Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres and its velocity is $\mathbf{v}$ m s^{-1}. The motion of P is modelled by the differential equation

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 3\mathbf{v}.$$

Given that when $t = 0, \mathbf{r} = 3\mathbf{i}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$\frac{d\mathbf{v}}{dt} - 3\mathbf{v} = \mathbf{0}$ Auxiliary equation is $\lambda - 3 = 0$

$\Rightarrow \lambda = 3$

General solution is $\mathbf{v} = \mathbf{A}e^{3t}$

When $t = 0, \mathbf{v} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = \mathbf{A}$

So, $\mathbf{v} = (\mathbf{i} - \mathbf{j})e^{3t}$

i.e. $\frac{d\mathbf{r}}{dt} = (\mathbf{i} - \mathbf{j})e^{3t}$

$\Rightarrow \mathbf{r} = \frac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t} + \mathbf{B}$

When $t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \frac{1}{3}(\mathbf{i} - \mathbf{j}) + \mathbf{B}$

$\Rightarrow \frac{8}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} = \mathbf{B}$

Hence, $\mathbf{r} = \frac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t} + \frac{8}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$

$= \frac{1}{3}\{(e^{3t} + 8)\mathbf{i} + (1 - e^{3t})\mathbf{j}\}$

Alternative method

$\frac{d\mathbf{v}}{dt} - 3\mathbf{v} = \mathbf{0}$

$\Rightarrow \frac{d^2\mathbf{r}}{dt^2} - 3\frac{d\mathbf{r}}{dt} = \mathbf{0}$

Auxiliary equation:

$\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda - 3) = 0$

$\Rightarrow \lambda = 0$ or 3

$\Rightarrow$ General Solution is $\mathbf{v} = \mathbf{A} + \mathbf{B}e^{3t}$

$t = 0, \mathbf{r} = 3\mathbf{i} \Rightarrow 3\mathbf{i} = \mathbf{A} + \mathbf{B}$ ①

$\mathbf{r} = \mathbf{A} + \mathbf{B}e^{3t} \Rightarrow \mathbf{v} = 3\mathbf{B}e^{3t}$

$t = 0, \mathbf{v} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = 3\mathbf{B}$

$\Rightarrow \frac{1}{3}(\mathbf{i} - \mathbf{j}) = \mathbf{B}$

Substitute into ①

$\mathbf{A} = \frac{8}{3}\mathbf{i} + \frac{1}{3}\mathbf{j}$

Hence, $\mathbf{r} = \frac{8}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{3}(\mathbf{i} - \mathbf{j})e^{3t}$ as before

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Exercise A, Question 2

Question:

The velocity $\mathbf{v}$ m s^{-1} of a particle P at time t seconds satisfies the differential equation
$\frac{d\mathbf{v}}{dt} + \mathbf{v} = \mathbf{0}$.
Given that the initial velocity of P is $(12\mathbf{i} + 6\mathbf{j})$, find the velocity of P at $t = \ln 3$.

Solution:

$\frac{d\mathbf{v}}{dt} + \mathbf{v} = \mathbf{0} \Rightarrow \lambda + 1 = 0 \Rightarrow \lambda = -1$

So, general solution is $\mathbf{v} = \mathbf{A}e^{-t}$

$t = 0, \mathbf{v} = 12\mathbf{i} + 6\mathbf{j} \Rightarrow \mathbf{v} = (12\mathbf{i} + 6\mathbf{j})e^{-t}$

When $t = \ln 3$, $\mathbf{v} = (12\mathbf{i} + 6\mathbf{j})e^{-\ln 3}$

$= 4\mathbf{i} + 2\mathbf{j}$

Velocity is $(4\mathbf{i} + 2\mathbf{j})$ m s^{-1}

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Exercise A, Question 3

Question:

The velocity $\mathbf{v}$ m s^{-1} at time t seconds of a particle moving in a plane satisfies the differential equation

$\frac{d\mathbf{v}}{dt} = 6\mathbf{v}$, where $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j}$ when $t = 0$.

Given that the particle starts, at $t = 0$, at the point with position vector $(\mathbf{i} + \mathbf{j})$m find

a the position vector of the particle P at time t seconds,

b the time when the magnitude of the acceleration of the particle P first equals 100 m s^{-2}.

Solution:

$$\frac{d\mathbf{v}}{dt} - 6\mathbf{v} = \mathbf{0} \Rightarrow \lambda - 6 = 0 \Rightarrow \lambda = 6$$

So, general solution is $\mathbf{v} = \mathbf{A}e^{6t}$

$$t = 0, \mathbf{v} = 4\mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{v} = (4\mathbf{i} + 2\mathbf{j})e^{6t}$$

$$\Rightarrow \mathbf{r} = \frac{1}{6}(4\mathbf{i} + 2\mathbf{j})e^{6t} + \mathbf{B}$$

$$t = 0, \mathbf{r} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = \frac{1}{6}(4\mathbf{i} + 2\mathbf{j}) + \mathbf{B}$$

$$\Rightarrow \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} = \mathbf{B}$$

a Hence, $\mathbf{r} = \frac{1}{3}(2\mathbf{i} + \mathbf{j})e^{6t} + \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j}$

b $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 6\mathbf{v} = (24\mathbf{i} + 12\mathbf{j})e^{6t}$

$$= 12e^{6t}(2\mathbf{i} + \mathbf{j})$$

$$|\mathbf{a}| = 12e^{6t}\sqrt{2^2 + 1^2}$$

$$= 12\sqrt{5}e^{6t}$$

So, $12\sqrt{5}e^{6t} = 100$

$$\Rightarrow t = \frac{1}{6}\ln\left(\frac{100}{12\sqrt{5}}\right) = 0.2195 \quad \text{(3 s.f.)}$$

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Exercise A, Question 4

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres and its velocity is $\mathbf{v}$ m s^{-1}. The motion of P is described by the differential equation

$$\frac{d\mathbf{v}}{dt} = 4\mathbf{v}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$,

a find $\mathbf{r}$ in terms of t,

b find the speed of P when $t = 2$,

c find the magnitude of the acceleration of P when $t = 2$.

Solution:

$$\frac{d\mathbf{v}}{dt} - 4\mathbf{v} = \mathbf{0} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} = 0 \Rightarrow \lambda^2 - 4\lambda = 0$$

$$\Rightarrow \lambda = 0 \text{ or } 4$$

So, general solution is $\mathbf{r} = \mathbf{A} + \mathbf{B}e^{4t}$

a $t = 0, \mathbf{r} = \mathbf{i} - \mathbf{k} \Rightarrow \mathbf{i} - \mathbf{k} = \mathbf{A} + \mathbf{B}$ ①

$$\mathbf{v} = 4\mathbf{B}e^{4t};\ t = 0, \mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{B} = \frac{1}{4}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\Rightarrow \mathbf{A} = \mathbf{i} - \mathbf{k} - \frac{1}{4}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$= \frac{3}{4}\mathbf{i} - \frac{1}{4}\mathbf{j} - \frac{5}{4}\mathbf{k} = \frac{1}{4}(3\mathbf{i} - \mathbf{j} - 5\mathbf{k})$$

So, $\mathbf{r} = \frac{1}{4}\left\{3\mathbf{i} - \mathbf{j} - 5\mathbf{k} + (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{4t}\right\}$

b $\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{4t}$

When $t = 2, \mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k})e^{8}$

$|\mathbf{v}| = e^8\sqrt{3}$ m s^{-1}

c As $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{v}$

At $t = 2, |\mathbf{a}| = 4e^8\sqrt{3}$ m s^{-2}

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Exercise A, Question 5

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres. The motion of P is described by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = 2\frac{d\mathbf{r}}{dt}.$$

Given that the initial velocity of P is $(2\mathbf{i} - \mathbf{j})\text{m s}^{-1}$, find the speed of P at $t = \ln 3$.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} = \mathbf{0} \Rightarrow \frac{d\mathbf{v}}{dt} - 2\mathbf{v} = \mathbf{0}$$

$$\Rightarrow \lambda - 2 = 0 \Rightarrow \lambda = 2$$

$$\Rightarrow \mathbf{v} = \mathbf{A}e^{2t}$$

$$t = 0, \mathbf{v} = 2\mathbf{i} - \mathbf{j} \Rightarrow \mathbf{v} = (2\mathbf{i} - \mathbf{j})e^{2t}$$

$$t = \ln 3, \quad \mathbf{v} = (2\mathbf{i} - \mathbf{j})e^{2\ln 3}$$

$$= (18\mathbf{i} - 9\mathbf{j})$$

$$\therefore |\mathbf{v}| = 9\sqrt{2^2 + (-1)^2} = 9\sqrt{5} \text{ m s}^{-1}$$

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Exercise A, Question 6

Question:

At time t seconds the position vector of a particle P is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = (15\mathbf{i} + 10\mathbf{j})e^{3t}.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} + \mathbf{j}$, find

a $\mathbf{r}$ in terms of t,

b the velocity of P when $t = \ln 4$.

