1. A raindrop falls vertically under gravity through a cloud. In a model of the motion the raindrop is assumed to be spherical at all times and the cloud is assumed to consist of stationary water particles. At time $t = 0$, the raindrop is at rest and has radius a. As the raindrop falls, water particles from the cloud condense onto it and the radius of the raindrop is assumed to increase at a constant rate λ. A time t the speed of the raindrop is v.

(a) Show that

$$\frac{dv}{dt} + \frac{3\lambda v}{(\lambda t + a)} = g.$$

(8)

(b) Find the speed of the raindrop when its radius is $3a$.

(7)
(Total 15 marks)

2. A spaceship is moving in a straight line in deep space and needs to increase its speed. This is done by ejecting fuel backwards from the spaceship at a constant speed c relative to the spaceship. When the speed of the spaceship is v, its mass is m.

(a) Show that, while the spaceship is ejecting fuel,

$$\frac{dv}{dm} = -\frac{c}{m}.$$

(5)

The initial mass of the spaceship is m_0 and at time t the mass of the spaceship is given by $m = m_0 (1 - kt)$, where k is a positive constant.

(b) Find the acceleration of the spaceship at time t.

(4)
(Total 9 marks)

3. At time $t = 0$ a rocket is launched from rest vertically upwards. The rocket propels itself upwards by expelling burnt fuel vertically downwards with constant speed U m s^{-1} relative to the rocket. The initial mass of the rocket is M_0 kg. At time t seconds, where $t < 2$, its mass is $M_0(1 - \frac{1}{2}t)$ kg, and it is moving upwards with speed v m s^{-1}.

(a) Show that

$$\frac{dv}{dt} = \frac{U}{(2-t)} - 9.8$$

(7)

(b) Hence show that $U > 19.6$

(2)

(c) Find, in terms of U, the speed of the rocket one second after its launch.

(5)

(Total 14 marks)

4. A motor boat of mass M is moving in a straight line, with its engine switched off, across a stretch of still water. The boat is moving with speed U when, at time $t = 0$, it develops a leak. The water comes in at a constant rate so that at time t, the mass of water in the boat is λt. At time t the speed of the boat is v and it experiences a total resistance to motion of magnitude $2\lambda v$.

(a) Show that.

$$(M + \lambda t)\frac{dv}{dt} + 3\lambda v = 0.$$

(6)

(b) Show that the time taken for the speed of the boat to reduce to $\frac{1}{2}U$ is $\frac{M}{\lambda}\left(2^{\frac{1}{3}} - 1\right)$.

(6)

The boat sinks when the mass of water inside the boat is M.

(c) Show that the boat does not sink before the speed of the boat is $\frac{1}{2}U$.

(2)
(Total 14 marks)

5. A space-ship is moving in a straight line in deep space and needs to reduce its speed from *U* to *V*. This is done by ejecting fuel from the front of the space-ship at a constant speed *k* relative to the space-ship. When the speed of the space-ship is *v* its mass is *m*.

(a) Show that, while the space-ship is ejecting fuel, $\frac{\mathrm{d}m}{\mathrm{d}v} = \frac{m}{k}$.

(6)

The initial mass of the space-ship is *M*.

(b) Find, in terms of *U*, *V*, *k* and *M*, the amount of fuel which needs to be used to reduce the speed of the space-ship from *U* to *V*.

(6)
(Total 12 marks)

6. At time $t = 0$, a small body is projected vertically upwards. While ascending it picks up small drops of moisture from the atmosphere. The drops of moisture are at rest before they are picked up. At time *t*, the combined body *P* has mass *m* and speed *v*.

(a) Show that, while *P* is moving upwards, $m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$.

(5)

The initial mass of *P* is *M*, and m = Me^{kt}, where *k* is a positive constant.

(b) Show that, while *P* is moving upwards, $\frac{\mathrm{d}}{\mathrm{d}t}\left(ve^{kt}\right) = -g\mathrm{e}^{kt}$.

(3)

Given that the initial projection speed of *P* is $\frac{g}{2k}$,

(c) find, in terms of *M*, the mass of *P* when it reaches its highest point.

