1. A bird leaves its nest at time $t = 0$ for a short flight along a straight line.

The bird then returns to its nest.

The bird is modelled as a particle moving in a straight horizontal line.

The distance, s metres, of the bird from its nest at time t seconds is given by

$$s = \frac{1}{10}(t^4 - 20t^3 + 100t^2), \quad \text{where } 0 \leqslant t \leqslant 10$$

(a) Explain the restriction, $0 \leqslant t \leqslant 10$

(3)

(b) Find the distance of the bird from the nest when the bird first comes to instantaneous rest.

(6)

(a) When $t=0$, $s = \frac{1}{10}(0^4 - 20(0)^3 + 100(0)^2)$

$\underline{s=0}$

When $t = 10$, $s = \frac{1}{10}(10^4 - 20(10)^3 + 100(10)^2)$

$= \frac{1}{10}(10000 - 20(1000) + 100(100))$

$= \frac{1}{10}(10000 - 20000 + 10000)$

$\underline{= 0}$

So when $t=0$ and $t=10$, $s=0$

Now, $s = \frac{1}{10}(t^4 - 20t^3 + 100t^2)$

$s = \frac{1}{10}t^2(t^2 - 20t + 100)$

$s = \frac{1}{10}t^2(t-10)(t-10)$

$s = \frac{1}{10} t^2 (t-10)^2$

$\therefore$ s is a perfect square, so $s > 0$ for $0 < t < 10$

(b) Differentiate displacement, s for velocity, v:

$$v = \frac{1}{10}(4t^3 - 60t^2 + 200t)$$

When the bird reaches rest, velocity $= 0 ms^{-1}$

$$\therefore \frac{1}{10}(4t^3 - 60t^2 + 200t) = 0$$

$$\frac{1}{10} t(4t^2 - 60t + 200) = 0$$

$$\frac{4}{10} t(t^2 - 15t + 50) = 0$$

$$\frac{2}{5} t(t-10)(t-5) = 0$$

Either $t=0$ or $t=5$ or $t=10$

When the bird first reaches instantaneous rest, $t=5$. It isn't $t=0$, because this is when the bird leaves the nest.

Now, substitute $t=5$ into

$$s=\frac{1}{10}(t^4-20t^3+100t^2):$$

$$s=\frac{1}{10}(5^4-20(5)^3+100(5)^2)$$

$$s=\frac{1}{10}(625-20(125)+100(25))$$

$$s=\frac{1}{10}(625-2500+2500)$$

$$s=\frac{1}{10}(625)$$

$$\boxed{s=62.5\text{ m}}$$

distance of the bird from the nest when it first comes to instantaneous rest is 62.5m

2. A particle, P, moves along the x-axis. At time t seconds, $t \geqslant 0$, the displacement, x metres, of P from the origin O, is given by $x = \frac{1}{2}t^2(t^2 - 2t + 1)$

(a) Find the times when P is instantaneously at rest. **(5)**

(b) Find the total distance travelled by P in the time interval $0 \leqslant t \leqslant 2$ **(3)**

(c) Show that P will never move along the negative x-axis. **(2)**

a) $\frac{dx}{dt} = 0$

$x = \frac{1}{2}t^4 - t^3 + \frac{1}{2}t^2$

$\frac{dx}{dt} = 2t^3 - 3t^2 + t$

$2t^3 - 3t^2 + t = 0$

$t(2t-1)(t-1) = 0$

$t = 0 \quad t = \frac{1}{2} \quad t = 1$

b) $t = 0, \quad x = 0$

$t = \frac{1}{2}, \quad x = \frac{1}{32}$ $\quad +\frac{1}{32}$

$t = 1, \quad x = 0$ $\quad -\frac{1}{32}$

$t = 2, \quad x = 2$ $\quad +2$

distance travelled $= 2\left(\frac{1}{32}\right) + 2$

$= \frac{33}{16}$

$= 2.0625$

$\simeq 2.06$

c) $x = \frac{1}{2}t^2(t^2 - 2t + 1)$

$= \frac{1}{2}t^2(t-1)^2$

For $t \geqslant 0$, x is always positive

3. A particle, P, moves along a straight line such that at time t seconds, $t \geqslant 0$, the velocity of P, $v\,\mathrm{m\,s^{-1}}$, is modelled as

$$v = 12 + 4t - t^2$$

Find

(a) the magnitude of the acceleration of P when P is at instantaneous rest,

(5)

(b) the distance travelled by P in the interval $0 \leqslant t \leqslant 3$

(3)

a rest $\Rightarrow v = 0$!