Solution:

a $\frac{d\mathbf{r}}{dt} + 2\mathbf{r} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} e^{3t}$

Integrating factor $= e^{\int 2dt} = e^{2t}$

$$\frac{d}{dt}\left(e^{2t}\mathbf{r}\right) = \begin{pmatrix} 15 \\ 10 \end{pmatrix} e^{5t}$$

$$e^{2t}\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{5t} + \mathbf{C}$$

$$t = 0, \mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mathbf{c} \Rightarrow \mathbf{c} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{-2t}$$

b $\mathbf{v} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} e^{3t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-2t}$

$$t = \ln 4, \mathbf{v} = \begin{pmatrix} 9 \\ 6 \end{pmatrix} \times 64 + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \times \frac{1}{16} = \begin{pmatrix} 576\frac{1}{8} \\ 384\frac{1}{8} \end{pmatrix}$$

velocity is $(576\frac{1}{8}\mathbf{i} + 384\frac{1}{8}\mathbf{j})$ m s^{-1}

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Exercise A, Question 7

Question:

The position vector $\mathbf{r}$ metres of a particle P at time t seconds satisfies the vector differential equation

$$\frac{d\mathbf{r}}{dt}+\mathbf{r}=\mathbf{k}e^{t}.$$

Given that when $t=0, \mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt}+\mathbf{r}=\mathbf{k}e^{t}$$

Integrating factor $= e^{\int 1 dt} = e^{t}$

$$\Rightarrow \frac{d}{dt}(\mathbf{r}e^{t}) = \mathbf{k}e^{2t}$$

$$\Rightarrow \mathbf{r}e^{t} = \int \mathbf{k}e^{2t} dt = \frac{1}{2}\mathbf{k}e^{2t}+\mathbf{c}$$

When $t=0, \mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}$

$$\Rightarrow (2\mathbf{i}+3\mathbf{j}+\mathbf{k}) = \frac{1}{2}\mathbf{k}+\mathbf{c}$$

$$\Rightarrow 2\mathbf{i}+3\mathbf{j}+\frac{1}{2}\mathbf{k}=\mathbf{c}$$

$$\Rightarrow \mathbf{r}=\frac{1}{2}\mathbf{k}e^{t}+(2\mathbf{i}+3\mathbf{j}+\frac{1}{2}\mathbf{k})e^{-t}$$

$$\mathbf{r} = 2e^{-t}\mathbf{i}+3e^{-t}\mathbf{j}+\frac{1}{2}(e^{t}+e^{-t})\mathbf{k}$$

Alternative method

$$\frac{d\mathbf{r}}{dt}+\mathbf{r} = \mathbf{0} \Rightarrow \lambda+1=0 \Rightarrow \lambda=-1$$

C.F. is $\mathbf{r} = \mathbf{A}e^{-t}$

P.I. Try $\mathbf{r}=\mathbf{B}e^{t} \Rightarrow \frac{d\mathbf{r}}{dt}=\mathbf{B}e^{t}$

So, $\mathbf{B}e^{t}+\mathbf{B}e^{t} \equiv \mathbf{k}e^{t} \Rightarrow \mathbf{B}=\frac{1}{2}\mathbf{k}$

So P.I. is $\mathbf{r}=\frac{1}{2}\mathbf{k}e^{t}$

General solution is C.F. + P.I.

i.e. $\mathbf{r}=\mathbf{A}e^{-t}+\frac{1}{2}\mathbf{k}\,e^{t}$

$t=0, \mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k} \Rightarrow 2\mathbf{i}+3\mathbf{j}+\mathbf{k}=\mathbf{A}+\frac{1}{2}\mathbf{k}$

$$\Rightarrow \quad 2\mathbf{i}+3\mathbf{j}+\frac{1}{2}\mathbf{k}=\mathbf{A}$$

So, $\mathbf{r}=(2\mathbf{i}+3\mathbf{j}+\frac{1}{2}\mathbf{k})e^{-t}+\frac{1}{2}\mathbf{k}e^{t}$ as before

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Exercise A, Question 8

Question:

At time t the velocity $\mathbf{v}$ of a particle P satisfies the vector differential equation
$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}+\frac{3\mathbf{v}}{T}=\mathbf{0}$, where T is a constant.
At time $t=0$ the position vector of P is $a(2\mathbf{i}+\mathbf{j})$ and its velocity is $\frac{3a(\mathbf{i}-\mathbf{j})}{T}$.
Find the position vector of P at time t.

Solution:

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}+\frac{3}{T}\mathbf{v}=\mathbf{0} \Rightarrow \lambda+\frac{3}{T}=0$$
$$\Rightarrow \lambda=\frac{-3}{T}$$
$$\therefore \mathbf{v}=\mathbf{A}\mathrm{e}^{\frac{-3}{T}t}$$
$$\Rightarrow \mathbf{r}=-\frac{T}{3}\mathbf{A}\mathrm{e}^{\frac{-3t}{T}}+\mathbf{B}$$
When $t=0, \mathbf{v}=\frac{3a}{T}(\mathbf{i}-\mathbf{j}) \Rightarrow \frac{3a}{T}(\mathbf{i}-\mathbf{j})=\mathbf{A}$

So, $\mathbf{r}=-a(\mathbf{i}-\mathbf{j})\mathrm{e}^{\frac{-3t}{T}}+\mathbf{B}$

When $t=0, \mathbf{r}=a(2\mathbf{i}+\mathbf{j})$
$$\Rightarrow a(2\mathbf{i}+\mathbf{j})=-a(\mathbf{i}-\mathbf{j})+\mathbf{B}$$
$$\Rightarrow a(3\mathbf{i})=\mathbf{B}$$
So, $\mathbf{r}=3a\mathbf{i}-a(\mathbf{i}-\mathbf{j})\mathrm{e}^{\frac{-3t}{T}}$

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Exercise A, Question 9

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the vector differential equation

$$\frac{d\mathbf{r}}{dt}+3\mathbf{r}=4e^{-t}\mathbf{j}.$$

Given that when $t=0, \mathbf{r}=2\mathbf{i}-\mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt}+3\mathbf{r}=4e^{-t}\mathbf{j} \quad \text{I.F.}=e^{\int 3dt}=e^{3t}$$

$$\Rightarrow \frac{d}{dt}(\mathbf{r}e^{3t}) = 4e^{2t}\mathbf{j}$$

$$\mathbf{r}e^{3t} = 2e^{2t}\mathbf{j}+\mathbf{A}$$

$$t=0, \mathbf{r}=2\mathbf{i}-\mathbf{j} \Rightarrow 2\mathbf{i}-\mathbf{j}=2\mathbf{j}+\mathbf{A}$$

$$\Rightarrow 2\mathbf{i}-3\mathbf{j}=\mathbf{A}$$

$$\Rightarrow \mathbf{r} = 2e^{-t}\mathbf{j}+(2\mathbf{i}-3\mathbf{j})e^{-3t}$$

$$= 2e^{-3t}\mathbf{i}+(2e^{-t}-3e^{-3t})\mathbf{j}$$

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Exercise A, Question 10

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the vector differential equation

$$\frac{d\mathbf{r}}{dt} - \frac{2}{t}\mathbf{r} = 4\mathbf{j}, t > 0.$$

Given that when $t = 1, \mathbf{r} = \mathbf{i} - \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d\mathbf{r}}{dt} - \frac{2}{t}\mathbf{r} = 4\mathbf{j}$$

$$\text{I.F. } e^{\int \frac{2}{t} dt} = e^{-2\ln t} = e^{\ln t^{-2}} = \frac{1}{t^2}$$

$$\frac{d}{dt}(\mathbf{r} \times \frac{1}{t^2}) = \frac{4}{t^2}\mathbf{j}$$

$$\frac{1}{t^2}\mathbf{r} = \int 4t^{-2}\mathbf{j}\, dt$$

$$= \frac{-4}{t}\mathbf{j} + \mathbf{c}$$

$$t = 1, \mathbf{r} = \mathbf{i} - \mathbf{j} \Rightarrow \mathbf{i} - \mathbf{j} = -4\mathbf{j} + \mathbf{c}$$

$$\mathbf{c} = \mathbf{i} + 3\mathbf{j}$$

$$\text{So } \mathbf{r} = (\mathbf{i} + 3\mathbf{j})t^2 - 4t\mathbf{j}$$

$$\text{i.e. } \Rightarrow \mathbf{r} = t^2\mathbf{i} + (3t^2 - 4t)\mathbf{j}$$

$$\textit{Check } \frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + (6t - 4)\mathbf{j}$$

$$\text{LHS} = 2t\mathbf{i} + (6t - 4)\mathbf{j} - \frac{2}{t}\left\{t^2\mathbf{i} + (3t^2 - 4t)\mathbf{j}\right\}$$

$$= 4\mathbf{j} = \text{RHS}$$

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Exercise B, Question 1

Question:

At time t seconds the position vector $\mathbf{r}$ metres of a particle P satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+8\mathbf{r}=\mathbf{0}.$$

At $t=0, \mathbf{r}=\mathbf{i}+\mathbf{j}$ and the velocity of P is $(2\mathbf{i}-4\mathbf{j})\text{m s}^{-1}$.

Find an expression for $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2}+4\frac{d\mathbf{r}}{dt}+8\mathbf{r}=\mathbf{0} \Rightarrow \lambda^2+4\lambda+8=0$$

$$\Rightarrow (\lambda+2)^2=-4$$

$$\Rightarrow \lambda=-2\pm 2i$$

General solution is $\mathbf{r}=e^{-2t}(\mathbf{A}\cos 2t+\mathbf{B}\sin 2t)$

$$t=0, \mathbf{r}=\mathbf{i}+\mathbf{j} \Rightarrow \mathbf{i}+\mathbf{j}=\mathbf{A}$$

$$\frac{d\mathbf{r}}{dt}=e^{-2t}(-2\mathbf{A}\sin 2t+2\mathbf{B}\cos 2t)-2e^{-2t}(\mathbf{A}\cos 2t+\mathbf{B}\sin 2t)$$

$$t=0, \mathbf{v}=2\mathbf{i}-4\mathbf{j} \Rightarrow 2\mathbf{i}-4\mathbf{j}=2\mathbf{B}-2\mathbf{A}$$

$$\Rightarrow \quad \mathbf{B}=\mathbf{i}-2\mathbf{j}+\mathbf{i}+\mathbf{j}=2\mathbf{i}-\mathbf{j}$$

So, $\mathbf{r}=e^{-2t}[(\mathbf{i}+\mathbf{j})\cos 2t+(2\mathbf{i}-\mathbf{j})\sin 2t]$

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Exercise B, Question 2

Question:

The position vector $\mathbf{r}$ metres of a particle P at time t seconds satisfies the vector differential equation

$$\frac{d^2\mathbf{r}}{dt^2}+3\frac{d\mathbf{r}}{dt}+2\mathbf{r}=\mathbf{0}.$$

At $t=0$, the particle is at the point with position vector $2\mathbf{j}$ m moving with velocity $(\mathbf{i}+\mathbf{j})\text{m s}^{-1}$.