(7)
(Total 15 marks)

7. A rocket-driven car moves along a straight horizontal road. The car has total initial mass M. It propels itself forwards by ejecting mass backwards at a constant rate λ per unit time at a constant speed U relative to the car. The car starts from rest at time $t = 0$. At time t the speed of the car is v. The total resistance to motion is modelled as having magnitude kv, where k is a constant.

Given that $t < \dfrac{M}{\lambda}$, show that

(a) $$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda U - kv}{M - \lambda t},$$

(7)

(b) $$v = \frac{\lambda U}{k}\left\{1 - \left(1 - \frac{\lambda t}{M}\right)^{\frac{k}{\lambda}}\right\}.$$

(6)
(Total 13 marks)

8. A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upward by ejecting burnt fuel vertically downwards at a constant speed u relative to the rocket. The initial mass of the rocket, including fuel, is M. At time t, before all the fuel has been used up, the mass of the rocket, including fuel, is $M(1 - kt)$ and the speed of the rocket is v.

(a) Show that $\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{ku}{1 - kt} - g$.

(7)

(b) Hence find the speed of the rocket when $t = \dfrac{1}{3k}$.

(3)
(Total 10 marks)

9. A rocket-driven car propels itself forwards in a straight line on a horizontal track by ejecting burnt fuel backwards at a constant rate λ kg s^{-1} and at a constant speed U m s^{-1} relative to the car. At time t seconds, the speed of the car is v m s^{-1} and the total resistance to the motion of the car has magnitude kv N, where k is a positive constant. When $t = 0$ the total mass of the car, including fuel, is M kg. Assuming that at time t seconds some fuel remains in the car,

(a) show that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda U - kv}{M - \lambda t},$$

(7)

(b) find the speed of the car at time t seconds, given that it starts from rest when $t = 0$ and that $\lambda = k = 10$.

(6)

(Total 13 marks)

10. A spaceship is moving in deep space with no external forces acting on it. Initially it has total mass M and is moving with speed V. The spaceship reduces its speed to $\frac{2}{3}V$ by ejecting fuel from its front end with a speed of c relative to itself and in the same direction as its own motion.

Find the mass of fuel ejected.

(Total 11 marks)

1.	(a)	$\frac{\mathrm{d}r}{\mathrm{d}t} = \lambda \Rightarrow r = \lambda t + a$	B1	
		$(m+\delta m)(v+\delta v) - mv = mg\delta t$	M1 A1	
		$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = g$	DM1 A1	
		$\frac{\mathrm{d}m}{\mathrm{d}t} = 4\pi r^2(\rho)\lambda$	A1 B1	
		$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3v}{4\pi r^3 \rho} 4\pi r^2 \rho\lambda = g \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3v\lambda}{r} = g$	DM1	
		$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{3v\lambda}{\lambda t + a} = g$ *	A1	8
	(b)	$R = \mathrm{e}^{\int \frac{3\lambda}{\lambda t + a}\mathrm{d}t} = \mathrm{e}^{3\ln(\lambda t + a)} = \mathrm{e}^{\ln(\lambda t + a)^3} = (\lambda t + a)^3$	M1 A1	
		$v(\lambda t + a)^3 = g\int (\lambda t + a)^3 \mathrm{d}t$	DM1	
		$v(\lambda t + a)^3 = \frac{1}{4\lambda} g(\lambda t + a)^4$	A1	
		$t = 0, v = 0 \Rightarrow C = -\frac{1}{4\lambda} ga^4$	DM1	
		$\lambda t + a = 3a$	DM1	
		$v = \frac{1}{4\lambda} g(3a) - \frac{1}{4\lambda}\frac{ga^4}{27a^3} = \frac{20ag}{27\lambda}$	A1	7

[15]