$-t^2 + 4t + 12 = 0$

$(6 - t)(t + 2) = 0$

$t = 6\,s$ (or $t = -2$, not valid)

$a = \frac{dv}{dt} = 4 - 2t$

at $t = 6$, $a = 4 - 2 \times 6$

$= -8\,ms^{-2}$

Magnitude $= 8\,ms^{-2}$

b. $s = \int v\,dt$

$= \int_0^3 (12 + 4t - t^2)\,dt$

$= \left[12t + 2t^2 - \frac{t^3}{3}\right]_0^3$

↓

$= 36 + 18 - 9 - 0$

$= 45 \text{ m}$

4. A particle P moves along a straight line such that at time t seconds, $t \geqslant 0$, after leaving the point O on the line, the velocity, $v\,\mathrm{m\,s^{-1}}$, of P is modelled as

$$v = (7 - 2t)(t + 2)$$

(a) Find the value of t at the instant when P stops accelerating. **(4)**

(b) Find the distance of P from O at the instant when P changes its direction of motion. **(5)**

In this question, solutions relying on calculator technology are not acceptable.

a) 'stops accelerating': $a = \frac{dv}{dt} = 0$

$V = (7-2t)(t+2) = 3t - 2t^2 + 14$

$\frac{dv}{dt} = 3 - (2 \times 2)t$

$= 3 - 4t$

so $3 - 4t = 0$

$t = \frac{3}{4}$ s

b) change of direction: $V = 0$

$3t - 2t^2 + 14 = 0 \Rightarrow (2t-7)(t+2) = 0$

$t = \frac{7}{2}$ or $t = -2$

$t > 0$ so $t = \frac{7}{2}$ find t @ which direction changes

$s = \int V dt$ integrate to find distance

$= \int_0^{\frac{7}{2}} (3t - 2t^2 + 14)\,dt$

$= \left[\frac{3}{2}t^2 - \frac{2t^3}{3} + 14t\right]_0^{\frac{7}{2}}$

$= \frac{3}{2}\left(\frac{7}{2}\right)^2 - \frac{2}{3}\left(\frac{7}{2}\right)^3 + 14\left(\frac{7}{2}\right)$

$= \frac{931}{24}$ m $= 38.79166\ldots$ m

5. At time t seconds, where $t \geqslant 0$, a particle P moves so that its acceleration $\mathbf{a}$ m s^{-2} is given by

$$\mathbf{a} = 5t\mathbf{i} - 15t^{\frac{1}{2}}\mathbf{j}$$

When $t = 0$, the velocity of P is $20\mathbf{i}$ m s^{-1}

Find the speed of P when $t = 4$

(6)

$\underline{a} = 5t\underline{i} - 15t^{1/2}\underline{j}$, $t = 0$, $\underline{v} = 20\underline{i}$ ms^{-1}

$\downarrow$ v?   Speed when $t = 4$?

Differentiate $\downarrow$ Displacement, Velocity, Acceleration $\uparrow$ Integrate

We can integrate our equation for acceleration with respect to t, to find an expression for the velocity.

