so we can use the vector SUVAT equations.

- You can consider the horizontal and vertical motion separately to solve vector projectile motion questions, or you can use the vector SUVAT equations.
- The acceleration of a projectile in vector form is $0i - gj$, or in column notation $\begin{pmatrix}0\\-g\end{pmatrix}$ if we take the upwards direction to be positive. This is because there is no acceleration in the horizontal direction and the particle is under the influence of only gravity.
- You can assume that **I** and **j** are the unit vectors horizontally and vertically unless told otherwise.

Example 2: A particle is projected from the origin with velocity $(4i + 5j)$ ms^{-1}. The particle moves freely under gravity.
Find: a) the position vector of P after t seconds.
b) the greatest height of the particle.



a) Using $r = ut + \frac{1}{2}at^2$: The acceleration can be expressed as 0i-gj.	$r = \begin{pmatrix}4\\5\end{pmatrix}(t) + \frac{1}{2}\begin{pmatrix}0\\-g\end{pmatrix}(t^2)$ $\Rightarrow r = \begin{pmatrix}4t\\5t - \frac{gt^2}{2}\end{pmatrix}$
b) The greatest height will be when the vertical component of the velocity will be 0. We can use $v = u + at$ to find an expression for the velocity at time t in vector form:	$v = u + at$ $v = \begin{pmatrix}4\\5\end{pmatrix} + \begin{pmatrix}0\\-g\end{pmatrix}t$ $v = \begin{pmatrix}4\\5 - gt\end{pmatrix}$
Equating the j component to 0 and solving for t:	$5 - gt = 0$ $\therefore t = \frac{5}{g} = 0.510s$

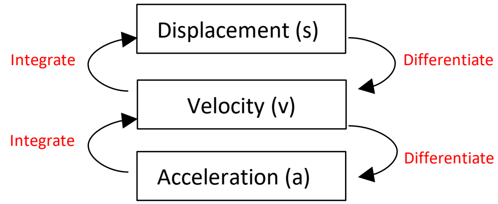
Note that we could have also treated the **I** and **j** components separately and used methods from Chapter 6 to solve this question.

Variable acceleration in one dimension

So far, we have only looked at problems involving particles moving with a constant acceleration. You can be expected to solve problems where the acceleration is variable.

- When a particle experiences variable acceleration, the acceleration will be given as a function of time.

You can be expected to use calculus with vectors to solve variable acceleration problems. The following relationship between displacement, velocity and acceleration that you covered in Pure Year 1 is important:



These functions will involve more complicated expressions, including trigonometric and exponential functions. You may need to use your knowledge of differentiation and integration from Pure Year 2 to solve such questions.

Example 3: A particle of mass 6kg is moving on the positive x-axis. At time t seconds the displacement, s of the particle from the origin O is given by $s = 2t^{\frac{3}{2}} + \frac{e^{-2t}}{3}$ m, where $t \geq 0$.
a) Find the velocity of the particle when $t = 1.5$.
b) Given that the particle is acted on by a single force of variable magnitude F N which acts in the direction of the positive x-axis, find the value of F when $t = 1.5$.

a) To find an expression for v, we can differentiate s :	$v = \frac{ds}{dt} = 3t^{\frac{1}{2}} - \frac{2}{3}e^{-2t}$
Substituting $t = 1.5$:	$v = 3\left(1.5^{\frac{1}{2}}\right) - \frac{2}{3}\left(e^{-2(1.5)}\right) = 3.64ms^{-1}$ to 3 s.f.
b) We first need to figure out the acceleration and substitute it into $F = ma$ to find the force. Differentiating v to find a:	$a = \frac{dv}{dt} = \frac{3}{2}t^{-\frac{1}{2}} + \frac{4}{3}e^{-2t}$
Substituting $t = 1.5$:	At $t = 1.5,\ a = \frac{3}{2}(1.5)^{-\frac{1}{2}} + \frac{4}{3}e^{-2(1.5)}$ $= 1.0850 \ldots ms^{-2}$
Substituting $m = 6,\ \ a = 1.0850..$ into $F = ma$:	$\Rightarrow F = 6 \times 1.0850.. = 6.51N$ (3 s.f.)