2.	(a)	$mv = (m+\delta m)(v+\delta v) - (-\delta m)(c - v)$	M1 A2	
		$mv = mv + m\delta v + v\delta m + c\delta m - v\delta m$		
		$-m\delta v = c\delta m$		
		$\frac{\mathrm{d}v}{\mathrm{d}m} = -\frac{c}{m}$ *	DM1 A1	5
	(b)	$\frac{\mathrm{d}m}{\mathrm{d}t} = -m_0 k$	B1	
		$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}m} \times \frac{\mathrm{d}m}{\mathrm{d}t}$	M1	
		$= -\frac{c}{m} \times -m_0 k$		
		$= -\frac{cm_0 k}{m_0(1-kt)}$	DM1	
		$= \frac{ck}{(1-kt)}$	A1	4

[9]

3.	(a)	$-mg\ \delta t = (m + \delta m)(v + \delta v) + \delta m(U - v) - mv$		
		$-mg\ \delta t = mv + m\delta v + v\delta m + U\ \delta m - v\delta m - mv$	M1 A2	
		$-mg = m\dfrac{dv}{dt} + U\dfrac{dm}{dt}$	A1	
		$m = M_0(1 - \frac{1}{2}t) \Rightarrow \dfrac{dm}{dt} = -\frac{1}{2}M_0$	B1	
		$-M_0 g(1 - \frac{1}{2}t) = M_0(1 - \frac{1}{2}t)\dfrac{dv}{dt} - \frac{1}{2}M_0 U$	M1	
		$U - g(2 - t) = (2 - t)\dfrac{dv}{dt}$		
		$\dfrac{U}{(2-t)} - 9.8 = \dfrac{dv}{dt}$*	A1	7
	(b)	$\dfrac{dv}{dt} > 0$ when $t = 0 \Rightarrow \dfrac{U}{2} - 9.8 > 0$	M1	
		$\Rightarrow U > 19.6$*	A1	2
	(c)	$v = \displaystyle\int \frac{U}{(2-t)} - 9.8\,dt$	M1	
		$= -U\ln(2 - t) - 9.8t + C$	A1	
		$t = 0, v = 0$: $0 = -U\ln 2 + C \Rightarrow C = U\ln 2$	M1	
		so, $v = U\ln\dfrac{2}{(2-t)} - 9.8t$		
		$t = 1$: $v = U\ln 2 - 9.8$	M1A1	5

[14]

4.	(a)	$(m + \delta m)(v + \delta v) - mv = -2\lambda v\delta t$	M1A1	
		$m\dfrac{dv}{dt} + v\dfrac{dm}{dt} = -2\lambda v$		
		$\dfrac{dm}{dt} = \lambda;\ m = M + \lambda t$	B1; B1	
		$(M + \lambda t)\dfrac{dv}{dt} + 3\lambda v = 0$*	DM1A1	6

(b) $-\int \frac{\mathrm{d}v}{3\lambda v} = \int \frac{\mathrm{d}t}{(M+\lambda t)}$ M1

$-\frac{1}{3\lambda}\left[\ln v\right]_u^{\frac{1}{2}u} = \frac{1}{\lambda}\left[\ln(M+\lambda t)\right]_0^T$ DM1A1

$\frac{1}{3}\ln 2 = \ln\frac{(M+\lambda T)}{M}$ DM1

$T = \frac{M}{\lambda}(2^{\frac{1}{3}} - 1)*$ DM1A1 6

(c) Sinks at $T_S = \frac{M}{\lambda}$

Reaches speed $\frac{1}{2}U$ at $T = \frac{M}{\lambda}(2^{\frac{1}{3}} - 1)$

Since $(2^{\frac{1}{3}} - 1) < 1$, $T < T_S$ M1

I.e. Reaches speed $\frac{1}{2}U$ before it sinks A1cso 2

[14]

5. (a) $mv \cong (m+\partial m)(v+\partial v) + (-\partial m)(\mathrm{K}+v+\partial v)$ M1 A3

$\cancel{mv} \cong \cancel{mv} + m\delta v + v\cancel{\delta m} - \mathrm{K}\delta m - v\cancel{\delta m}$ M1