Find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2}-3\frac{d\mathbf{r}}{dt}+2\mathbf{r}=\mathbf{0}\Rightarrow \lambda^2+3\lambda+2=0$$

$$\Rightarrow(\lambda+2)(\lambda+1)=0$$

$$\Rightarrow \lambda=-2 \text{ or } -1$$

General solution is

$$\mathbf{r}=\mathbf{A}e^{-2t}+\mathbf{B}e^{-t}$$

$$t=0, \mathbf{r}=2\mathbf{j}\Rightarrow 2\mathbf{j}=\mathbf{A}+\mathbf{B} \qquad ①$$

$$\frac{d\mathbf{r}}{dt}=-2\mathbf{A}e^{-2t}-\mathbf{B}e^{-t}$$

$$t=0, \mathbf{v}=\mathbf{i}+\mathbf{j}\Rightarrow \mathbf{i}+\mathbf{j}=-2\mathbf{A}-\mathbf{B} \qquad ②$$

① + ②: $\mathbf{i}+3\mathbf{j}=-\mathbf{A}\Rightarrow\mathbf{A}=-\mathbf{i}-3\mathbf{j}$

$$\Rightarrow \mathbf{B}=\mathbf{i}+5\mathbf{j}$$

So, $\mathbf{r}=(-\mathbf{i}-3\mathbf{j})e^{-2t}+(\mathbf{i}+5\mathbf{j})e^{-t}$

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Exercise B, Question 3

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{j}$, find the distance of P from the origin O when $t = 2$.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1 \text{(twice)}$$

General solution is $\mathbf{r} = e^t(\mathbf{A} + \mathbf{B}t)$

$t = 0, \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{i} = \mathbf{A}$

$$\frac{dr}{dt} = e^t\mathbf{B} + e^t(\mathbf{A} + \mathbf{B}t)$$

$t = 0, \mathbf{v} = \mathbf{j} \Rightarrow \mathbf{j} = \mathbf{B} + \mathbf{A} \Rightarrow \mathbf{B} = -\mathbf{i} + \mathbf{j}$

$\mathbf{r} = e^t[\mathbf{i} + t(-\mathbf{i} + \mathbf{j})]$

When $t = 2, \mathbf{r} = e^2(\mathbf{i} - 2\mathbf{i} + 2\mathbf{j})$

$= e^2(-\mathbf{i} + 2\mathbf{j})$

$|\mathbf{r}| = e^2\sqrt{(-1)^2 + 2^2} = e^2\sqrt{5}$ m

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Exercise B, Question 4

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}+\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}+\mathbf{r}=\mathbf{0}.$$

Given that when $t=0, \mathbf{r}=-2\mathbf{i}$ and $\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathbf{i}+\sqrt{3}\mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}+\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}+\mathbf{r}=\mathbf{0} \Rightarrow \lambda^2+\lambda+1=0$$

$$\Rightarrow \left(\lambda+\frac{1}{2}\right)^2=-\frac{3}{4}$$

$$\Rightarrow \lambda=\frac{-1}{2}\pm \mathrm{i}\frac{\sqrt{3}}{2}$$

General solution is $\mathbf{r}=\mathrm{e}^{-\frac{1}{2}t}(\mathbf{A}\cos\frac{\sqrt{3}}{2}t+\mathbf{B}\sin\frac{\sqrt{3}}{2}t)$

$t=0, \mathbf{r}=-2\mathbf{i} \Rightarrow -2\mathbf{i}=\mathbf{A}$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathrm{e}^{-\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\mathbf{A}\sin\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{2}\mathbf{B}\cos\frac{\sqrt{3}}{2}t\right)-\frac{1}{2}\mathrm{e}^{-\frac{1}{2}t}(\mathbf{A}\cos\frac{\sqrt{3}}{2}t+\mathbf{B}\sin\frac{\sqrt{3}}{2}t)$$

$$t=0, \mathbf{v}=\mathbf{i}+\sqrt{3}\mathbf{j} \Rightarrow \mathbf{i}+\sqrt{3}\mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{B}-\frac{1}{2}\mathbf{A}$$

$$\Rightarrow \mathbf{i}+\sqrt{3}\mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{B}+\mathbf{i} \Rightarrow \mathbf{B}=2\mathbf{j}$$

So, $\mathbf{r}=\mathrm{e}^{-\frac{1}{2}t}(-2\cos\frac{\sqrt{3}}{2}t\mathbf{i}+2\sin\frac{\sqrt{3}}{2}t\mathbf{j})$

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Exercise B, Question 5

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} = \mathbf{0}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{0}$ and $\frac{d\mathbf{r}}{dt} = 4\mathbf{i}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + 2\frac{d\mathbf{r}}{dt} = \mathbf{0} \Rightarrow \lambda^2 + 2\lambda = 0$$

$$\Rightarrow \lambda(\lambda + 2) = 0$$

$$\Rightarrow \lambda = 0 \text{ or } -2$$

General Solution is $\mathbf{r} = \mathbf{A} + \mathbf{B}e^{-2t}$

$t = 0, \mathbf{r} = \mathbf{0} \Rightarrow \mathbf{0} = \mathbf{A} + \mathbf{B}$ ①

$$\frac{d\mathbf{r}}{dt} = -2\mathbf{B}e^{-2t}$$

$$t = 0, \frac{d\mathbf{r}}{dt} = 4\mathbf{i} \Rightarrow 4\mathbf{i} = -2\mathbf{B} \Rightarrow \mathbf{B} = -2\mathbf{i}$$

$$\Rightarrow \mathbf{A} = 2\mathbf{i}$$

So, $\mathbf{r} = 2\mathbf{i} - 2\mathbf{i}e^{-2t}$

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Exercise B, Question 6

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} + \mathbf{r} = 10e^{2t}\mathbf{i}.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i}$ and $\frac{d\mathbf{r}}{dt} = 2\mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} + \mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = \mathbf{A}\cos t + \mathbf{B}\sin t$$

For P.I. try $\mathbf{r} = \mathbf{C}e^{2t}$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = 2\mathbf{C}e^{2t} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} = 4\mathbf{C}e^{2t}$$

So $4\mathbf{C}e^{2t} + \mathbf{C}e^{2t} \equiv 10e^{2t}\mathbf{i}$

$$\Rightarrow \mathbf{C} = 2\mathbf{i}$$

$$\text{P.I. is } \mathbf{r} = 2\mathbf{i}e^{2t}$$

$\therefore$ general solution is $\mathbf{r} = \mathbf{A}\cos t + \mathbf{B}\sin t + 2e^{2t}\mathbf{i}$

$$t = 0, \mathbf{r} = \mathbf{i} \Rightarrow \mathbf{i} = \mathbf{A} + \mathbf{0} + 2\mathbf{i} \Rightarrow \mathbf{A} = -\mathbf{i}$$

$$\frac{d\mathbf{r}}{dt} = -\mathbf{A}\sin t + \mathbf{B}\cos t + 4e^{2t}\mathbf{i}$$

$$t = 0, \frac{d\mathbf{r}}{dt} = 2\mathbf{j} \Rightarrow 2\mathbf{j} = \mathbf{B} + 4\mathbf{i} \Rightarrow \mathbf{B} = -4\mathbf{i} + 2\mathbf{j}$$

So $\mathbf{r} = -\cos t\mathbf{i} + \sin t(-4\mathbf{i} + 2\mathbf{j}) + 2e^{2t}\mathbf{i}$

i.e. $\mathbf{r} = (2e^{2t} - \cos t - 4\sin t)\mathbf{i} + 2\sin t\mathbf{j}$

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Exercise B, Question 7

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + 5\mathbf{r} = 10\sin t\mathbf{i}.$$

Given that when $t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} + 5\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda + 5 = 0$$

$$\Rightarrow (\lambda - 1)^2 = -4$$

$$\Rightarrow \lambda = 1 \pm 2\mathrm{i}$$

$$\Rightarrow \text{C.F. is } \mathbf{r} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t)$$

For P.I. try

$$\mathbf{r} = \mathbf{p}\sin t + \mathbf{q}\cos t$$

$$\Rightarrow \dot{\mathbf{r}} = \mathbf{p}\cos t - \mathbf{q}\sin t$$

$$\Rightarrow \ddot{\mathbf{r}} = -p\sin t - \mathbf{q}\cos t$$

So, $-\mathbf{p}\sin t - \mathbf{q}\cos t - 2(\mathbf{p}\cos t - \mathbf{q}\sin t) + 5(\mathbf{p}\sin t + \mathbf{q}\cos t) \equiv 10\sin t\mathbf{i}$

$$4\mathbf{p}\sin t + 2\mathbf{q}\sin t + 4\mathbf{q}\cos t - 2\mathbf{p}\cos t \equiv 10\sin t\mathbf{i}$$

$$4\mathbf{p} + 2\mathbf{q} = 10\mathbf{i} \quad \text{and} \quad 4\mathbf{q} - 2\mathbf{p} = \mathbf{0} \Rightarrow \mathbf{p} = 2\mathbf{q}$$

$$10\mathbf{q} = 10i$$

$$\mathbf{q} = \mathbf{i} \Rightarrow \mathbf{p} = 2\mathbf{i}$$

So P.I. is $\mathbf{r} = 2\mathbf{i}\sin t + \mathbf{i}\cos t$

General solution is

$$\mathbf{r} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t) + 2\mathbf{i}\sin t + \mathbf{i}\cos t$$

$$t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{j} \Rightarrow 2\mathbf{i} - \mathbf{j} = \mathbf{A} + \mathbf{i} \Rightarrow \mathbf{A} = \mathbf{i} - \mathbf{j}$$

$$\frac{d\mathbf{r}}{dt} = e^t(\mathbf{A}\cos 2t + \mathbf{B}\sin 2t) + e^t(-2\mathbf{A}\sin 2t + 2\mathbf{B}\cos 2t) + 2\mathbf{i}\cos t - \mathbf{i}\sin t$$

$$t = 0, \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{i} + \mathbf{j} = \mathbf{A} + 2\mathbf{B} + 2\mathbf{i}$$

So $\mathbf{i} + \mathbf{j} = \mathbf{i} - \mathbf{j} + 2\mathbf{B} + 2\mathbf{i}$

$\mathbf{j} - \mathbf{i} = \mathbf{B}$

So $\mathbf{r} = e^t[(\mathbf{i} - \mathbf{j})\cos 2t + (-\mathbf{i} + \mathbf{j})\sin 2t)] + 2\mathbf{i}\sin t + \mathbf{i}\cos t$

i.e. $\mathbf{r} = \mathbf{i}\left(e^t\cos 2t - e^t\sin 2t + 2\sin t + \cos t\right) + \mathbf{j}(e^t\sin 2t - e^t\cos 2t)$

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Exercise B, Question 8

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} + 4\mathbf{r} = 8\mathbf{i}.$$

At $t = 0$, the particle is at the point with position vector $(2\mathbf{i} - \mathbf{k})$m moving with velocity $(\mathbf{i} + 2\mathbf{j})$ m s^{-1}.

