Solutionbank M5
Edexcel AS and A Level Modular Mathematics

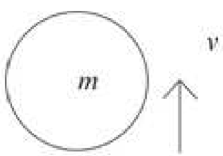
Exercise A, Question 1

Question:

A rocket is launched vertically upwards under gravity from rest at time $t = 0$. The rocket propels itself upwards by ejecting burnt fuel vertically downwards at a constant speed u relative to the rocket. At time t seconds after the launch the rocket has velocity v and mass $(M - kt)$. Derive the equation of motion for the rocket. Ignore air resistance.

Solution:

At time t



After an interval δt :

Change in momentum: $(m + \delta m)(v + \delta v) + (-\delta m)(v - u) - mv = -mg\delta t$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} + u\frac{\mathrm{d}m}{\mathrm{d}t} = -mg$$

$$m = M - kt \Rightarrow \frac{\mathrm{d}m}{\mathrm{d}t} = -k, (M - kt)\frac{\mathrm{d}v}{\mathrm{d}t} - ku = -(M - kt)g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{ku}{M - kt} - g$$

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Exercise A, Question 2

Question:

A spaceship is moving in deep space with no external forces acting on it. At time t the spaceship has total mass m and is moving with velocity v. The spaceship reduces its speed by ejecting fuel from its front end with a speed c relative to itself and in the same direction as its own motion.

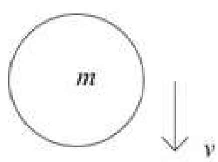
a Show that $\frac{\mathrm{d}v}{\mathrm{d}m} = \frac{c}{m}$.

Initially the spaceship is moving with speed V and has total mass M. Its speed is reduced to $\frac{1}{2}V$.

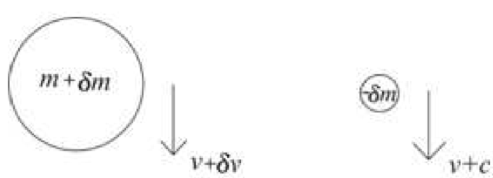
b Find the mass of fuel ejected.

Solution:

a At time t



After interval δt



Change in momentum $\Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v+c)-mv = 0$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m-c\delta m-mv = 0$$

$$m\delta v+\delta m\delta v-c\delta m = 0$$

$$\Rightarrow m\frac{\delta v}{\delta m}+\delta v-c = 0, \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}m} = \frac{c}{m}$$

b $\frac{\mathrm{d}v}{\mathrm{d}m} = \frac{c}{m} \Rightarrow \int_V^{\frac{V}{2}} 1\,\mathrm{d}v = c\int_M^m \frac{1}{m}\mathrm{d}m$

$$-\frac{V}{2} = c\left[\ln m\right]_M^m = c\ln\left(\frac{m}{M}\right)$$

$$\Rightarrow -\frac{V}{2c} = \ln\left(\frac{m}{M}\right), e^{-\frac{V}{2c}} = \frac{m}{M}, m = Me^{-\frac{V}{2c}}$$

Mass of fuel ejected $= M\left(1-e^{-\frac{V}{2c}}\right)$

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Exercise A, Question 3

Question:

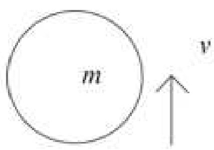
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1000 kg. The rocket burns fuel at the rate of 20 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after 30 seconds. At time t seconds $(t < 30)$ after the launch, the speed of the rocket is $v\text{ m s}^{-1}$. Air resistance may be assumed to be negligible.

a Show that $-g(50-t) = (50-t)\frac{\mathrm{d}v}{\mathrm{d}t} - 2000$.

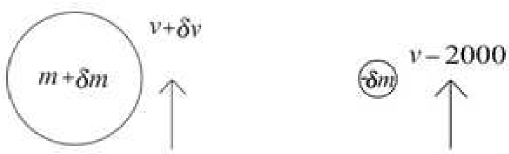
b Find the speed of the rocket when the burning stops.

Solution:

a At time t



After interval δt



Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m=1000-20t$

$$\Rightarrow (1000-20t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times-20=-(1000-20t)g$$

Dividing by $20 \Rightarrow (50-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(50-t)$

b $\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{50-t}$

$$\Rightarrow \int_0^V 1\,\mathrm{d}v=\int_0^{30} -g+\frac{2000}{50-t}\,\mathrm{d}t$$

$$V=[-gt-2000\ln(50-t)]_0^{30}$$

$$=-30g-2000\ln 20+0+2000\ln 50\approx 1540\text{ m s}^{-1}.$$

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Exercise A, Question 4

Question:

A rocket is launched vertically upwards from rest. The rocket expels burnt fuel vertically downwards with speed u relative to the rocket. Initially the rocket has mass M. At time t the rocket has speed v and mass $M\left(1-\frac{1}{3}t\right)$. Ignore air resistance.

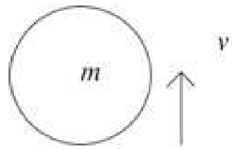
a Show that $\frac{dv}{dt} = \frac{u}{3-t} - g$.

b Find the speed of the rocket when $t = 1$.

c Find the height of the rocket above the launch site when $t = 1$.