$\Rightarrow \underline{a} = 5t\underline{i} - 15t^{1/2}\underline{j}$ $\int 5t \, dt = \frac{5t^2}{2}$ and $\int 15t^{1/2} dt = 10t^{3/2}$

$\Rightarrow \int 5t\underline{i} - 15t^{1/2}\underline{j} \, dt = \frac{5t^2}{2}\underline{i} - 10t^{3/2}\underline{j} + C = \underline{V}$ (1) (1)

When $t = 0$, $\underline{V} = 20\underline{i}$ $\Rightarrow \frac{5(0)^2}{2}\underline{i} - 10(0)^{3/2}\underline{j} + C = 20\underline{i} \Rightarrow C = 20\underline{i}$

$\Rightarrow \underline{v} = \frac{5t^2}{2}\underline{i} - 10t^{3/2}\underline{j} + 20\underline{i}$ (1)

When $t = 4$, we need to find velocity $\Rightarrow \underline{V} = \frac{5(4)^2}{2}\underline{i} - 10(4)^{3/2}\underline{j} + 20\underline{i} = 40\underline{i} - 80\underline{j} + 20\underline{i}$

$\Rightarrow \underline{V} = 60\underline{i} - 80\underline{j}$ (1) (Velocity expression for $t=4$)

We can now find the speed by working out the magnitude since speed is the scalar equivalent of velocity.

Speed $= |\underline{V}| = \sqrt{(60)^2 + (-80)^2} = \underline{\underline{100 \text{ ms}^{-1}}}$ (1) (1)

6. Unless otherwise stated, whenever a numerical value of g is required, take $g = 9.8\,\text{m}\,\text{s}^{-2}$ and give your answer to either 2 significant figures or 3 significant figures.

At time t seconds, where $t \geqslant 0$, a particle P moves in the x-y plane in such a way that its velocity $\mathbf{v}\,\text{m}\,\text{s}^{-1}$ is given by

$$\mathbf{v} = t^{-\frac{1}{2}}\mathbf{i} - 4t\mathbf{j}$$

When $t = 1$, P is at the point A and when $t = 4$, P is at the point B.

Find the exact distance AB.

(6)

$\underline{v} = \dfrac{d\underline{s}}{dt}$

$\therefore\ \underline{s} = \int \underline{v}.dt$

$\underline{s} = \int (t^{-\frac{1}{2}}\underline{i} - 4t\underline{j})dt$ - 1

$\underline{s} = 2t^{\frac{1}{2}}\underline{i} - 2t^2\underline{j} + \underline{c}$ - 1

At A: - substitute t=1

$\underline{s} = 2(1)^{\frac{1}{2}}\underline{i} - 2(1)^2\underline{j} + \underline{c}$ - 1

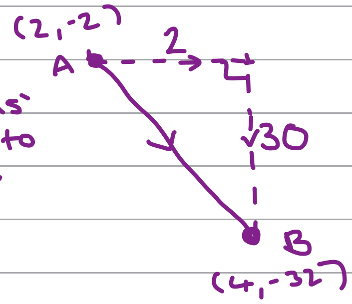
$\underline{s} = 2\underline{i} - 2\underline{j} + \underline{c}$ - 1

(As we are finding the length of the vertical and horizontal lines, the $\underline{c}$'s cancel out and can therefore be left out.)

At B: - substitute t=4

$\underline{s} = 2(4)^{\frac{1}{2}}\underline{i} - 2(4)^2\underline{j} + \underline{c}$

$\underline{s} = 4\underline{i} - 32\underline{j} + \underline{c}$



$AB = \sqrt{(2)^2 + (30)^2}$ - 1

use Pythagoras' theorem to find AB.

$= \sqrt{904}$

$= 2\sqrt{226}$ m - 1

[*In this question position vectors are given relative to a fixed origin O*]

7. At time t seconds, where $t \geqslant 0$, a particle, P, moves so that its velocity $\mathbf{v}\,\mathrm{m\,s^{-1}}$ is given by

$$\mathbf{v} = 6t\mathbf{i} - 5t^{\frac{3}{2}}\mathbf{j}$$

When $t = 0$, the position vector of P is $(-20\mathbf{i} + 20\mathbf{j})\,\mathrm{m}$.