Find $\mathbf{r}$ in terms of t.

Solution:

$\frac{d^2\mathbf{r}}{dt^2} - 4\frac{d\mathbf{r}}{dt} + 4\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 4\lambda + 4 = 0$

$\Rightarrow (\lambda - 2)^2 = 0$

$\Rightarrow \lambda = 2$ (twice)

$\Rightarrow$ C.F. is $\mathbf{r} = e^{2t}(\mathbf{A} + \mathbf{B}t)$

For P.I. try $\mathbf{r} = \mathbf{C}$

$\dot{\mathbf{r}} = \mathbf{0}$

$\ddot{\mathbf{r}} = \mathbf{0}$

So $4\mathbf{C} = 8\mathbf{i} \Rightarrow \mathbf{C} = 2\mathbf{i}$

G.S. is $\mathbf{r} = e^{2t}(\mathbf{A} + \mathbf{B}t) + 2\mathbf{i}$

$t = 0, \mathbf{r} = 2\mathbf{i} - \mathbf{k} \Rightarrow 2\mathbf{i} - \mathbf{k} = \mathbf{A} + 2\mathbf{i} \Rightarrow \mathbf{A} = -\mathbf{k}$

$\frac{d\mathbf{r}}{dt} = e^{2t}\mathbf{B} + 2e^{2t}(\mathbf{A} + \mathbf{B}t)$

$t = 0, \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{i} + 2\mathbf{j} = \mathbf{B} + 2\mathbf{A} \Rightarrow \mathbf{B} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

So $\mathbf{r} = e^{2t}[-\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})] + 2\mathbf{i}$

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Exercise B, Question 9

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres and satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 4\mathbf{r} = 12t\mathbf{i} - 2\mathbf{j}.$$

At $t = 0$, the particle is at the point with position vector $(\mathbf{i}+\mathbf{k})$m moving with velocity $2\mathbf{j}$ m s^{-1}.

Find $\mathbf{r}$ in terms of t.

Solution:

$\frac{d^2\mathbf{r}}{dt^2} - 4\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 4 = 0 \Rightarrow \lambda = \pm 2$

$\Rightarrow$ C.F. is $\mathbf{r} = \mathbf{A}e^{2t} + \mathbf{B}e^{-2t}$

For P.I. try $\mathbf{r} = \mathbf{C}t + \mathbf{D}$

$\dot{\mathbf{r}} = \mathbf{C}$

$\ddot{\mathbf{r}} = \mathbf{0}$

So $\mathbf{0} - 4(\mathbf{C}t + \mathbf{D}) \equiv 12t\mathbf{i} - 2\mathbf{j}$

$\Rightarrow -4\mathbf{C} = 12\mathbf{i} \Rightarrow \mathbf{c} = -3\mathbf{i}$

$-4\mathbf{D} = -2\mathbf{j} \Rightarrow \mathbf{D} = \frac{1}{2}\mathbf{j}$

So P.I. is $\mathbf{r} = -3t\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\therefore$ G.S. is $\mathbf{r} = \mathbf{A}e^{2t} + \mathbf{B}e^{-2t} - 3t\mathbf{i} + \frac{1}{2}\mathbf{j}$

$t = 0, \mathbf{r} = \mathbf{i} + \mathbf{k} \Rightarrow \mathbf{i} + \mathbf{k} = \mathbf{A} + \mathbf{B} + \frac{1}{2}\mathbf{j}$

$\Rightarrow \mathbf{i} - \frac{1}{2}\mathbf{j} + \mathbf{k} = \mathbf{A} + \mathbf{B}$ ①

$\frac{d\mathbf{r}}{dt} = 2\mathbf{A}e^{2t} - 2\mathbf{B}e^{-2t} - 3\mathbf{i}$

$t = 0, \frac{d\mathbf{r}}{dt} = 2\mathbf{j} \Rightarrow 2\mathbf{j} = 2\mathbf{A} - 2\mathbf{B} - 3\mathbf{i}$

$\Rightarrow 3\mathbf{i} + 2\mathbf{j} = 2\mathbf{A} - 2\mathbf{B}$ ②

$2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = 2\mathbf{A} + 2\mathbf{B}$ ①

Add,

$5\mathbf{i} + \mathbf{j} + 2\mathbf{k} = 4\mathbf{A}$

$\frac{5}{4}\mathbf{i} + \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k} = \mathbf{A}$

$\Rightarrow \frac{-1}{4}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{2}\mathbf{k} = \mathbf{B}$

$$\text{So, } \mathbf{r} = \begin{pmatrix} \frac{5}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} e^{2t} + \begin{pmatrix} -\frac{1}{4} \\ -\frac{3}{4} \\ \frac{1}{2} \end{pmatrix} e^{-2t} + \begin{pmatrix} -3t \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

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Exercise B, Question 10

Question:

The position vector of a particle P at time t seconds is $\mathbf{r}$ metres. The motion of P is modelled by the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} - 8\mathbf{r} = (9\mathbf{i} + 18\mathbf{j})e^t.$$

Given that when $t = 0, \mathbf{r} = \mathbf{i} + 2\mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j}$, find $\mathbf{r}$ in terms of t.

Solution:

$$\frac{d^2\mathbf{r}}{dt^2} - 2\frac{d\mathbf{r}}{dt} - 8\mathbf{r} = \mathbf{0} \Rightarrow \lambda^2 - 2\lambda - 8 = 0$$

$\Rightarrow (\lambda - 4)(\lambda + 2) = 0$

$\Rightarrow \lambda = 4 \text{ or } -2$

$\Rightarrow$ C.F. is $\mathbf{r} = \mathbf{A}e^{4t} + e^{-2t}\mathbf{B}$

For P.I. try

$\mathbf{r} = \mathbf{C}e^t$

$\dot{\mathbf{r}} = \mathbf{C}e^t$

$\ddot{\mathbf{r}} = \mathbf{C}e^t$

$\mathbf{C}e^t - 2\mathbf{C}e^t - 8\mathbf{C}e^t \equiv (9\mathbf{i} + 18\mathbf{j})e^t$

$-9\mathbf{C} = 9\mathbf{i} + 18\mathbf{j} \Rightarrow \mathbf{C} = -\mathbf{i} - 2\mathbf{j}$

So P.I. is $\mathbf{r} = (-\mathbf{i} - 2\mathbf{j})e^t$

$\therefore$ G.S. is $\mathbf{r} = \mathbf{A}e^{4t} + \mathbf{B}e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$

$t = 0, \mathbf{r} = \mathbf{i} + 2\mathbf{j} \Rightarrow \mathbf{i} + 2\mathbf{j} = \mathbf{A} + \mathbf{B} + (-\mathbf{i} - 2\mathbf{j})$

$\Rightarrow 2\mathbf{i} + 4\mathbf{j} = \mathbf{A} + \mathbf{B}$ ①

$\frac{d\mathbf{r}}{dt} = 4\mathbf{A}e^{4t} - 2\mathbf{B}e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$

$t = 0, \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + \mathbf{j} \Rightarrow 2\mathbf{i} + \mathbf{j} = 4\mathbf{A} - 2\mathbf{B} - \mathbf{i} - 2\mathbf{j}$

$\Rightarrow 3\mathbf{i} + 3\mathbf{j} = 4\mathbf{A} - 2\mathbf{B}$ ②

$2\mathbf{i} + 4\mathbf{j} = \mathbf{A} + \mathbf{B}$ ①

$3\mathbf{i} + 3\mathbf{j} = 4\mathbf{A} - 2\mathbf{B}$ ②

$4\mathbf{i} + 8\mathbf{j} = 2\mathbf{A} + 2\mathbf{B}$ ② × ①

$7\mathbf{i} + 11\mathbf{j} = 6\mathbf{A}$

$\frac{7}{6}\mathbf{i} + \frac{11}{6}\mathbf{j} = \mathbf{A} \Rightarrow \mathbf{B} = \frac{5}{6}\mathbf{i} + \frac{13}{6}\mathbf{j}$

$$\mathbf{r} = \left(\frac{7}{6}\mathbf{i} + \frac{11}{6}\mathbf{j}\right)e^{4t} + \left(\frac{5}{6}\mathbf{i} + \frac{13}{6}\mathbf{j}\right)e^{-2t} + (-\mathbf{i} - 2\mathbf{j})e^t$$

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Exercise C, Question 1

Question:

In each of the following cases calculate the work done by the force $\mathbf{F}$ as it moves its point of application through the displacement $\mathbf{d}$:

a $\mathbf{F} = (3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{N}, \mathbf{d} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{m}$.

b $\mathbf{F} = (-4\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}, \mathbf{d} = (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})\text{m}$.

c $\mathbf{F} = (\mathbf{i} - 2\mathbf{k})\text{N}, \mathbf{d} = (\mathbf{j} - 3\mathbf{k})\text{m}$.