Solution:

a At time t



After interval δt

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv=-mg\,\delta t$$

$$m\delta v+\delta m\delta v+u\delta m \ =-mg\,\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} \ =-mg$$

$$m \ = M\left(1-\frac{1}{3}t\right)$$

$$\Rightarrow M\left(1-\frac{1}{3}t\right)\frac{\mathrm{d}v}{\mathrm{d}t}+u\left(-\frac{1}{3}M\right) \ =-M\left(1-\frac{1}{3}t\right)g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\frac{1}{3}u}{1-\frac{1}{3}t}-g \ =\frac{u}{3-t}-g$$

b $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{u}{3-t}-g\Rightarrow v=\int\frac{u}{3-t}-g\,\mathrm{d}t=-u\ln|3-t|-gt+C$

$t=0, v=0\Rightarrow 0=-u\ln 3+C; v=u\ln\left|\frac{3}{3-t}\right|-gt=u\ln\frac{3}{2}-g$ when $t=1$

c $v=\frac{\mathrm{d}x}{\mathrm{d}t}=u\ln\left(\frac{3}{3-t}\right)-gt, t<3$

Using integration by parts

$$\Rightarrow x=\int u\ln\left(\frac{3}{3-t}\right)-gt\,\mathrm{d}t=\int u\ln 3-u\ln(3-t)-gt\,\mathrm{d}t$$

$$=(u\ln 3)t+u(3-t)\ln(3-t)-u(3-t)-\frac{1}{2}gt^2+C$$

$$t=0, x=0\Rightarrow 0 \ =0+3u\ln 3-3u+C, C=3u-3u\ln 3$$

When $t=1, x=u\ln 3+2u\ln 2-2u-\frac{g}{2}+3u-3u\ln 3=2u\ln\frac{2}{3}+u-\frac{g}{2}$

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Exercise A, Question 5

Question:

A spherical hailstone is falling under gravity in still air. At time t the hailstone has speed v. The radius r increases by condensation. Given that $\frac{dr}{dt} = kr$, where k is a constant, and neglecting air resistance,

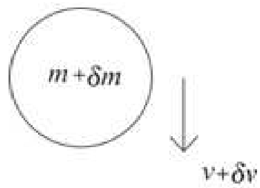
a show that $\frac{dv}{dt} = g - 3kv$,

b find the time taken for the speed of the hailstone to increase from $\frac{g}{9k}$ to $\frac{g}{6k}$.

Solution:

a At time t

After time δt :



$$\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=(m+\delta m)g\delta t$$

$$\Rightarrow m\frac{\delta v}{\delta t}+v\frac{\delta m}{\delta t}+\frac{\delta m\delta v}{\delta t}\ =mg+g\delta m$$

$$\text{so } m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}\ =mg$$

The mass of the hailstone is $\lambda\times\frac{4}{3}\pi r^3$

$$\Rightarrow\frac{\mathrm{d}m}{\mathrm{d}t}=4\lambda\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}=4\lambda\pi r^2\times kr=4k\lambda\pi r^3$$

$$\Rightarrow\lambda\times\frac{4}{3}\pi r^3\frac{\mathrm{d}v}{\mathrm{d}t}+v\times 4k\lambda\pi r^3\ =\lambda\times\frac{4}{3}\pi r^3 g$$

And therefore $\frac{\mathrm{d}v}{\mathrm{d}t}=g-3kv$

b $$\frac{\mathrm{d}v}{\mathrm{d}t}=g-3kv\Rightarrow t=\int_{\frac{g}{9k}}^{\frac{g}{6k}}\frac{1}{g-3kv}\mathrm{d}v=\left[-\frac{1}{3k}\ln\left|g-3kv\right|\right]_{\frac{g}{9k}}^{\frac{g}{6k}}$$

$$=-\frac{1}{3k}\ln\left(\frac{g-\frac{3kg}{6k}}{g-\frac{3kg}{9k}}\right)=-\frac{1}{3k}\ln\frac{g\times 3}{2\times 2g}=\frac{1}{3k}\ln\frac{4}{3}$$

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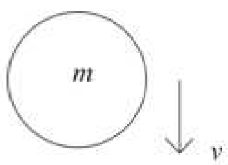
Exercise A, Question 6

Question:

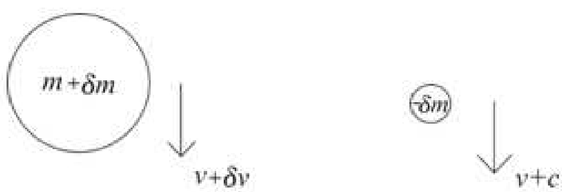
A spaceship is moving in deep space with no external forces acting on it. Initially it has total mass M and is moving with velocity V. The spaceship reduces its speed to $\frac{3}{5}V$ by ejecting fuel from its front end with a speed c relative to itself and in the same direction as its own motion. Find the mass of fuel ejected.

Solution:

At time t



After interval δt



Change in momentum $\Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v+c)-mv=0$

$$mv+m\delta v+v\delta m+\delta m\delta v-v\delta m-c\,\delta m-mv=0$$

$$m\delta v+\delta m\delta v-c\,\delta m=0$$

$$\Rightarrow m\frac{\delta v}{\delta m}+\delta v-c=0, \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}m}=\frac{c}{m}$$

Speed reduced from V to $\frac{3}{5}V$: $\int_V^{\frac{3V}{5}} 1\,\mathrm{d}v = c\int_M^m \frac{1}{m}\mathrm{d}m$

$$-\frac{2V}{5}=c\,[\ln m]_M^m = c\ln\left(\frac{m}{M}\right)$$

$$\Rightarrow -\frac{2V}{5c}=\ln\left(\frac{m}{M}\right), c^{-\frac{2V}{5c}}=\frac{m}{M}, m=Me^{-\frac{2V}{5c}}$$

Mass of fuel ejected $= M\left(1-e^{-\frac{2V}{5c}}\right)$

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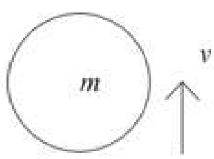
Exercise A, Question 7

Question:

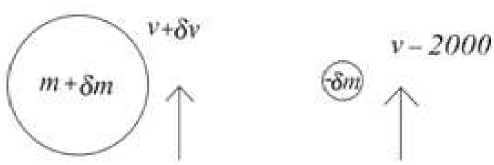
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1500 kg. The rocket burns fuel at the rate of 15 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after 60 seconds. Air resistance may be assumed to be negligible.
Find the speed of the rocket when the burning stops.

