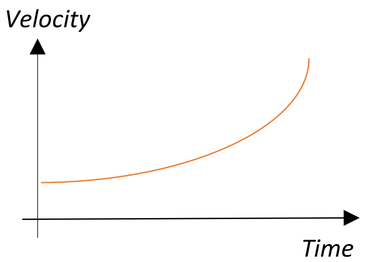
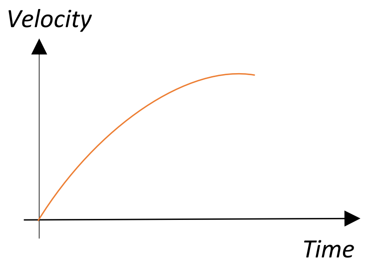
Variable acceleration

Functions of time

- If acceleration of a moving particle is variable, it changes with time and can be expressed as a function of time.
- Velocity and displacement can also be expressed as functions of time



- *Increasing acceleration*
- *The gradient of the curve increases over time*



- *Decreasing acceleration*
- *Gradient of the curve decreases over time*

Example 1: A body moves in a straight line, such that its displacement, *s* metres, from a point O at time *t* seconds is given by $s = 2t^3 - 3t$ for *t* > 0. Find:

a. *s* when *t* = 2
$s = 2 \times 2^3 - 3 \times 2$
$= 16 - 6 = 10$ metres

b. the time taken for the particle to return to O
$2t^3 - 3t = 0$
$t\,(2t^2 - 3) = 0 \quad \Rightarrow \quad$ either $t = 0$ or $2t^2 = 3$

$\Rightarrow t^2 = \frac{3}{2}$ so $t = \pm\sqrt{\frac{3}{2}}$ seconds

Only take $+\sqrt{\frac{3}{2}}$ seconds because equation is only valid for *t* > 0

Time taken to return to O $= \sqrt{\frac{3}{2}}$ seconds

Using differentiation

Velocity is the rate of change of displacement.

- If the displacement, *s*, is expressed as a function of *t*, then the velocity, *v*, can be expressed as $v = \frac{ds}{dt}$

Acceleration is the rate of change of velocity.

- If the velocity, *v*, is expressed as a function of *t*, then the acceleration, *a*, can be expressed as $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Example 2: A particle *P* is moving on the x-axis. At time *t* seconds, the displacement x metres from O is given by $x = t^4 - 32t + 12$. Find:

a. The velocity of *P* when $t = 3$
$x = t^4 - 32t + 12$

$v = \frac{dx}{dt} = 4t^3 - 32$

When $t = 3$,
$v = 4 \times 3^3 - 32 = 76$
The velocity of *P* when $t = 3$ is 76 ms^{-1} in the direction of x increasing.

b. The value of *t* for which *P* is instantaneously at rest
$v = 4t^3 - 32 = 0$

$t^3 = \frac{32}{4} = 8$

$t = 2$

c. The acceleration of *P* when $t = 1.5$.
$v = 4t^3 - 32$

$a = \frac{dv}{dt} = 12t^2$

When $t = 1.5$,
$a = 12 \times (1.5)^2 = 27$
The acceleration of *P* when $t = 1.5$ is 27 ms^{-2}.

Maxima and minima problems

You can use calculus to determine maximum and minimum values of displacement, velocity and acceleration.

Example 3: A child is playing with a yo-yo. The yo-yo leaves the child's hand at time t=0 and travels vertically in a straight line before returning to the child's hand. The distance, *s* m, of the yo-yo from the child's hand after time t seconds is given by:

$$s = 0.6t + 0.4t^2 - 0.2t^3, \quad 0 \le t \le 3$$

Find the maximum distance of the yo-yo from the child's hand, correct to 3 s.f. *(Note that maximum value always occur at turning point, where* $\frac{ds}{dt} = 0$*).*

$\frac{ds}{dt} = 0.6 + 0.8t - 0.6t^2 \qquad \Rightarrow \frac{ds}{dt} = 0$

$0.6 + 0.8t - 0.6t^2 = 0$
$3t^2 - 4t - 3 = 0$

$t = \frac{4 \pm \sqrt{52}}{6} = 1.8685 \;\; or \;\; -0.5351$

Only take the positive value of t

$s = 0.6\,(1.8685) + 0.4(1.8685)^2 - 0.2(1.8685)^3 = 1.21$ m (3 s.f.)

Using integration

You can integrate acceleration with respect to time to find velocity, and you can integrate velocity with respect to time to find displacement.

Example 4: A particle is moving on the x-axis. At time t=0, the particle is at the point where x = 5. The velocity of the particle at time *t* seconds (where $t \ge 0$) is $(6t - t^2)$ ms^{-1}. Find:

a. An expression for the displacement of the particle from O at time *t* seconds
$x = \int v \; dt$

$= 3t^2 - \frac{t^3}{3} + c.$ where c is a constant of integration.

When $t = 0, x = 5$

$5 = 3 \times 0^2 - \frac{(0)^3}{3} + c. \qquad \Longrightarrow c = 5$

The displacement of the particle from O after t seconds is $\left(3t^2 - \frac{t^3}{3} + 5\right)$ m

b. The distance of the particle from its starting point when *t* = 6.

When *t* = 6

$\Longrightarrow 3 \times 6^2 - \frac{(6)^3}{3} + 5 = 41$

The distance from the starting point is $(41 - 5)$ m $= 36$ m.

Constant acceleration formulae

You can use calculus to derive the formulae for motion with constant acceleration.

Example 5: A particle moves in a straight line with constant acceleration, a ms^{-2}. Given that its initial velocity is u ms^{-1} and its initial displacement is 0 m, prove that its velocity, v ms^{-1} at time *t* seconds is given by $v = u + at$

$v = \int a \; dt$

$= at + c$

When $t = 0, v = u,$

So $u = a \times 0 + c$

$u = c$

Hence $v = u + at$