Solution:

a $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 3 + 1 + 4 = 8\text{ J}$

b $\begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = -12 + 1 + 8 = -3\text{ J}$

c $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = 6\text{ J}$

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Exercise C, Question 2

Question:

In each of the following cases calculate the work done by the force $\mathbf{F}$ as it moves its point of application from the point A with position vector $\mathbf{r}_A$ to the point B with position vector $\mathbf{r}_B$:

a $\mathbf{F} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{N}, \mathbf{r}_A = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{ m}, \mathbf{r}_B = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k})\text{ m}$.

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j} + \mathbf{k})\text{ m}, \mathbf{r}_B = (4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})\text{ m}$.

c $\mathbf{F} = (\mathbf{i} - \mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j})\text{ m}, \mathbf{r}_B = (3\mathbf{i} - \mathbf{j} + \mathbf{k})\text{ m}$.

Solution:

a $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = 1 - 4 - 6 = -9\text{ J}$

b $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = 4 + 2 - 9 = -3\text{ J}$

c $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 + 0 - 1 = 0\text{ J}$

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Exercise C, Question 3

Question:

In each of the following cases a particle P of mass 0.5 kg is moved from the point A with position vector $\mathbf{r}_A$ to the point B with position vector $\mathbf{r}_B$ by a force $\mathbf{F}$. Assuming that in each case there are no other forces, apart from $\mathbf{F}$, doing work on P and that the speed of P at the point A is 4 m s^{-1}, find in each case the speed of P when it reaches the point B:

a $\mathbf{F} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}, \mathbf{r}_A = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \text{ m}, \mathbf{r}_B = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \text{ m}$.

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j} - 3\mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) \text{ m}, \mathbf{r}_B = (4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) \text{ m}$.

c $\mathbf{F} = (\mathbf{i} - \mathbf{k})\text{N}, \mathbf{r}_A = (2\mathbf{i} - \mathbf{j}) \text{ m}, \mathbf{r}_B = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \text{ m}$.

Solution:

a work done = K.E. gain

$$\begin{pmatrix} 1 \\ -1 \\ +2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$1 + 4 + 6 = \frac{1}{4}(v^2 - 16)$$

$$60 = v^2$$

$$\sqrt{60} \text{ m s}^{-1} = v$$

b

$$\begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$4 + 2 + 9 = \frac{1}{4}(v^2 - 16)$$

$$76 = v^2$$

$$\sqrt{76} \text{ m s}^{-1} = v$$

c

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \times 0.5(v^2 - 4^2)$$

$$1 + 0 - 1 = \frac{1}{4}(v^2 - 16)$$

$$16 = v^2$$

$$4 \text{ m s}^{-1} = v$$

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Exercise C, Question 4

Question:

Forces of magnitudes 6 N, 7 N and 9 N act in the directions $2\mathbf{i}+2\mathbf{j}+\mathbf{k}$, $6\mathbf{i}-3\mathbf{j}+2\mathbf{k}$, and $7\mathbf{i}+4\mathbf{j}-4\mathbf{k}$, respectively. The three forces act on a particle causing a displacement of $(34\mathbf{i}+10\mathbf{j})\text{m}$.

a Find the work done by each force.

b Verify that the total work done by all three forces is equal to the work done by the resultant force.

Solution:

$$\mathbf{F}_1 = 6\times\frac{1}{\sqrt{2^2+2^2+1^2}}\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}4\\4\\2\end{pmatrix}$$

$$\mathbf{F}_2 = 7\times\frac{1}{\sqrt{6^2+(-3)^2+2^2}}\begin{pmatrix}6\\-3\\2\end{pmatrix}=\begin{pmatrix}6\\-3\\2\end{pmatrix}$$

$$\mathbf{F}_3 = 9\times\frac{1}{\sqrt{7^2+4^2+(-4)^2}}\begin{pmatrix}7\\4\\-4\end{pmatrix}=\begin{pmatrix}7\\4\\-4\end{pmatrix}$$

a $\begin{pmatrix}4\\4\\2\end{pmatrix}\cdot\begin{pmatrix}34\\10\\0\end{pmatrix}=136+40=176\text{ J}$

$\begin{pmatrix}6\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}34\\10\\0\end{pmatrix}=204-30=174\text{ J}$

$\begin{pmatrix}7\\4\\-4\end{pmatrix}\cdot\begin{pmatrix}34\\10\\0\end{pmatrix}=238+40=278\text{ J}$

b Total work done $=628\text{ J}$

$$\rho=\begin{pmatrix}4\\4\\2\end{pmatrix}+\begin{pmatrix}6\\-3\\2\end{pmatrix}+\begin{pmatrix}7\\4\\-4\end{pmatrix}=\begin{pmatrix}17\\5\\0\end{pmatrix}$$

$$\begin{pmatrix}17\\5\\0\end{pmatrix}\cdot\begin{pmatrix}34\\10\\0\end{pmatrix}=578+50=628\text{ J}$$

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Exercise C, Question 5

Question:

In each of the following cases the force $\mathbf{F}$ acts through the point with position vector $\mathbf{r}$ relative to the origin O. Find the vector moment of $\mathbf{F}$ about O.

a $\mathbf{F} = (\mathbf{i} + 2\mathbf{j})\text{N}, \mathbf{r} = (\mathbf{i} - \mathbf{j})$ m .

b $\mathbf{F} = (\mathbf{i} + 2\mathbf{k})\text{N}, \mathbf{r} = (2\mathbf{i} - \mathbf{k})$ m .

c $\mathbf{F} = (\mathbf{i} - \mathbf{j})\text{N}, \mathbf{r} = 3\mathbf{k}$ m .

d $\mathbf{F} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{N}, \mathbf{r} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$ m .

Solution:

a $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \text{Nm}$

b $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} \text{Nm}$

c $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \text{Nm}$

d $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix} \text{Nm}$

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Exercise C, Question 6

Question:

In each of the following cases the force $\mathbf{F}$ acts through the point with position vector $\mathbf{r}$ relative to the origin O. Find the vector moment of $\mathbf{F}$ about the point A.

a $\mathbf{F} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\text{N}, \mathbf{r} = (3\mathbf{i} - 2\mathbf{j})\text{ m}, A(0,1,0)$,

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j})\text{N}, \mathbf{r} = (\mathbf{i} + 2\mathbf{j})\text{ m}, A(0,1)$,

c $\mathbf{F} = (\mathbf{i} + 2\mathbf{k})\text{N}, \mathbf{r} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\text{ m}, A(2,0,2)$.

Solution:

a $\begin{pmatrix} 3-0 \\ -2-1 \\ 0-0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix}$ Nm

b $\begin{pmatrix} 1-0 \\ 2-1 \\ 0-0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$ Nm

c $\begin{pmatrix} 1-2 \\ 1-0 \\ -1-2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ Nm

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Exercise C, Question 7

Question:

A force $\mathbf{F}$ acts through a point with position vector $\mathbf{p}$. Find, in terms of $\mathbf{F}$, $\mathbf{p}$ and $\mathbf{q}$, the vector moment of $\mathbf{F}$ about the point with position vector $\mathbf{q}$.

Solution:

$$(\mathbf{p}-\mathbf{q})\times\mathbf{F}=\mathbf{p}\times\mathbf{F}-\mathbf{q}\times\mathbf{F}$$

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Exercise C, Question 8

Question:

In each of the following cases the force **F** has vector moment **n** about the origin O.
Find a vector equation of the line of action of **F** if

a $\mathbf{F} = (\mathbf{i} + \mathbf{j})\text{N}, \mathbf{n} = 4\mathbf{k}$ Nm ,

b $\mathbf{F} = (2\mathbf{i} - \mathbf{j})\text{N}, \mathbf{n} = (\mathbf{i} + 2\mathbf{j})$ Nm ,

c $\mathbf{F} = (\mathbf{i} + \mathbf{j} - \mathbf{k})\text{N}, \mathbf{n} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ Nm .

Solution:

a $\mathbf{r}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix}$ i.e. $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}-z\\z\\x-y\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix}$$

$$\Rightarrow \quad z=0, x-y=4$$

Let $y=0, x=4$

Equation is $\mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$

b $\mathbf{r}\times\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$ i.e. $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}z\\2z\\-x-2y\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$\Rightarrow z=1, -x-2y=0$$

Let $y=0, x=0$

equation is $\mathbf{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\0\end{pmatrix}$

c $\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$ i.e. $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}-y-z\\z+x\\x-y\end{pmatrix}=\begin{pmatrix}3\\-2\\1\end{pmatrix}$$

$$\Rightarrow -y-z=3; z+x=-2; x-y=1$$

Let $y=0 \Rightarrow z=-3$ and $x=1$

Equation is $\mathbf{r}=\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$

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Exercise C, Question 9

Question:

In each of the following cases the force **F** acts through a point P. Find a vector equation of the axis through the point Q about which the moment of **F** is calculated.

a $\mathbf{F} = (\mathbf{i} - 2\mathbf{j})\text{N}, P(0,1,0), Q(0,0,0)$,

b $\mathbf{F} = (\mathbf{j} + 2\mathbf{k})\text{N}, P(\mathbf{i} + \mathbf{k})\text{m}, Q(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\text{m}$,

c $\mathbf{F} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\text{N}, P(\mathbf{i} + \mathbf{j} + 2\mathbf{k})\text{m}, Q(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})\text{m}$.

Solution:

a $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ is equation of axis.

b $\begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix}$ is equation of axis.

c $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix}$

$\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 8 \\ 4 \end{pmatrix}$ is equation of axis.