Solution:

At time t



After interval δt



Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m=1500-15t$

$$\Rightarrow (1500-15t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times -15=-(1500-15t)g$$

Dividing by $15 \Rightarrow (100-t)\frac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(100-t)$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{100-t}$$

$$\Rightarrow \int_0^V 1\,\mathrm{d}v=\int_0^{60}-g+\frac{2000}{100-t}\,\mathrm{d}t$$

$$V=\left[-gt-2000\ln(100-t)\right]_0^{60}$$

$$=-60g-2000\ln 40+0+2000\ln 100\approx 1240\text{ m s}^{-1}.$$

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Exercise A, Question 8

Question:

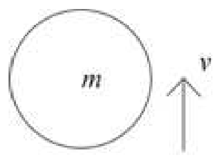
A rocket is launched vertically upwards from rest. The initial mass of the rocket and its fuel is 1200 kg. The rocket burns fuel at the rate of 24 kg s^{-1}. The burnt fuel is ejected vertically downwards with a speed of 2000 m s^{-1} relative to the rocket, and burning stops after 30 seconds. Air resistance may be assumed to be negligible.

a Find the speed of the rocket when the burning stops.

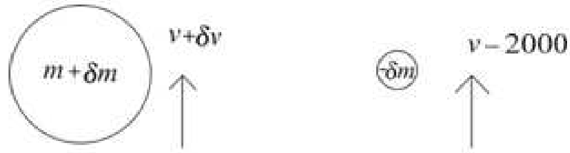
b Find the height of the rocket above the launch pad when the burning stops.

Solution:

a At time t



After interval δt



Considering the change in momentum:

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=-mg\delta t$$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

At time $t, m=1200-24t$

$$\Rightarrow (1200-24t)\frac{\mathrm{d}v}{\mathrm{d}t}+2000\times -24=-(1200-24t)g$$

Dividing by 24 $\Rightarrow (50-t)\dfrac{\mathrm{d}v}{\mathrm{d}t}-2000=-g(50-t)$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=-g+\frac{2000}{50-t}\Rightarrow \int_0^V 1\,\mathrm{d}v=\int_0^{30}-g+\frac{2000}{50-t}\,\mathrm{d}t$$

$$V=\left[-gt-2000\ln(50-t)\right]_0^{30}$$

$$=-30g-2000\ln 20+0+2000\ln 50\approx 1540\text{ m s}^{-1}$$

b After time t,

$$v=\left[-gt-2000\ln(50-t)\right]_0^t=-gt-2000\ln\left(\frac{50-t}{50}\right)=-gt-2000\ln\left(1-\frac{t}{50}\right)$$

so, using integration by parts,

$$x=\int_0^{30}-gt-2000\ln\left(1-\frac{t}{50}\right)\mathrm{d}t=\left[-\frac{g}{2}t^2+100\,000\left(1-\frac{t}{50}\right)\ln\left(1-\frac{t}{50}\right)-100\,000\left(1-\frac{t}{50}\right)\right]_0^{30}$$

$$=-\frac{900g}{2}+100\,000\times\frac{2}{5}\ln\frac{2}{5}-100\,000\times\frac{2}{5}+0-0+100\,000\approx 18\,900\text{ m}$$

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Exercise A, Question 9

Question:

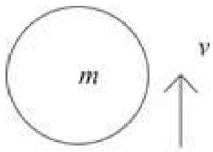
A rocket is launched vertically upwards from rest. The rocket expels burnt fuel vertically downwards with speed u relative to the rocket. Initially the rocket has mass M. At time t the rocket has speed v and mass $M\left(1-\frac{1}{4}t\right)$. Ignore air resistance.

a Find the speed of the rocket when $t = 2$.

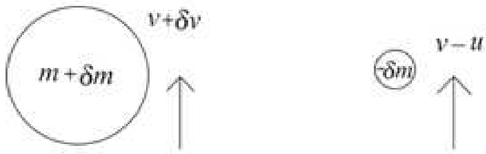
b Find the height of the rocket above the launch site when $t = 2$.

Solution:

a At time t



After interval δt



$$
\begin{aligned}
(m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv &= -mg\delta t\\
m\delta v+\delta m\delta v+u\delta m &= -mg\delta t\\
\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} &= -mg\\
m &= M\left(1-\frac{1}{4}t\right)\\
\Rightarrow M\left(1-\frac{1}{4}t\right)\frac{\mathrm{d}v}{\mathrm{d}t}+u\left(-\frac{1}{4}M\right) &= -M\left(1-\frac{1}{4}t\right)g\\
\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\frac{1}{4}u}{1-\frac{1}{4}t}-g &= \frac{u}{4-t}-g\\
\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{u}{4-t}-g\Rightarrow v=\int\frac{u}{4-t}-g\,\mathrm{d}t &= -u\ln|4-t|-gt+C\\
t=0,\ v=0\Rightarrow 0 &= -u\ln 4+C\\
v=u\ln\left|\frac{4}{4-t}\right|-gt &= u\ln\frac{4}{2}-2g=u\ln 2-2g \text{ when } t=2
\end{aligned}
$$

b $v=\frac{\mathrm{d}x}{\mathrm{d}t}=u\ln\left(\frac{4}{4-t}\right)-gt, t<4$

Using integration by parts

$$\Rightarrow x=\int u\ln\left(\frac{4}{4-t}\right)-gt\,\mathrm{d}t=\int u\ln 4-u\ln(4-t)-gt\,\mathrm{d}t$$

$$=(u\ln 4)t+u(4-t)\ln(4-t)-u(4-t)-\frac{1}{2}gt^2+C$$

$t=0, x=0\Rightarrow 0=0+4u\ln 4-4u+C, C=4u-4u\ln 4$

When $t=2, x=2\times u\ln 4+2u\ln 2-2u-2g+4u-4u\ln 4=-2u\ln 2+2u-2g$

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Exercise A, Question 10

Question:

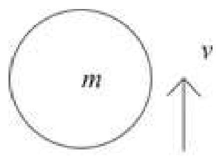
A rocket uses fuel at a rate λ kg s^{-1}. The rocket moves forwards by expelling used fuel backwards from the rocket with speed 2500 m s^{-1} relative to the rocket. At time t the rocket is moving with speed v and the combined mass of the rocket and its fuel is m. The rocket starts from rest at time $t = 0$ with a total mass 10 000 kg and reaches a final speed 5000 m s^{-1} after 200 seconds. Given that no external forces act on the rocket

a show that $m\frac{dv}{dt} = 2500\lambda$,

b find the value of λ kg s^{-1}.