(a) Find the acceleration of P when $t = 4$

(3)

(b) Find the position vector of P when $t = 4$

(3)

a) $\underline{v} = \int \underline{a}.dt$

$\underline{a} = \frac{d\underline{v}}{dt} = 6\underline{i} - \frac{15}{2} t^{\frac{1}{2}} \underline{j}$ $-$ 2

When $t = 4$

$\underline{a} = 6\underline{i} - \frac{15}{2}(4)^{\frac{1}{2}}\underline{j}$

$\underline{a} = (6\underline{i} - 15\underline{j})\ \mathrm{ms^{-2}}$ $-$ 1

b) $v = \frac{ds}{dt}$

$\underline{s} = \int (6t\underline{i} - 5t^{\frac{3}{2}}\underline{j})dt$ $-$ 1

$\underline{s} = 3t^2\underline{i} - 2t^{\frac{5}{2}}\underline{j} + \underline{c}$ $-$ 1

when $t=0$:

$3(0)^2\underline{i} - 2(0)^{\frac{5}{2}}\underline{j} + \underline{c} = -20\underline{i} + 20\underline{j}$

$\underline{c} = -20\underline{i} + 20\underline{j}$

$\underline{s} = (3t^2 - 20)\underline{i} + (20 - 2t^{\frac{5}{2}})\underline{j}$

when $t=4$:

$\underline{s} = (3(4)^2 - 20)\underline{i} + (20 - 2(4)^{\frac{5}{2}})\underline{j}$

$\underline{s} = (28\underline{i} - 44\underline{j})\ \mathrm{m}$ $-$ 1

8. A particle, P, moves with constant acceleration $(2\mathbf{i} - 3\mathbf{j})\,\mathrm{m\,s^{-2}}$

At time $t = 0$, the particle is at the point A and is moving with velocity $(-\mathbf{i} + 4\mathbf{j})\,\mathrm{m\,s^{-1}}$

At time $t = T$ seconds, P is moving in the direction of vector $(3\mathbf{i} - 4\mathbf{j})$

$\hookrightarrow \lambda(3\underline{i} - 4\underline{j})$

$= 3\lambda\underline{i} - 4\lambda\underline{j}$

(a) Find the value of T.

(4)

At time $t = 4$ seconds, P is at the point B.

(b) Find the distance AB.

(4)

a) $\underline{a} = \frac{d\underline{v}}{dt}$

$\underline{v} = \int (2\underline{i} - 3\underline{j})\,dt$

$= 2t\underline{i} - 3t\underline{j} + \underline{c}$ – 1

when $t = 0$;

$2(0)\underline{i} - 3(0)\underline{j} + \underline{c} = -\underline{i} + 4\underline{j}$

$\underline{c} = -\underline{i} + 4\underline{j}$

$\underline{v} = (2t - 1)\underline{i} + (4 - 3t)\underline{j}$ – 1

when $t = T$;

$(2T - 1)\underline{i} + (4 - 3T)\underline{j} = 3\lambda\underline{i} - 4\lambda\underline{j}$

$2T - 1 = 3\lambda$ $\qquad$ $4 - 3T = -4\lambda$

$\lambda = \frac{2T - 1}{3}$ – 1 $\qquad$ $\lambda = \frac{-(4 - 3T)}{4}$ – 2

1 = 2

$\frac{2T - 1}{3} = \frac{-4 + 3T}{4}$ – 1

$4(2T - 1) = 3(-4 + 3T)$

$8T - 4 = -12 + 9T$

$T = 8$ – 1

b) $\underline{v} = \frac{d\underline{s}}{dt}$

$\underline{s} = \int ((2t - 1)\underline{i} + (4 - 3t)\underline{j})\,dt$

b) $\underset{\sim}{s} = (t^2 - t)\underset{\sim}{i} + (4t - \frac{3}{2}t^2)\underset{\sim}{j} + \underset{\sim}{d}$ – 1

when $t = 0$, $\underset{\sim}{s} = 0$

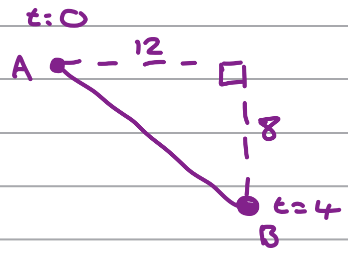
$\therefore \underset{\sim}{d} = 0$

$\underset{\sim}{s} = (t^2 - t)\underset{\sim}{i} + (4t - \frac{3}{2}t^2)\underset{\sim}{j}$ – 1

when $t = 4$,

$\underset{\sim}{s} = ((4)^2 - 4)\underset{\sim}{i} + (4(4) - \frac{3}{2}(4)^2)\underset{\sim}{j}$