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Exercise C, Question 10

Question:

The moment of a non-zero force **F** about a point P is equal to its moment about another point Q. Show that the line of action of **F** is parallel to the line PQ.

Solution:

Let **T** be the position vector of a point on the line of action of **F**.

Let $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$

Then the moment of **F** about **P** is $(\mathbf{T} - \mathbf{P}) \times \mathbf{F}$ and above Q is $(\mathbf{r} - \mathbf{q}) \times \mathbf{F}$

Therefore $(\mathbf{r} - \mathbf{p}) \times \mathbf{F} = (\mathbf{r} - \mathbf{q}) \times \mathbf{F} \Rightarrow \mathbf{T} \times \mathbf{F} - \mathbf{p} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} - \mathbf{q} \times \mathbf{F}$

$$\Rightarrow \mathbf{p} \times \mathbf{F} = \mathbf{q} \times \mathbf{F}$$

$$\Rightarrow \mathbf{p} \times \mathbf{F} - \mathbf{q} \times \mathbf{F} = \mathbf{0}$$

$$\Rightarrow (\mathbf{p} - \mathbf{q}) \times \mathbf{F} = \mathbf{0}$$

$$\Rightarrow \overrightarrow{QP} \times \mathbf{F} = \mathbf{0}$$

Since $\mathbf{F} \neq \mathbf{0}$, $\overrightarrow{QP}$ must be parallel to **F**

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Exercise D, Question 1

Question:

Prove that the following system of forces reduces to a couple:

$\mathbf{F_1} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_1} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{m}$,

$\mathbf{F_2} = (-3\mathbf{i} + \mathbf{j} - 3\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_2} = (3\mathbf{i} + \mathbf{k})\text{m}$,

$\mathbf{F_3} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_3} = (2\mathbf{i} + \mathbf{j} - \mathbf{k})\text{m}$,

$\mathbf{F_4} = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_4} = (\mathbf{j} - 2\mathbf{k})\text{m}$.

Solution:

$$\sum \mathbf{F}_i = \begin{pmatrix}1\\2\\-3\end{pmatrix} + \begin{pmatrix}-3\\1\\-3\end{pmatrix} + \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}1\\-1\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\-3\end{pmatrix} + \begin{pmatrix}3\\0\\1\end{pmatrix} \times \begin{pmatrix}-3\\1\\-3\end{pmatrix} + \begin{pmatrix}2\\1\\-1\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} + \begin{pmatrix}0\\1\\-2\end{pmatrix} \times \begin{pmatrix}1\\-2\\4\end{pmatrix}$$

$$= \begin{pmatrix}-1\\5\\3\end{pmatrix} + \begin{pmatrix}-1\\6\\3\end{pmatrix} + \begin{pmatrix}1\\-5\\-3\end{pmatrix} + \begin{pmatrix}0\\-2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\4\\2\end{pmatrix}$$

Hence system is a couple of vector moment $(-\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})\text{Nm}$.

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Exercise D, Question 2

Question:

Prove that the following system of forces is in equilibrium:

$\mathbf{F}_1 = (\mathbf{i} - \mathbf{j})$N acting at the point with position vector $\mathbf{r}_1 = (\mathbf{i} + \mathbf{k})$m ,

$\mathbf{F}_2 = (2\mathbf{j} + \mathbf{k})$N acting at the point with position vector $\mathbf{r}_2 = (\mathbf{i} - 2\mathbf{j})$m ,

$\mathbf{F}_3 = (-2\mathbf{i} - \mathbf{j})$N acting at the point with position vector $\mathbf{r}_3 = (3\mathbf{i} + \mathbf{j} + \mathbf{k})$m ,

$\mathbf{F}_4 = (\mathbf{i} - \mathbf{k})$N acting at the point with position vector $\mathbf{r}_4 = 2\mathbf{i}$ m .

Solution:

$$\sum \mathbf{F}_i = \begin{pmatrix}1\\-1\\0\end{pmatrix} + \begin{pmatrix}0\\2\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\-2\\0\end{pmatrix} \times \begin{pmatrix}0\\2\\1\end{pmatrix} + \begin{pmatrix}3\\1\\1\end{pmatrix} \times \begin{pmatrix}-2\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\-1\end{pmatrix} + \begin{pmatrix}-2\\-1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\\-1\end{pmatrix} + \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\end{pmatrix}$$ Hence system is in equilibrium.

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Exercise D, Question 3

Question:

A system consists of three forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_3$. The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act at the points $(2, -1, 0)$ and $(2, 0, 1)$ respectively. $\mathbf{F}_1 = (\mathbf{i} - 2\mathbf{j})\text{N}; \mathbf{F}_2 = (\mathbf{j} + \mathbf{k})\text{N}; \mathbf{F}_3$ has magnitude $\sqrt{11}\,\text{N}$ and acts along the line whose equation is $\mathbf{r} = 5\mathbf{i} - \mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} - \mathbf{k})$.
Prove that the system reduces to a single force and find a vector equation for its line of action.

Solution:

$$\mathbf{F}_3 = \frac{1}{\sqrt{3^2+1^2+(-1)^2}}\cdot\sqrt{11}\begin{pmatrix}3\\1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$$

$$\sum \mathbf{F}_i = \begin{pmatrix}1\\-2\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}3\\1\\-1\end{pmatrix}=\begin{pmatrix}4\\0\\0\end{pmatrix}$$

$$\sum \mathbf{r}_i\times\mathbf{F}_i = \begin{pmatrix}2\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\-2\\0\end{pmatrix}+\begin{pmatrix}2\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}+\begin{pmatrix}5\\0\\-1\end{pmatrix}\times\begin{pmatrix}3\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\-3\end{pmatrix}+\begin{pmatrix}-1\\-2\\2\end{pmatrix}+\begin{pmatrix}1\\2\\5\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\4\end{pmatrix}$$

$$\sum \mathbf{r}_i\times\mathbf{F}_i\cdot\sum\mathbf{F}_i = \begin{pmatrix}0\\0\\4\end{pmatrix}\cdot\begin{pmatrix}4\\0\\0\end{pmatrix}=0$$

Hence the resultant is coplanar with the couple
Hence system reduces to a single force.

$$\mathbf{r}\times\begin{pmatrix}4\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix} \text{ for a point } \mathbf{r} \text{ on the line of action}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}4\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix}$$

$$\begin{pmatrix}0\\4z\\-4y\end{pmatrix}=\begin{pmatrix}0\\0\\4\end{pmatrix}\Rightarrow z=0,\ y=-1\quad x=\lambda \text{ (anything)}$$

$$\therefore \mathbf{r} = \begin{pmatrix}0\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix} \text{ is vector equation.}$$

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Exercise D, Question 4

Question:

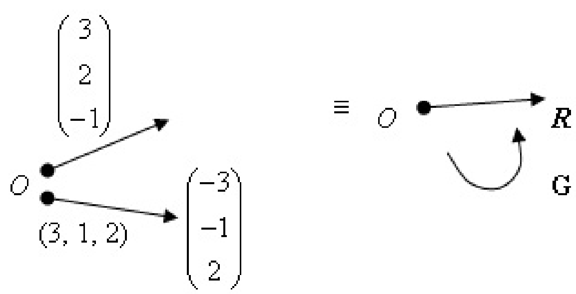
The line of action of a force $(3\mathbf{i}+2\mathbf{j}-\mathbf{k})$N passes through the origin O and the line of action of a force $(-3\mathbf{i}-\mathbf{j}+2\mathbf{k})$ N passes through the point with position vector $(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$m .

a Reduce the system of two forces to a single force acting through the origin O together with a couple.

b Find the magnitude of the couple.

Solution:

a



$$\mathbf{R}=\begin{pmatrix}3\\2\\-1\end{pmatrix}+\begin{pmatrix}-3\\-1\\2\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Comparing moments about O,

$$\begin{pmatrix}3\\1\\2\end{pmatrix}\times\begin{pmatrix}-3\\-1\\2\end{pmatrix}=\mathbf{G}$$

$$\begin{pmatrix}4\\-12\\0\end{pmatrix}=\mathbf{G}$$

Single force is $(\mathbf{j}+\mathbf{k})$N

Couple has vector moment $(4\mathbf{i}-12\mathbf{j})$Nm

b $|\mathbf{G}|=\sqrt{4^2+(-12)^2}=\sqrt{160}=4\sqrt{10}$Nm

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Exercise D, Question 5

Question:

Two forces $4\mathbf{j}$ N and $3\mathbf{k}$ N act through the points with position vectors $(\mathbf{i}+\mathbf{j})$ m and $(\mathbf{j}+\mathbf{k})$ m respectively. A third force acts through the point with position vector $(\mathbf{i}+\mathbf{k})$ m and is such that the three forces are equivalent to a couple. Find the vector moment and the magnitude of this couple.