Solution:

a At time t



After an interval δt

$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2500)-mv=0$$
$$\Rightarrow m\delta v+\delta m\delta v+\delta m 2500 \;=0$$
$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2500\frac{\mathrm{d}m}{\mathrm{d}t}\;=0$$

but we are told that $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$

so $m\frac{\mathrm{d}v}{\mathrm{d}t}-2500\lambda=0, m\frac{\mathrm{d}v}{\mathrm{d}t}=2500\lambda$

b The initial mass is 10 000 and $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$, so

$(10\,000-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}=2500\lambda$

Separating the variables

$$\Rightarrow \int_0^{5000} 1\mathrm{d}v=\int_0^{200}\frac{2500\lambda}{10\,000-\lambda t}\mathrm{d}t$$

$$5000=\left[-2500\ln(10\,000-\lambda t)\right]_0^{200}=-2500\ln\left(\frac{10\,000-200\lambda}{10\,000}\right)$$

$$\Rightarrow 2=\ln\left(\frac{50}{50-\lambda}\right), e^2=\frac{50}{50-\lambda}, 50-\lambda=50e^{-2},$$

$$\lambda=50\left(1-e^{-2}\right)\approx 43.2$$

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Exercise A, Question 11

Question:

A rocket uses fuel at a rate λ. The rocket moves forwards by expelling used fuel backwards from the rocket with speed 2000 m s^{-1} relative to the rocket. At time t the rocket is moving with speed v and the combined mass of the rocket and its fuel is m. The rocket starts from rest at time $t = 0$ with a total mass 12 000 kg and reaches a speed of 5000 m s^{-1} after 3 minutes.

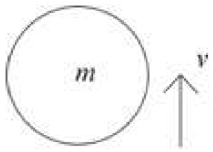
Given that no external forces act on the rocket

a show that $m\frac{dv}{dt} = 2000\lambda$,

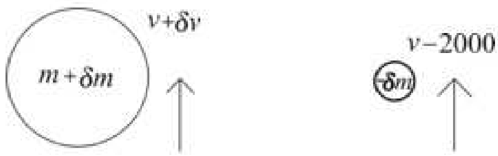
b find the greatest and the least acceleration of the vehicle during these three minutes.

Solution:

a At time t



After an interval δt



$$(m+\delta m)(v+\delta v)+(-\delta m)(v-2000)-mv=0$$
$$\Rightarrow m\delta v+\delta m\delta v+\delta m 2000 \;=0$$
$$m\frac{\mathrm{d}v}{\mathrm{d}t}+2000\frac{\mathrm{d}m}{\mathrm{d}t}\;=0$$

but we are told that $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$

so $m\frac{\mathrm{d}v}{\mathrm{d}t}-2000\lambda=0, m\frac{\mathrm{d}v}{\mathrm{d}t}=2000\lambda$

b The initial mass is 12 000 and $\frac{\mathrm{d}m}{\mathrm{d}t}=-\lambda$, so

$$(12000-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}=2000\lambda$$
$$\int_0^{5000}1\mathrm{d}v=\int_0^{180}\frac{2000\lambda}{12\,000-\lambda t}\mathrm{d}t$$
$$5000=\left[-2000\ln(12\,000-\lambda t)\right]_0^{180}=-2000\ln\left(\frac{12\,000-180\lambda}{12\,000}\right)$$
$$\Rightarrow\frac{5}{2}=\ln\left(\frac{200}{200-3\lambda}\right), \mathrm{e}^{\frac{5}{2}}=\frac{200}{200-3\lambda}, 200-3\lambda=200\mathrm{e}^{-\frac{5}{2}},$$
$$\lambda=\frac{200}{3}\left(1-\mathrm{e}^{-\frac{5}{2}}\right)\approx 61.2$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{2000\lambda}{m}\Rightarrow \text{min acceleration }=\frac{2000\times 61.2}{12\,000}=10.2\text{ m s}^{-2}$$
$$\text{max acceleration }=\frac{2000\times 61.2}{12\,000-180\times 61.2}=124\text{ m s}^{-2}$$

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Exercise A, Question 12

Question:

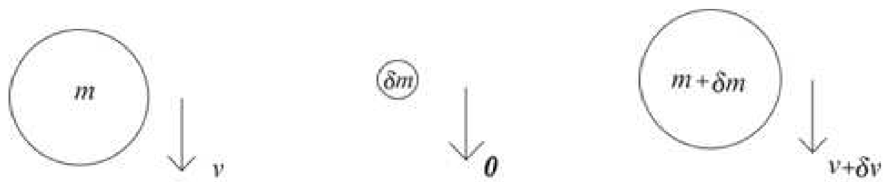
A particle falls from rest under gravity through a stationary cloud. At time t the particle has fallen a distance x, has mass m and speed v. The mass of the particle increases by accretion from the cloud at a rate of kmv, where k is a constant. Ignore air resistance. Show that

a $kv^2 = g\left(1 - e^{-2kx}\right)$,

b $x = \frac{1}{k}\ln\left[\cosh\left(\sqrt{kg}\, t\right)\right]$.