$\underset{\sim}{s} = 12\underset{\sim}{i} - 8\underset{\sim}{j}$



$AB = \sqrt{12^2 + 8^2}$ – 1

$= 4\sqrt{13}$

distance $AB = (4\sqrt{13})$ m – 1

9. A particle P moves with acceleration $(4\mathbf{i} - 5\mathbf{j})\,\mathrm{m\,s^{-2}}$

At time $t = 0$, P is moving with velocity $(-2\mathbf{i} + 2\mathbf{j})\,\mathrm{m\,s^{-1}}$ → Initial Condition

(a) Find the velocity of P at time $t = 2$ seconds.

(2)

At time $t = 0$, P passes through the origin O. → Initial Condition.

At time $t = T$ seconds, where $T > 0$, the particle P passes through the point A.

The position vector of A is $(\lambda\mathbf{i} - 4.5\mathbf{j})$m relative to O, where λ is a constant.

(b) Find the value of T.

(4)

(c) Hence find the value of λ

(2)

a)

$\underline{v} = \int \underline{a}\, dt \Rightarrow \underline{v} = \int 4\underline{i} - 5\underline{j}\, dt$

$= 4t\underline{i} - 5t\underline{j} + \underline{c}$

$\underline{v}(t) = 4t\underline{i} - 5t\underline{j} + \underline{c}$

$\underline{v}(0) = \cancel{4(0)\underline{i}} - \cancel{5(0)\underline{j}} + \underline{c} = -2\underline{i} + 2\underline{j}$

$\underline{c} = -2\underline{i} + 2\underline{j}$

$\underline{v}(t) = 4t\underline{i} - 5t\underline{j} - 2\underline{i} + 2\underline{j}$ ✓

$\underline{v}(2) = 4(2)\underline{i} - 5(2)\underline{j} - 2\underline{i} + 2\underline{j}$

$= 8\underline{i} - 10\underline{j} - 2\underline{i} + 2\underline{j}$

$= (6\underline{i} - 8\underline{j})\ \text{ms}^{-1}$ ✓

b)

$\underline{v}(t) = 4t\underline{i} - 5t\underline{j} - 2\underline{i} + 2\underline{j}$

$\underline{r} = \int \underline{v}\, dt \Rightarrow \underline{r} = \int 4t\underline{i} - 5t\underline{j} - 2\underline{i} + 2\underline{j}\, dt$

$= 2t^2\underline{i} - \frac{5t^2}{2}\underline{j} - 2t\underline{i} + 2t\underline{j} + \underline{c}$

$\underline{r}(t) = 2t^2\underline{i} - \frac{5t^2}{2}\underline{j} - 2t\underline{i} + 2t\underline{j} + \underline{c}$

$\underline{r}(0) = \cancel{2(0)^2\underline{i}} - \cancel{\frac{5(0)^2}{2}\underline{j}} - \cancel{2(0)\underline{i}} + \cancel{2(0)\underline{j}} + \underline{c} = 0$

$\underline{c} = 0$

$\underline{r}(t) = 2t^2\underline{i} - \frac{5t^2}{2}\underline{j} - 2t\underline{i} + 2t\underline{j}$ ✓

$r(T) = 2T^2\underline{i} - \frac{5T^2}{2}\underline{j} - 2T\underline{i} + 2T\underline{j} = \lambda\underline{i} - 4.5\underline{j}$

Equating $\underline{j}$ component terms:

$-\frac{5T^2}{2}\underline{j} + 2T\underline{j} = -4.5\underline{j}$ ✓

$-5T^2\underline{j} + 4T\underline{j} = -9\underline{j}$

$-5T^2 + 4T + 9 = 0$ ✓

$\rightarrow T = 1.8$ or $T = -1$

$\hookrightarrow -1 < 0$ (invalid)

$\therefore T = 1.8$ ✓

c)