Solution:

$$4\mathbf{j}+3\mathbf{k}+\mathbf{F}_3=\mathbf{0} \quad \text{(since system reduces to a couple)}$$

$$\Rightarrow \mathbf{F}_3=(-4\mathbf{j}-3\mathbf{k})\text{N}$$

$$\mathbf{G}=\sum \mathbf{r}_1 \times \mathbf{f}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\4\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}0\\0\\3\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\-4\\-3\end{pmatrix}$$

$$=\begin{pmatrix}0\\0\\4\end{pmatrix}+\begin{pmatrix}3\\0\\0\end{pmatrix}+\begin{pmatrix}4\\3\\-4\end{pmatrix}$$

$$=\begin{pmatrix}7\\3\\0\end{pmatrix}\text{Nm}=(7\mathbf{i}+3\mathbf{j})\text{Nm}$$

$$|\mathbf{G}|=\sqrt{7^2+3^2}=\sqrt{58}\text{ Nm}$$

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Exercise D, Question 6

Question:

In each of the following cases find the simplest system of forces which is equivalent to the given system:

a $\mathbf{F}_1 = (-2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$N acting at the point with position vector $\mathbf{r}_1 = (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$m ,
$\mathbf{F}_2 = (-2\mathbf{j} - \mathbf{k})$N acting at the point with position vector $\mathbf{r}_2 = 3\mathbf{j}$m ,
$\mathbf{F}_3 = (\mathbf{i} - \mathbf{j} + \mathbf{k})$N acting at the point with position vector $\mathbf{r}_3 = (\mathbf{i} - 2\mathbf{k})$m .

b $\mathbf{F}_1 = (2\mathbf{i} - \mathbf{j})$N acting at the point with position vector $\mathbf{r}_1 = (\mathbf{i} + 2\mathbf{j})$m ,
$\mathbf{F}_2 = (3\mathbf{i} + \mathbf{j})$N acting at the point with position vector $\mathbf{r}_2 = (2\mathbf{i} + 3\mathbf{j})$m ,
$\mathbf{F}_3 = (-\mathbf{i} + \mathbf{j})$N acting at the point with position vector $\mathbf{r}_3 = -2\mathbf{j}$m .

c Forces **PQ**, **QR** and **RP** where the points P, Q and R have position vectors **p**, **q** and **r** respectively.

d Forces **AB**, **BC**, **CD** and **DA** where the $ABCD$ is a regular tetrahedron.

e Force $(\mathbf{i} + 3\mathbf{j})$N acting at the origin O and a couple of vector moment $3\mathbf{k}$ Nm.

Solution:

a $\sum \mathbf{F}_i = \begin{pmatrix}-2\\3\\1\end{pmatrix} + \begin{pmatrix}0\\-2\\-1\end{pmatrix} + \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}-1\\2\\-1\end{pmatrix} \times \begin{pmatrix}-2\\3\\1\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix} \times \begin{pmatrix}0\\-2\\-1\end{pmatrix} + \begin{pmatrix}1\\0\\-2\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}+5\\3\\1\end{pmatrix} + \begin{pmatrix}-3\\0\\0\end{pmatrix} + \begin{pmatrix}-2\\-3\\-1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence system reduces to a single force $(-\mathbf{i}+\mathbf{k})\text{N}$ acting through the origin O.

b $\sum \mathbf{F}_i = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}3\\1\\0\end{pmatrix} + \begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}4\\1\\0\end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}2\\3\\0\end{pmatrix} \times \begin{pmatrix}3\\1\\0\end{pmatrix} + \begin{pmatrix}0\\-2\\0\end{pmatrix} \times \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\-5\end{pmatrix} + \begin{pmatrix}0\\0\\-7\end{pmatrix} + \begin{pmatrix}0\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\0\\-14\end{pmatrix}$$

$$\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}4\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\0\\-14\end{pmatrix} = 0$$

Hence resultant force is coplanar with the couple.
Point on line of action given by

$$\mathbf{r} \times \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-14\end{pmatrix} \text{ i.e. } \begin{pmatrix}x\\y\\z\end{pmatrix} \times \begin{pmatrix}4\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-14\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}-z\\4z\\x-4y\end{pmatrix} = \begin{pmatrix}0\\0\\-14\end{pmatrix} \qquad \Rightarrow \begin{cases} z = 0 \\ x - 4y = -14 \end{cases}$$

Let $y = 0 \Rightarrow x = -14$
Single force $(4\mathbf{i}+\mathbf{j})\text{N}$

$\therefore$ Line of action has equation $\mathbf{r} = \begin{pmatrix} -14 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

c $\sum \mathbf{F}_i = \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \overrightarrow{PR} + \overrightarrow{RP} = \mathbf{0}$ Hence couple or equilibrium

$$\begin{aligned} \sum \mathbf{r}_i \times \mathbf{F}_i &= (\mathbf{p} \times \overrightarrow{PQ}) + (\mathbf{q} \times \overrightarrow{QR}) + (\mathbf{r} \times \overrightarrow{RP}) \\ &= \mathbf{p} \times (\mathbf{q} - \mathbf{p}) + \mathbf{p} \times (\mathbf{r} - \mathbf{q}) + \mathbf{r} \times (\mathbf{p} - \mathbf{r}) \\ &= \mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{r} + \mathbf{r} \times \mathbf{p} \end{aligned}$$

Hence system reduces to a couple of vector moment $(\mathbf{p} \times \mathbf{q}) + (\mathbf{q} + \mathbf{r}) + (\mathbf{r} \times \mathbf{p})$

d $\sum \mathbf{F}_i = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = \mathbf{0}$

$$\begin{aligned} \sum \mathbf{r}_i \times \mathbf{F}_i &= \mathbf{a} \times \overrightarrow{AB} + \mathbf{b} \times \overrightarrow{BC} + \mathbf{c} \times \overrightarrow{CD} + \mathbf{p} \times \overrightarrow{DA} \\ &= \mathbf{a} \times (\mathbf{b} - \mathbf{a}) + \mathbf{b} \times (\mathbf{c} - \mathbf{b}) + \mathbf{c} \times (\mathbf{d} - \mathbf{c}) + \mathbf{d} \times (\mathbf{a} - \mathbf{d}) \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a} \end{aligned}$$

Hence system reduces to a couple of vector moment $(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{d} + \mathbf{d} \times \mathbf{a})$.

e $\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0} + 3\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Since $\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0$ force is coplanar with the couple.

Point on time of action given by

$$\mathbf{r} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \quad \text{i.e.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} -3z \\ z \\ 3x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$\Rightarrow z = 0, 3x - y = 3$

Let $x = 0, y = -3$

$\therefore$ Equation of line of action is $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

Hence system reduces to a force $(\mathbf{i} + 3\mathbf{j})$N acting at the point $(0, -3, 0)$.

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Exercise D, Question 7

Question:

$\mathbf{F}_1 = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_1 = (3\mathbf{i} - \mathbf{k})\text{m}$,

$\mathbf{F}_2 = (-\mathbf{i} - 4\mathbf{j} + \mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_2 = (2\mathbf{i} - 4\mathbf{j})\text{m}$,

$\mathbf{F}_3 = (\mathbf{i} + \mathbf{j} - 2\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_3 = (-3\mathbf{j} + 5\mathbf{k})\text{m}$.

When a fourth force $\mathbf{F}_4$ is added the system is in equilibrium.

a Find the force $\mathbf{F}_4$.

b Find a vector equation of its line of action.

Solution:

a $\sum \mathbf{F}_i = \mathbf{0} \Rightarrow \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mathbf{F}_4 = 0$

$$\Rightarrow \mathbf{F}_4 = \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} \text{N}$$

i.e. $\mathbf{F}_4 = (-3\mathbf{i} + 4\mathbf{j} - \mathbf{k})\text{N}$

b $\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ -9 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \\ -12 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 \\ -6 \\ -12 \end{pmatrix} + \mathbf{r} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -y - 4z \\ -3z + x \\ 4x + 3y \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}$$

$$\Rightarrow -y - 4z = 4$$
$$-3z + x = 6$$
$$4x + 3y = 12$$

Let $x = 0 \Rightarrow y = 4$ and $z = -2$

$\therefore$ Equation of line of action is

$$\mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$$

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Exercise D, Question 8

Question:

A force $(\mathbf{i}-\mathbf{j}+2\mathbf{k})\text{N}$ acts through the point $(-1,-1,1)$. Show that this force is equivalent to an equal force acting through the origin together with a couple. Find the magnitude of this couple.

Solution:

$\begin{pmatrix}-1\\-1\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\2\end{pmatrix}=\mathbf{O}+\mathbf{G}$, comparing moments about O.

$\begin{pmatrix}-1\\3\\2\end{pmatrix}=\mathbf{G}$ so $|\mathbf{G}|=\sqrt{(-1)^2+3^2+2^2}$

$=\sqrt{14}\text{ Nm}$

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Exercise D, Question 9

Question:

Prove that the following system of forces can be reduced to a force together with a non-coplanar couple.

$\mathbf{F_1} = (\mathbf{i}+\mathbf{j})\text{N}$ acting at the point with position vector $\mathbf{r_1} = (3\mathbf{i}+\mathbf{j}+\mathbf{k})$ m ,

$\mathbf{F_2} = (\mathbf{i}+\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_2} = \mathbf{i}$ m ,

$\mathbf{F_3} = (2\mathbf{j}-\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r_3} = 2\mathbf{j}$ m .

Solution:

$$\sum \mathbf{F}_i = \begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\3\\0\end{pmatrix}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}3\\1\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$$=\begin{pmatrix}-1\\1\\2\end{pmatrix}+\begin{pmatrix}0\\-1\\0\end{pmatrix}+\begin{pmatrix}-2\\0\\0\end{pmatrix}$$

$$=\begin{pmatrix}-3\\0\\2\end{pmatrix}$$

$$\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}2\\3\\0\end{pmatrix}\cdot\begin{pmatrix}-3\\0\\2\end{pmatrix} = -6 \neq 0$$

Hence, resultant force is *not* coplanar with couple.
So system can be reduced to a force $(2\mathbf{i}+3\mathbf{j})\text{N}$ acting through the origin O together with a couple of vector moment $(-3\mathbf{i}+2\mathbf{k})\text{Nm}$.

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Exercise D, Question 10

Question:

A force $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$N acts through the point (2, 0, 0) together with a couple of vector moment $(2\mathbf{i}-\mathbf{k})$Nm .

a Show that this system cannot be reduced.

b Find an equivalent system where the force acts through the point $(1, -3, 4)$.