Solution:

a At time t After an interval δt :



$$[(m+\delta m)(v+\delta v)]-[mv+\delta m\times 0]=(m+\delta m)g\,\delta t$$

$$\Rightarrow m\frac{dv}{dt}+v\frac{dm}{dt}=mg$$

But we are told that $\frac{dm}{dt}=mkv$, so $m\frac{dv}{dt}+v\times mkv=mg$

$$\frac{dv}{dt}=g-kv^2, \Rightarrow v\frac{dv}{dx}=g-kv^2$$

$$\int\frac{v}{g-kv^2}\,dv=\int 1\,dx \quad \Rightarrow -\frac{1}{2k}\ln\left(g-kv^2\right)=x+C$$

$$x=0, v=0\Rightarrow -\frac{1}{2k}\ln g=C\Rightarrow x=-\frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right)$$

$$e^{2kx}=\frac{g}{g-kv^2}, \quad \left(g-kv^2\right)e^{2kx}=g, \quad kv^2=g\left(1-e^{-2kx}\right)$$

b $$v^2=\frac{g}{k}\left(1-e^{-2kx}\right), v=\sqrt{\frac{g}{k}\left(1-e^{-2kx}\right)}=\frac{dx}{dt}$$

$$\int\sqrt{\frac{g}{k}}\,dt=\int\frac{1}{\sqrt{1-e^{-2kx}}}\,dx=\int\frac{e^{kx}}{\sqrt{e^{2kx}-1}}\,dx$$

$\Rightarrow$ by using the substitution $\cosh u=e^{kx}, \sinh u.\frac{du}{dx}=ke^{kx}$

$$\sqrt{\frac{g}{k}}\,t=\int\frac{e^{kx}}{\sqrt{e^{2kx}-1}}\,dx=\frac{1}{k}\int\frac{\sinh u}{\sqrt{\cosh^2 u-1}}\,du=\frac{1}{k}\int 1\,du=\frac{u}{k}+C$$

$$t=0, x=0\Rightarrow \cosh u=1, u=\cosh^{-1}1=0, \Rightarrow C=0$$

$$\Rightarrow \sqrt{kg}\,t=u, \ \cosh(\sqrt{kg}\,t)=e^{kx}, \ kx=\ln[\cosh(\sqrt{kg}\,t)], x=\frac{1}{k}\ln\left[\cosh(\sqrt{kg}\,t)\right]$$

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Exercise A, Question 13

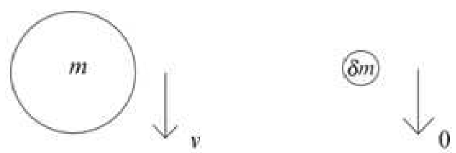
Question:

A raindrop falls through a stationary cloud. When the raindrop has fallen distance x it has mass m and speed v. The mass increases uniformly by accretion so that $m = M(1+kx)$.

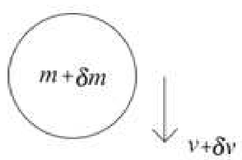
Given that $v = 0$ when $x = 0$, find an expression, in terms of M, k and x for the kinetic energy of the raindrop when it has fallen a distance x. Ignore air resistance.

Solution:

At time t



After an interval δt :



Impulse momentum: $[(m+\delta m)(v+\delta v)] - [mv] = (m+\delta m)g\delta t$

$$m\frac{\delta v}{\delta x} + v\frac{\delta m}{\delta x} + \frac{\delta m \delta v}{\delta x} = mg\frac{\delta t}{\delta x} + \delta mg\frac{\delta t}{\delta x}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}m}{\mathrm{d}x} = mg\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{mg}{v}, mv\frac{\mathrm{d}v}{\mathrm{d}x} + v^2\frac{\mathrm{d}m}{\mathrm{d}x} = mg$$

Substituting for m:

$$M(1+kx)v\frac{\mathrm{d}v}{\mathrm{d}x} + v^2kM = M(1+kx)g$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} + v^2\frac{k}{1+kx} = g, 2v\frac{\mathrm{d}v}{\mathrm{d}x} + \frac{2k}{1+kx}v^2 = 2g$$

A linear differential equation in v^2.

Integrating factor $e^{\int \frac{2k}{1+kx}\mathrm{d}x} = e^{2\ln(1+kx)} = (1+kx)^2$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left[v^2(1+kx)^2\right] = 2g(1+kx)^2, \quad v^2(1+kx)^2 = \frac{2g}{3k}(1+kx)^3 + C$$

$$x = 0, v = 0 \Rightarrow 0 = \frac{2g}{3k} + C, \quad v^2 = \frac{2g}{3k}(1+kx) - \frac{2g}{3k(1+kx)^2},$$

$$\text{so K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}M(1+kx)\left[\frac{2g}{3k}(1+kx) - \frac{2g}{3k(1+kx)^2}\right]$$

$$= \frac{Mg}{3k}\left[(1+kx)^2 - \frac{1}{(1+kx)}\right]$$

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Exercise A, Question 14

Question:

A rocket is on the ground facing vertically upwards. When launched it propels itself by ejecting mass backwards with speed u relative to the rocket at a constant rate k per unit time. The initial mass of the rocket is M. Ignore air resistance.

a Explain why it is necessary for $ku > Mg$.

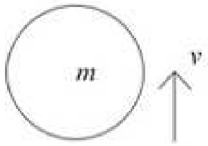
Given that $ku > Mg$,

b show that the velocity of the rocket after time t is $-u\ln\left(1-\frac{kt}{M}\right)-gt$,

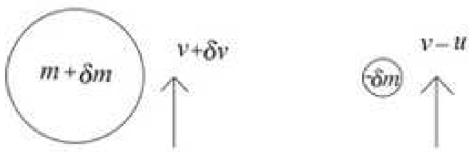
c find the height of the rocket above the ground when the mass of the rocket has reduced by one third of its initial value.

Solution:

a At time t



After an interval δt



Change in momentum $= (m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv=-mg\delta t$

$$\Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

$$m=M-kt\Rightarrow (M-kt)\frac{\mathrm{d}v}{\mathrm{d}t}+u(-k)=-(M-kt)g$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{ku}{M-kt}-g$$

If the rotor is to be able to launch then when $t=0,\frac{dv}{dt}>0$

$\frac{ku}{M}-g>0$, i.e. $ku>Mg$

b $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{ku}{M-kt}-g\Rightarrow v=-u\ln(M-kt)-gt+C$

$t=0, v=0\Rightarrow 0=-u\ln M+C$

$$\Rightarrow v=-u\ln(M-kt)-gt+u\ln M=-u\ln\left(\frac{M-kt}{M}\right)-gt$$

$$=-u\ln\left(1-\frac{kt}{M}\right)-gt$$

c $v=-u\ln\left(1-\frac{kt}{M}\right)-gt=\frac{\mathrm{d}x}{\mathrm{d}t}$

$$\Rightarrow x=\int -u\ln\left(1-\frac{kt}{M}\right)-gt\,\mathrm{d}t=\frac{uM}{k}\left[\left(1-\frac{kt}{M}\right)\ln\left(1-\frac{kt}{M}\right)-\left(1-\frac{kt}{M}\right)\right]-\frac{gt^2}{2}+C$$