$\mathrm{K}\delta m \simeq m\delta v$

In the limit, as $\partial t \to 0$,

$\frac{dm}{dv} = \frac{m}{\mathrm{K}}$ A1 6

(b) $\int_M^{m_1} \frac{dm}{m} = \int_U^V \frac{dv}{K}$ M1

$\ln m_1 - \ln M = \frac{1}{K}(V - U)$ A1

$\ln \frac{m_1}{M} = \frac{1}{K}(V - U)$ M1

$m_1 = Me^{\left(\frac{V-U}{K}\right)}$ A1

Amount of fuel = $M - m_1 = M\left(1 - e^{\left(\frac{V-U}{K}\right)}\right)$ M1 A1 6

[12]

6. O↑ δm ↑$v + \delta v$

v↑ m $m + \delta m$

(a) $(m + \delta m)(v + \delta v) - mv = -mg\delta t$ M1 A2, 1, 0

$\cancel{mv} + m\delta v + v\delta m + \cancel{\delta m \delta v} - \cancel{mv} = -mg\delta t$

$m\frac{dv}{dt} + v\frac{dm}{dt} = -mg$ M1 A1 5

(b) $m = Me^{kt}$ $\Rightarrow$ $\frac{dm}{dt} = kMe^{kt}$ B1

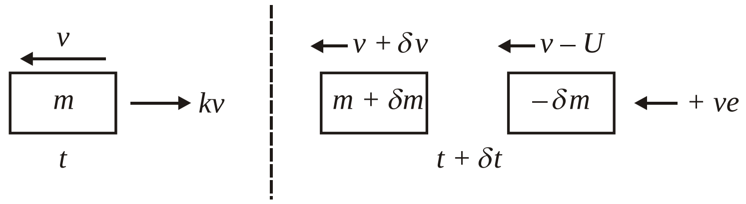
Hence $Me^{kt}\frac{dv}{dt} + kv \,.\, kMe^{kt} = -Me^{kt}g$ M1

$\Rightarrow$ $\underline{\frac{d}{dt}(ve^{kt}) = -ge^{kt}}$ A1 3

	(c)	$ve^{kt} = -g\int e^{kt}\,\mathrm{d}t$	M1	
		$= -\dfrac{g}{k}e^{kt}$ (+c)	A1	
		$t = o,\ v = \dfrac{g}{2k} \Rightarrow c = \dfrac{3g}{2k} = 0$	M1	
		$\Rightarrow e^{kt} = \dfrac{3}{2}$	A1	
		Hence $m = \mathrm{M}e^{kt} = \dfrac{3\mathrm{M}}{2}$	M1 A1	7

[15]

7. (a)

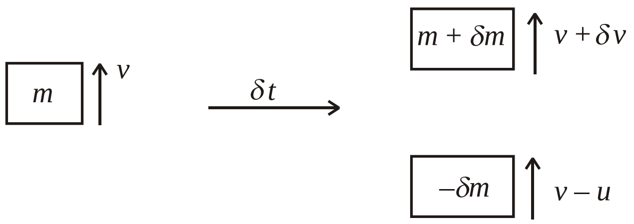


		Impulse – momentum:		
		$-kv\delta t = (m + \delta m)(v + \delta v) + (-\delta m)(v - \mu) - mv$	M1 A3	
		$-kv\delta t = mv + m\delta v + \delta mv - \delta m\mu + \delta m\mu - mv$		
		$-kv = m\dfrac{\delta v}{\delta t} + \mu\dfrac{\delta m}{\delta t}$		
		limit as $\delta t \to 0$: $-kv = m\dfrac{\mathrm{d}v}{\mathrm{d}t} - \mu\dfrac{\mathrm{d}m}{\mathrm{d}t}$		
		$\dfrac{\mathrm{d}m}{\mathrm{d}t} = -\lambda \Leftrightarrow m = M - \lambda t$	B1	
		$-kv = (M - \lambda)\dfrac{\mathrm{d}v}{\mathrm{d}t} - \lambda\mu$	M1	
		$\underline{\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\lambda\mu - kv}{M - \lambda t}}$ (*)	A1	7