$T = 1.8$ $\quad \underline{r}(T) = 2T^2\underline{i} - \frac{5T^2}{2}\underline{j} - 2T\underline{i} + 2T\underline{j} = \lambda\underline{i} - 4.5\underline{j}$

Equating $\underline{i}$ component:

$$2T^2\underline{i} - 2T\underline{i} = \lambda\underline{i}$$

$$2T^2 - 2T = \lambda$$

$$2(1.8)^2 - 2(1.8) = \lambda$$

$$\lambda = 2.88$$

10. (i) At time t seconds, where $t \geqslant 0$, a particle P moves so that its acceleration $\mathbf{a}\,\text{m}\,\text{s}^{-2}$ is given by

$$\mathbf{a} = (1 - 4t)\,\mathbf{i} + (3 - t^2)\,\mathbf{j}$$

At the instant when $t = 0$, the velocity of P is $36\mathbf{i}\,\text{m}\,\text{s}^{-1}$ → Initial condition

(a) Find the velocity of P when $t = 4$

(3)

(b) Find the value of t at the instant when P is moving in a direction perpendicular to $\mathbf{i}$

→ No $\underline{i}$ component

(3)

(ii) At time t seconds, where $t \geqslant 0$, a particle Q moves so that its position vector $\mathbf{r}$ metres, relative to a fixed origin O, is given by

$$\mathbf{r} = (t^2 - t)\,\mathbf{i} + 3t\,\mathbf{j}$$

Find the value of t at the instant when the speed of Q is $5\,\text{m}\,\text{s}^{-1}$

↳ Speed = $|\underline{v}|$

(6)

i) a)

$\underline{v} = \int \underline{a}\, dt \quad \Rightarrow \quad \underline{v} = \int (1-4t)\underline{i} + (3-t^2)\underline{j}\, dt$

$= \int (1-4t)\underline{i}\, dt + \int (3-t^2)\underline{j}\, dt$

$\underline{v} = (t - 2t^2)\underline{i} + (3t - \frac{t^3}{3})\underline{j} + c$

When $t = 0$, $\underline{v} = 36\underline{i}$

$36\underline{i} = (\cancel{0 - 2(0)^2})\underline{i} + (\cancel{3(0) - \frac{0^3}{3}})\underline{j} + c$

$\therefore c = 36\underline{i}$

$\underline{v} = (t - 2t^2 + 36)\underline{i} + (3t - \frac{t^3}{3})\underline{j}$

$\underline{v}(4) = (4 - 32 + 36)\underline{i} + (12 - \frac{64}{3})\underline{j}$

$= (8\underline{i} - \frac{28}{3}\underline{j})\ \text{ms}^{-1}$

i) b)

$\underline{v} = (t - 2t^2 + 36)\underline{i} + (3t - \frac{t^3}{3})\underline{j}$

$t - 2t^2 + 36 = 0$

$2t^2 - t - 36 = 0 \rightarrow t = 4.5, \quad t = -4$

$\downarrow$

$-4 < 0$ (invalid)

$\therefore t = 4.5$

ii)

$\underline{v} = \frac{d}{dt}(\underline{r}) \Rightarrow \underline{v} = \frac{d}{dt}\left((t^2 - t)\underline{i} + 3t\underline{j}\right)$

$\underline{v} = \frac{d}{dt}\left((t^2 - t)\underline{i}\right) + \frac{d}{dt}\left((3t)\underline{j}\right)$

$\underline{v} = (2t - 1)\underline{i} + 3\underline{j}$

for a vector $a\underline{i} + b\underline{j}$, $|a\underline{i} + b\underline{j}| = \sqrt{a^2 + b^2}$

Speed $= \sqrt{(2t-1)^2 + (3)^2} = 5$

$(2t-1)^2 + (3)^2 = 25$

$(2t-1)^2 = 16$

$(4t^2 - 4t + 1) = 16 \Rightarrow 4t^2 - 4t - 15 = 0$

$t = 2.5$ or $t = -1.5$

$\downarrow$

$-1.5 < 0$

$\therefore$ invalid.

$\therefore t = 2.5$