(using integration by parts of $\ln\left(1-\frac{kt}{M}\right)$)

$t=0, x=0\Rightarrow 0=\frac{uM}{k}\times -1+C$

$m=\frac{2}{3}M=M-kt, t=\frac{M}{3k}$

$$x=\frac{uM}{k}\left[\frac{2}{3}\ln\frac{2}{3}-\frac{2}{3}\right]-\frac{gM^2}{18k^2}+\frac{uM}{k}=\frac{uM}{3k}\left[2\ln\frac{2}{3}+1-\frac{gM}{6ku}\right]$$

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Exercise A, Question 15

Question:

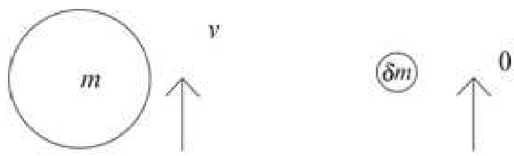
At time $t = 0$ a particle is projected vertically upwards. Initially the particle has mass M and speed gT, where T is a constant. At time t the speed of the particle is v and its mass is $Me^{\frac{t}{2T}}$. Ignore air resistance. If the added material is at rest when it is acquired, show that

a $\frac{\mathrm{d}}{\mathrm{d}t}\left(Mve^{\frac{t}{2T}}\right) = -Mge^{\frac{t}{2T}}$,

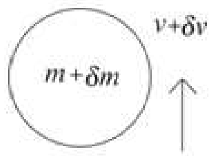
b the particle has mass $\frac{3M}{2}$ at its highest point.

Solution:

a At time t v



After an interval δt



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\,\delta t$

Taking the limit as $\delta t\to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=-mg,\ \text{i.e.}\ \frac{\mathrm{d}}{\mathrm{d}t}(mv)=-mg$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Mv\mathrm{e}^{\frac{t}{2T}}\right)=-M\mathrm{e}^{\frac{t}{2T}}g=-Mg\mathrm{e}^{\frac{t}{2T}}$$

b $\frac{\mathrm{d}}{\mathrm{d}t}\left(Mv\mathrm{e}^{\frac{t}{2T}}\right)=-M\mathrm{e}^{\frac{t}{2T}}g,\left(Mv\mathrm{e}^{\frac{t}{2T}}\right)=\int -Mg\mathrm{e}^{\frac{t}{2T}}dt$

$$\Rightarrow Mv\mathrm{e}^{\frac{t}{2T}}=-2MgT\mathrm{e}^{\frac{t}{2T}}+C$$

$$t=0, v=gT, C=3MgT$$

$$\Rightarrow Mv\mathrm{e}^{\frac{t}{2T}}=-2MgT\mathrm{e}^{\frac{t}{2T}}+3MgT,\Rightarrow v\mathrm{e}^{\frac{1}{2T}}=-2gT\mathrm{e}^{\frac{t}{2T}}+3gT$$

At the highest point, $v=0$, so $0=-2gT\mathrm{e}^{\frac{t}{2T}}+3gT,\ \mathrm{e}^{\frac{t}{2T}}=\frac{3}{2}$ and

$$\text{mass}=M\mathrm{e}^{\frac{t}{2T}}=\frac{3M}{2}$$

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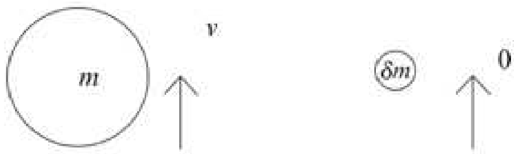
Exercise A, Question 16

Question:

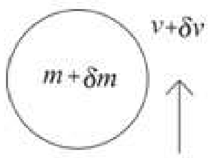
At time $t = 0$ a particle is projected vertically upwards from the ground. Initially the particle has mass M and speed $2gT$, where T is a constant. At time t the mass of the particle is $Me^{\frac{t}{T}}$. If the added material is at rest when it is acquired, show that the highest point reached by the particle is $gT^2(2 - \ln 3)$ above the ground. Ignore air resistance.

Solution:

At time t



After an interval δt



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\,\delta t$

Taking the limit as $\delta t \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=-mg,\text{ i.e. }\frac{\mathrm{d}}{\mathrm{d}t}(mv)=-mg$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Mve^{\frac{t}{T}}\right)=-Me^{\frac{t}{T}}g=-Mge^{\frac{t}{T}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Mve^{\frac{t}{T}}\right)=-Me^{\frac{t}{T}}g,\quad \left(Mve^{\frac{t}{T}}\right)=\int -Mge^{\frac{t}{T}}dt$$

$$\Rightarrow Mve^{\frac{t}{T}}=-MgTe^{\frac{t}{T}}+C$$

$$t=0, v=2gT, C=3MgT$$

$$\Rightarrow Mve^{\frac{t}{T}}=-MgTe^{\frac{t}{T}}+3MgT, \Rightarrow v=-gT+3gTe^{-\frac{t}{T}}$$

$$\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=-gT+3gTe^{-\frac{t}{T}}, x=-gTt-3gT^2e^{-\frac{t}{T}}+C$$

$$t=0, x=0, \Rightarrow C=3gT^2, x=-gTt-3gT^2e^{-\frac{t}{T}}+3gT^2$$

At the highest point, $v=0, \Rightarrow e^{-\frac{t}{T}}=\frac{1}{3}, -\frac{t}{T}=\ln\frac{1}{3}, t=T\ln 3$

$$\Rightarrow x=-gT.T\ln 3-3gT^2.\frac{1}{3}+3gT^2$$

$$=gT^2(2-\ln 3)$$

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Exercise A, Question 17

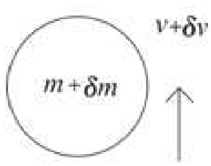
Question:

At time $t = 0$ a particle is projected vertically upwards. Initially the particle has mass M and speed gT, where T is a constant. At time t the mass of the particle is $Me^{\frac{t}{T}}$. If the added material is falling with constant speed gT when it is acquired, show that the particle has mass $\frac{3M}{2}$ at its highest point. Ignore air resistance.