(b)	$\int_0^t \frac{\mathrm{d}t}{M-\lambda t} = \int_0^v \frac{\mathrm{d}v}{\lambda\mu - kv}$	M1 A1	
	$-\frac{1}{\lambda}[\ln(M-\lambda t)]_0^t = 1\frac{1}{k}[\ln(\lambda\mu - kv)]_0^{\mu}$	A1	
	$\frac{k}{\lambda}(\ln(M-\lambda t) - \ln M) = \ln(\lambda\mu - kv) - \ln\lambda\mu$	M1	
	$\left(\frac{M-\lambda t}{M}\right)^{k/\lambda} = \frac{\lambda\mu - kv}{\lambda\mu}$	M1	
	$v = \frac{\lambda\mu}{k}\left[1-(1-\frac{\lambda t}{M})^{k/\lambda}\right]$ (*)	A1	6

[13]

8. (a)



	$(m + \delta m)(v + \delta m) + (-\delta m)(v - u) - mv = -mg\delta t$	M1 A2 (–1ee)	
	$mv + m\delta v + v\delta m - v\delta m + u\delta m - mv = -mg\delta t$		
	$m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$	A1	
	$m = M(1-kt) \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = -kM$	B1	
	$M(1-kt)\frac{\mathrm{d}v}{\mathrm{d}t} + u(-kM) = -M(1-kt)g$	M1	
	$\underline{\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{ku}{1-kt} - g}$ (*)	A1	7
(b)	$v = \int_0^{1/3k} \frac{ku}{1-kt} - g\,\mathrm{d}t$	M1	
	$= \left[-u\ln(1-kt) - gt\right]_0^{1/3k}$	A1	
	$= \underline{u\ln\left(\frac{3}{2}\right) - \frac{g}{3k}}$	A1	3

[10]

9. (a) $(m + \delta m)(v + \delta v) + (-\delta m)(v - U) - \text{mv} = -kv\delta$ M1 A1 A1

$\Rightarrow\ m\dfrac{\text{d}v}{\text{d}t} + U\dfrac{\text{d}m}{\text{d}t} = -kv$ A1

$m = M - \lambda t$ B1

$\Rightarrow (M - \lambda t)\dfrac{\text{d}v}{\text{d}t} = \lambda U - kv$ M1

$\Rightarrow\ \dfrac{\text{d}v}{\text{d}t} = \dfrac{\lambda U - kv}{M - \lambda t}$ (*) (no incorrect working seen) A1 cso 7

(b) Separating variables: $\displaystyle\int \frac{\text{d}v}{U - v} = \int \frac{10}{M - 10t}\text{d}t$ or equivalent M1

Integrating: $\ln(U - v) = \ln(M - 10t)\ (+ c)$ M1 A1

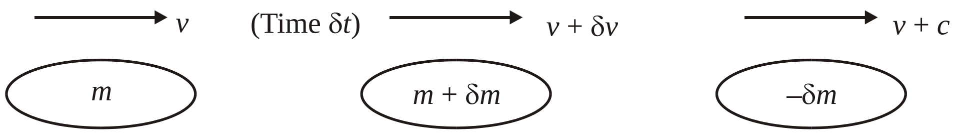
Using limits correctly: $[\ ]_v^0 = [\ ]_t^0$ applied or $t = 0$, $v = 0$ to find "c" M1

$[c = \ln\left(\dfrac{U}{M}\right)]$

Complete method to find v $[\ \ln\left(\dfrac{U}{U - v}\right) = \ln\left(\dfrac{M}{M - 10t}\right)]$ M1

$v = \dfrac{10Ut}{M}$ A1 6

[13]

10. 

$(m + \delta m)(v + \delta v) + (-\delta m)(v + c) - mv = 0$ M1 A2, 1, 0

$\Rightarrow m\,\text{d}v - c\,\text{d}m = 0$

reducing equation M1

$\displaystyle\int_V^{\frac{2}{3}V} \text{d}v = c\int_M^m \frac{\text{d}m}{m}$

separating variables M1

$$\left[\, v \,\right]_{V}^{\frac{2}{3}V} = c\left[\, \ln m \,\right]_{M}^{m}$$

integration M1 A1

$$-\tfrac{1}{3}v = c\ln\left(\frac{m}{M}\right)$$

applying limits M1

$$\Rightarrow m = Me^{-\frac{V}{3c}}$$

eliminating ln and m = M1

$\therefore$ Fuel used = $M - m = M(1 - e^{-\frac{V}{3c}})$ M1 A1 11

[11]

1. Many candidates knew exactly what to do here and there were a good number of fully correct solutions.

Part (a) required a standard technique which most candidates followed successfully to the given answer. It was pleasing to see the steps required were clearly set out. A few were unable to make a correct start and then attempted to fudge the given answer.