Solution:

At time t

After an interval δt



Change in momentum:

$$\left[(m+\delta m)(v+\delta v)\right]-\left[mv-\delta m\times gT\right]=-(m+\delta m)g\delta t$$

$$\Rightarrow v\frac{\mathrm{d}m}{\mathrm{d}t}+m\frac{\mathrm{d}v}{\mathrm{d}t}=-mg-gT\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$m=Me^{\frac{t}{T}}\Rightarrow\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{M}{T}e^{\frac{t}{T}},\ \frac{\mathrm{d}}{\mathrm{d}t}(mv)=-mg-gMe^{\frac{t}{T}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Me^{\frac{t}{T}}v\right)=-Me^{\frac{t}{T}}g-gMe^{\frac{t}{T}}=-2Mge^{\frac{t}{T}}$$

$$\Rightarrow Me^{\frac{t}{T}}v=\int-2gMe^{\frac{t}{T}}\,\mathrm{d}t=-2gTMe^{\frac{t}{T}}+C$$

$$t=0,v=gT\Rightarrow MgT=-2MgT+C,\ C=3MgT$$

$$\Rightarrow Me^{\frac{t}{T}}v=-2gTMe^{\frac{t}{T}}+3MgT$$

$$v=0,\Rightarrow 2gTMe^{\frac{t}{T}}=3MgT,\ e^{\frac{t}{T}}=\frac{3}{2},\Rightarrow m=\frac{3M}{2}$$

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Exercise A, Question 18

Question:

A particle of mass M is projected vertically upwards in a cloud. During the motion the particle absorbs moisture from the stationary cloud so that when the particle is at distance x above the point of projection, moving with speed v, it has mass $M(1+\alpha x)$, where α is a constant. The initial speed of the particle is $\sqrt{2gk}$. Ignore air resistance.

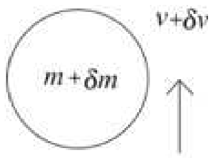
a Show that $2v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2\alpha}{1+\alpha x}v^2=-2g$.

b Show that at the greatest height, $h, (1+\alpha h)^3=1+3k\alpha$.

Solution:

a At time t

After an interval δt :



Change in momentum: $\left[(m+\delta m)(v+\delta v)\right]-\left[mv+\delta m\times 0\right]=-(m+\delta m)g\,\delta t$

Taking the limit as $\delta t \to 0$

$$m\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\mathrm{d}m}{\mathrm{d}t}=-mg$$

$m=M(1+\alpha x)\Rightarrow M(1+\alpha x)\frac{\mathrm{d}v}{\mathrm{d}t}+vM\alpha\frac{\mathrm{d}x}{\mathrm{d}t}=-M(1+\alpha x)g$

Using $\frac{\mathrm{d}v}{\mathrm{d}t}=v\frac{\mathrm{d}v}{\mathrm{d}x}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=v$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+v\frac{\alpha}{(1+\alpha x)}\frac{\mathrm{d}x}{\mathrm{d}t}=-g,\ v\frac{\mathrm{d}v}{\mathrm{d}x}+v^2\frac{\alpha}{(1+\alpha x)}=-g$$

$$\Rightarrow 2v\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{2\alpha}{(1+\alpha x)}v^2=-2g$$

b Multiply through the differential equation by the integrating factor (since the differential equation is linear differential equation in v^2)

I.F. $= \mathrm{e}^{\int\frac{2\alpha}{(1+\alpha x)}\mathrm{d}x}=\mathrm{e}^{2\ln(1+\alpha x)}=(1+\alpha x)^2$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}\left[v^2(1+\alpha x)^2\right]=-2g(1+\alpha x)^2,\ v^2(1+\alpha x)^2=\int -2g(1+\alpha x)^2\,\mathrm{d}x$$

$$=-\frac{2g}{3\alpha}(1+\alpha x)^3+C$$

$$x=0, v=\sqrt{2gk}\Rightarrow 2gk=-\frac{2g}{3\alpha}+C,\quad C=2g\left(k+\frac{1}{3\alpha}\right)$$

At the highest point, $v=0\Rightarrow 0=-\frac{2g}{3\alpha}(1+\alpha h)^3+2g\left(k+\frac{1}{3\alpha}\right)$

$\frac{1}{3\alpha}(1+\alpha h)^3=\left(k+\frac{1}{3\alpha}\right)$

and therefore $(1+\alpha h)^3=1+3k\alpha$

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Exercise A, Question 19

Question:

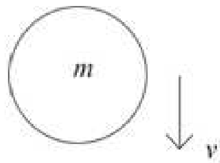
A body of mass $3M$ contains combustible and non-combustible material in the ratio 2 : 1. The body is initially at rest and falls freely under gravity. At time t the body has speed v.
The combustible part burns at a constant rate of λM per second, where λ is a constant. The burning material is ejected vertically upwards with constant speed u relative to the body. Assuming that air resistance may be neglected,

a show that $\frac{dv}{dt} = \frac{\lambda u}{3 - \lambda t} + g$,

b find how far the body has fallen when all the combustible material has been used up.

Solution:

a At time t



After an interval δt :

$$\Rightarrow (m+\delta m)(v+\delta v)+(-\delta m)(v-u)-mv = mg\,\delta t$$

$$m\delta v+\delta m\delta v+\delta m u = mg\,\delta t, \quad m\frac{\mathrm{d}v}{\mathrm{d}t}+u\frac{\mathrm{d}m}{\mathrm{d}t} = mg$$

$$m = M(3-\lambda t), \Rightarrow M(3-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}+u(-\lambda M) = M(3-\lambda t)g$$

$$(3-\lambda t)\frac{\mathrm{d}v}{\mathrm{d}t}-\lambda u = (3-\lambda t)g, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda u}{3-\lambda t}+g$$

b $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\lambda u}{3-\lambda t}+g \Rightarrow v = \int \frac{\lambda u}{3-\lambda t}+g\mathrm{d}t = -u\ln(3-\lambda t)+gt+C, (\lambda t < 3)$

$$t=0, v=0 \Rightarrow 0 = -u\ln 3+C$$

$$\Rightarrow v = -u\ln\left(\frac{3-\lambda t}{3}\right)+gt = -u\ln\left(1-\frac{\lambda t}{3}\right)+gt = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$\Rightarrow x = \int -u\ln\left(1-\frac{\lambda t}{3}\right)+gt\mathrm{d}t = \frac{3u}{\lambda}\left[\left(1-\frac{\lambda t}{3}\right)\ln\left(1-\frac{\lambda t}{3}\right)-\left(1-\frac{\lambda t}{3}\right)\right]+\frac{gt^2}{2}+C$$

$$t=0, x=0 \Rightarrow 0 = -\frac{3u}{\lambda}+C$$

All combustible material used $\Rightarrow m = M(3-\lambda t) = M, \quad t = \frac{2}{\lambda}$

$$\Rightarrow x = \frac{3u}{\lambda}\left[\left(1-\frac{\lambda t}{3}\right)\ln\left(1-\frac{\lambda t}{3}\right)-\left(1-\frac{\lambda t}{3}\right)\right]+\frac{gt^2}{2}+\frac{3u}{\lambda}$$

$$= \frac{3u}{\lambda}\left[\left(1-\frac{\lambda 2}{3\lambda}\right)\ln\left(1-\frac{\lambda 2}{3\lambda}\right)-\left(1-\frac{\lambda 2}{3\lambda}\right)\right]+\frac{g4}{2\lambda^2}+\frac{3u}{\lambda}$$

$$= \frac{3u}{\lambda}\left[\frac{1}{3}\ln\frac{1}{3}-\frac{1}{3}\right]+\frac{2g}{\lambda^2}+\frac{3u}{\lambda} = \frac{3u}{\lambda}\left[\frac{1}{3}\ln\frac{1}{3}+\frac{2}{3}\right]+\frac{2g}{\lambda^2}$$

$$= \frac{u}{\lambda}\left[\ln\frac{1}{3}+2\right]+\frac{2g}{\lambda^2} = \frac{u}{\lambda}(2-\ln 3)+\frac{2g}{\lambda^2}$$

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Exercise A, Question 20

Question:

A spherical hailstone falls vertically through a stationary cloud from rest under gravity. The initial radius of the hailstone is a. As the hailstone falls its volume increases through condensation. When the radius of the hailstone is r, the rate of increase of volume is $4\pi r^2\lambda$ and the hailstone is falling with speed v. Ignore air resistance.

a Show that, at time $t, r = a + \lambda t$.

b Show that $\frac{dv}{dt} = g - \frac{3\lambda v}{r}$.

c Find the speed of the particle when $t = \frac{a}{2\lambda}$.

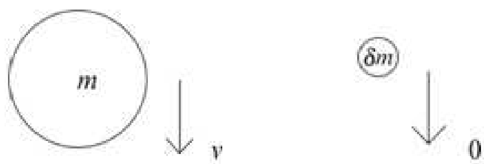
Solution:

a For the sphere, $\frac{dV}{dt} = 4\pi r^2 \lambda$, but

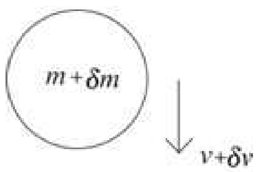
$$V = \frac{4}{3}\pi r^3 \Rightarrow 4\pi r^2 \frac{dr}{dt} = 4\pi r^2 \lambda$$

$$\Rightarrow \frac{dr}{dt} = \lambda, r = a + \lambda t$$

b At time t



After time δt :



$$\left[(m+\delta m)(v+\delta v)\right] - \left[mv + \delta m \times 0\right] = (m+\delta m)g\,\delta t$$

$$\Rightarrow m\frac{\delta v}{\delta t} + v\frac{\delta m}{\delta t} + \frac{\delta m \delta v}{\delta t} = mg + g\,\delta m \text{, so } m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

The mass of the hailstone is $\rho \times \frac{4}{3}\pi r^3$ since mass is proportional to volume

$$\Rightarrow \frac{dm}{dt} = 4\rho\pi r^2 \frac{dr}{dt} = 4\rho\pi r^2 \times \lambda$$

$$\Rightarrow \rho \times \frac{4}{3}\pi r^3 \frac{dv}{dt} + v \times 4\rho\lambda\pi r^2 = \rho \times \frac{4}{3}\pi r^3 g$$

and therefore $\frac{dv}{dt} = g - \frac{3\lambda v}{r}$

c $\frac{dv}{dt} = g - \frac{3\lambda v}{r} = g - \frac{3\lambda v}{a+\lambda t}, \frac{dv}{dt} + \frac{3\lambda v}{a+\lambda t} = g$

Using the integrating factor $e^{\int \frac{3\lambda}{a+\lambda t} dt} = e^{3\ln(a+\lambda t)} = (a+\lambda t)^3$:

$$v(a+\lambda t)^3 = \int g(a+\lambda t)^3\,dt = \frac{g}{4\lambda}(a+\lambda t)^4 + C$$

$$t = 0, v = 0, 0 = \frac{ga^4}{4\lambda} + C, \quad v(a+\lambda t)^3 = \frac{g}{4\lambda}(a+\lambda t)^4 - \frac{ga^4}{4\lambda}$$

$$v = \frac{g(a+\lambda t)}{4\lambda} - \frac{ga^4}{4\lambda(a+\lambda t)^3}$$

$$t = \frac{a}{2\lambda} \Rightarrow v = \frac{g\left(a+\lambda\frac{a}{2\lambda}\right)}{4\lambda} - \frac{ga^4}{4\lambda\left(a+\lambda\frac{a}{2\lambda}\right)^3} = \frac{3ag}{8\lambda} - \frac{2ga}{27\lambda} = \frac{65ag}{216\lambda}$$