Many completely correct solutions to part (b) were seen. Most were able to find the correct Integrating Factor, solve the differential equation correctly, remember the constant and use initial conditions to find v correctly. Some candidates attempted to put in the initial conditions at the start before attempting to solve their differential equation, a costly error, as this lost them all of the marks in this part of the question.

2. The majority of candidates recognise that they must start variable mass questions by considering the change in momentum over a small time interval (in this question the change in momentum was zero!). They must put the total mass at the beginning of the interval as m and that of the rocket at the end of the interval as $m + \delta m$ (in the case of rocket questions, δm is negative). Candidates who simply try to memorise the solution to one version of this question and then try to adapt it are almost invariably unsuccessful.

Although a significant number struggled with the first part, there was more success with part (b). The easiest method was to use the chain rule to split up dv/dt, but some candidates successfully integrated the result from part (a) by separating the variables and then differentiating the result with respect to t.

3. In order to gain all of the marks in part (a), it was necessary to consider the change in momentum in a time δt. Some candidates obviously consider this to be a standard piece of bookwork, whereas others are clearly trying to work it out in the exam, in some cases considering M_0 to be the mass at a general time t. Candidates trying to use the given answer to correct their signs should beware of altering the wrong ones!

In part (b), many candidates gave the very simple explanation involving the acceleration at time $t = 0$ being positive. Others considered the general expression for acceleration and struggled to produce the given inequality. Most candidates were successful with part (c).

4. Candidates who were successful in part (a) could either consider the change of momentum in a time δt or else use Newton's Second Law and consider the rate of change of momentum i.e. $\frac{\mathrm{d}(mv)}{\mathrm{d}t}$. Some candidates tried to do this part from memory, often apparently considering the alternative problem of a rocket ejecting fuel. Other difficulties included taking $\frac{\mathrm{d}m}{\mathrm{d}t}$ as λt or else putting U in the general equation for v and t.
Part (b) was usually successfully done, mostly by separating the variables. Many candidates argued part (c) correctly, but others compared appropriate values of t or m or v but did not then state any conclusion.

5. Candidates who attempted part (a) of this question from first principles were often successful, with the possible exception of their signs. Those who tried to fit it to a solution that they had met before had problems, often caused by the inclusion of an impulse term which should not have been there. Many candidates gained four marks for the second part of the question, starting from the printed answer but missing out the subtraction of the final mass from M. Some however worked with the answer that they had obtained for part (a). Others got confused over which speed went with which time and often introduced a speed of zero.

6. This was found to be reasonable straightforward as a question on variable mass. The derivation of the given differential equation was generally good with nearly all correctly considering a small time interval appropriately. Parts (b) and (c) were also generally well done. In most cases, the question proved to be a good source of marks.

7. In part (a), many candidates proved the result correctly, deriving the result from first principles as required. Those who did not either omitted terms or confused signs, writing δm instead of ($-\delta m$) and $\mathrm{d}m/\mathrm{d}t = \lambda$ (but then writing $m = M - \lambda t$ since that was in the given answer) In part (b), since the differential equation was given, candidates were able to progress and the standard of integration was pleasing, with many correct solutions seen.

8. This was for the most part done extremely well. It was very pleasing to see the derivation of the equation of motion done from first principles accurately (with the 'δm' term having the correct sign). Also the solution of the differential equation in part (b) was also generally done fully accurately.

9. The derivation of the differential equation was often very good, with the majority of candidates working, often successfully, from first principles. In part (b) which was well answered by many of the candidates who had a correct strategy, a significant number of candidates made it much harder for themselves by not substituting for λ and k immediately. Some candidates, too, used an integrating factor approach to solve the differential equation; in this case leaving λ and k made it quite a challenge but it was good to see some succeed.

10. No Report available for this question.