Solution:

a $\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\sum \mathbf{F}_i \cdot \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 3 \neq 0$$

Hence, resultant force is non-coplanar with the couple $\therefore$ no further reduction possible.

b Force $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ at $(2, 0, 0)$ + Couple $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \equiv$ Force $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ at $(1, -3, 4)$ + Couple $\mathbf{G}$

$(2,0,0)$ $(1,-3,4)$

Comparing moments about O,

From **a**,

$$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mathbf{G}$$

$$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \mathbf{G}$$

$$\begin{pmatrix} 7 \\ -3 \\ -2 \end{pmatrix} = \mathbf{G}$$

Equivalent system is $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$N acting at $(1, -3, 4)$ plus a couple of moment $(7\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$Nm

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Exercise E, Question 1

Question:

The vertices of a tetrahedron $PQRS$ have position vectors **p**, **q**, **r** and **s** respectively, where

$\mathbf{p} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \mathbf{q} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\mathbf{r} = 4\mathbf{i} + \mathbf{k} \quad \mathbf{s} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Forces of magnitude 30 and $\sqrt{117}$ act along RQ and RS respectively. A third force acts at P. Given that the system reduces to a couple,

a find the magnitude of this couple,

b find the force acting at P,

c find a unit vector along the axis of the couple.

Solution:

Find the third force $\mathbf{F}_3$ first.

a $\overrightarrow{RQ} = \mathbf{q} - \mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix}$

$\mathbf{F}_1 = \frac{1}{5}\begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} \times 30 = \begin{pmatrix} 0 \\ 24 \\ -18 \end{pmatrix}$

$\overrightarrow{RS} = \mathbf{s} - \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}$

$\mathbf{F}_2 = \frac{1}{\sqrt{13}}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix}\sqrt{117} = \frac{1}{\sqrt{13}}\begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} 3\sqrt{13} = \begin{pmatrix} -9 \\ -6 \\ 0 \end{pmatrix}$

$\sum \mathbf{F}_i = \begin{pmatrix} 0 \\ 24 \\ -18 \end{pmatrix} + \begin{pmatrix} -9 \\ -6 \\ 0 \end{pmatrix} + \mathbf{F}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \mathbf{F}_3 = \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$

Sum of moments about R:

$$\overrightarrow{RP} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix} = \left\{\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}\right\} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$$

$$= \begin{pmatrix} -72 \\ 18 \\ 54 \end{pmatrix} = 18\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore$ Magnitude $= 18\sqrt{26}$ Nm

b $\begin{pmatrix} 9 \\ -18 \\ 18 \end{pmatrix}$

c unit vector along axis $\frac{1}{\sqrt{26}}\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

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Exercise E, Question 2

Question:

Two forces $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$N and $(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$N act at the points $D(4,-1,1)$ and $E(3, 1, 6)$ respectively.

a Find the force through the origin O and the couple which together are equivalent to these two forces.

b Find the magnitude of the couple.

c Show that the lines of action of the forces through D and E meet and find the position vector of the point of intersection.

Solution:

a $\sum \mathbf{F}_i = \begin{pmatrix}3\\2\\1\end{pmatrix} + \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}4\\4\\4\end{pmatrix}$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}4\\-1\\1\end{pmatrix} \times \begin{pmatrix}3\\2\\1\end{pmatrix} + \begin{pmatrix}3\\1\\6\end{pmatrix} \times \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}-3\\-1\\11\end{pmatrix} + \begin{pmatrix}-9\\-3\\5\end{pmatrix}$$
$$= \begin{pmatrix}-12\\-4\\16\end{pmatrix} = \mathbf{G}$$

Hence force is $(4\mathbf{i}+4\mathbf{j}+4\mathbf{k})$N through O plus a couple of vector moment $(-12\mathbf{i}-4\mathbf{j}+16\mathbf{k})$Nm .

b $|\mathbf{G}| = 4\sqrt{(-3)^2+(-1)^2+4^2} = 4\sqrt{26}$ Nm

c $\begin{pmatrix}4\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\6\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}$

$\Rightarrow 4+3\lambda = 3+\mu$ ①

$-1+2\lambda = 1+2\mu$ ②

$1+\lambda = 6+3\mu$ ③

②−①: $0+3\lambda = 7+5\mu$ ④

① −④: $4 = -4-4\mu$

$-2 = \mu \Rightarrow \lambda = -1$

Point of intersection has position vector $(\mathbf{i}-3\mathbf{j})$m .

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Exercise E, Question 3

Question:

A system of five forces consists of the forces $\mathbf{F}_1 = (\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})\text{N}, \mathbf{F}_2 = (2\mathbf{i} + 6\mathbf{k})\text{N}$ and $\mathbf{F}_3 = (\mathbf{i} - 2\mathbf{j} - 4\mathbf{k})\text{N}$, all acting through the origin O, together with a force $\mathbf{F}_4 = (\mathbf{i} - 2\mathbf{j} - \mathbf{k})\text{N}$ acting through the point $(-2, 4, 2)$ and a force $\mathbf{F}_5 = (-\mathbf{i} - 2\mathbf{j} - 7\mathbf{k})\text{N}$ acting through the point $(1, -2, -1)$.

a Reduce the system to a force $\mathbf{R}$ acting at O together with a couple $\mathbf{G}$.

b Hence, or otherwise, verify that the system is equivalent to a single force $(4\mathbf{i} - 8\mathbf{j} - 4\mathbf{k})\text{N}$ acting through the point $(1, -1, 1)$.

Solution:

a $$\mathbf{R} = \sum \mathbf{F}_i = \begin{pmatrix}1\\-2\\2\end{pmatrix} + \begin{pmatrix}2\\0\\6\end{pmatrix} + \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \begin{pmatrix}1\\-2\\-1\end{pmatrix} + \begin{pmatrix}-1\\-2\\-7\end{pmatrix} = \begin{pmatrix}4\\-8\\-4\end{pmatrix}$$

$$\mathbf{G} = \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}-2\\4\\2\end{pmatrix} \times \begin{pmatrix}1\\-2\\-1\end{pmatrix} + \begin{pmatrix}1\\-2\\-1\end{pmatrix} \times \begin{pmatrix}-1\\-2\\-7\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}12\\8\\-4\end{pmatrix}$$

$$= \begin{pmatrix}12\\8\\-4\end{pmatrix}$$

$$\mathbf{R} = (4\mathbf{i} - 8\mathbf{j} - 4\mathbf{k})\text{N}; \mathbf{G} = (12\mathbf{i} + 8\mathbf{j} - 4\mathbf{k})\text{Nm}$$

b $$\begin{pmatrix}1\\-1\\1\end{pmatrix} \times \begin{pmatrix}4\\-8\\-4\end{pmatrix} = \begin{pmatrix}12\\8\\-4\end{pmatrix} = \mathbf{G} \quad \text{QED}$$

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Exercise E, Question 4

Question:

A system consists of three forces:

$\mathbf{F}_1 = (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_1 = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$ m ,

$\mathbf{F}_2 = (\mathbf{i} + 3\mathbf{j} - \mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_2 = (\mathbf{j} + 2\mathbf{k})$ m ,

$\mathbf{F}_3 = (s\mathbf{i} + t\mathbf{j} + 2\mathbf{k})\text{N}$ acting at the point with position vector $\mathbf{r}_3 = \mathbf{k}$ m .

a Obtain, in terms of s and t, the equivalent system consisting of a single force $\mathbf{F}$ acting through the origin and a couple of moment $\mathbf{G}$.

b Determine the values of s and t such that $\mathbf{G}$ is parallel to $\mathbf{F}$.

Solution:

a

$$\sum \mathbf{F}_i = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} s \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+s \\ 2+t \\ 3 \end{pmatrix} = \mathbf{F}$$

$$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} s \\ t \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -5 \\ -2 \end{pmatrix} + \begin{pmatrix} -7 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -t \\ s \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -8-t \\ -3+s \\ -3 \end{pmatrix} = \mathbf{G}$$

b $\mathbf{G}$ parallel to $\mathbf{F} \Rightarrow -(2+s) = -8-t \Rightarrow 6 = s-t$

$$-(2+t) = -3+s \Rightarrow 1 = s+t$$

$$\Rightarrow s = \frac{7}{2}; t = \frac{-5}{2}$$

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Exercise E, Question 5

Question:

A force $\mathbf{F_1}$ of magnitude 26 N, acts along the direction of the vector $(4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k})$.
Given that the line of action of $\mathbf{F_1}$ passes through the point $(2, 1, -1)$,

a find the moment of $\mathbf{F_1}$ about O.

A bead moves along a smooth straight wire from the point $P(3, -2, 1)$ to the point $Q(5, -22, 2)$, under the influence of $\mathbf{F_1}$ and the reaction from the wire only.

b Find the work done by $\mathbf{F_1}$ in this motion.

Solution:

a $\mathbf{F}_1 = \frac{1}{13}\begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} \times 26 = \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 18 \\ -56 \\ -20 \end{pmatrix} \text{Nm} = (18\mathbf{i} - 56\mathbf{j} - 20\mathbf{k})\,\text{Nm}$$

b $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -20 \\ 1 \end{pmatrix}$

$$\text{work done} = \begin{pmatrix} 8 \\ -6 \\ 24 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -20 \\ 1 \end{pmatrix} = 16 + 120 + 24 = 160\text{ J}$$

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Exercise E, Question 6

Question:

A system consists of three forces:

$\mathbf{F}_1 = (4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$N acting at the point with position vector $\mathbf{r}_1 = (6\mathbf{i} + 4\mathbf{j} + \mathbf{k})$ m ,

$\mathbf{F}_2 = (\mathbf{i} - 2\mathbf{j} + \mathbf{k})$N acting at the point with position vector $\mathbf{r}_2 = (\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$ m ,

$\mathbf{F}_3 = (-5\mathbf{i} + \mathbf{j} - 3\mathbf{k})$N acting at the point with position vector $\mathbf{r}_3 = (\mathbf{i} + \mathbf{j} + \mathbf{k})$ m .

a Show that this system reduces to a couple and find its magnitude.

The force $\mathbf{F}_3$ is now removed from the system and replaced by the force $\mathbf{F}_4$ such that the forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{F}_4$ are in equilibrium. Find

b the magnitude of $\mathbf{F}_4$,

c a vector equation for the line of action of $\mathbf{F}_4$.

Solution:

a $\sum \mathbf{F}_i = \begin{pmatrix}4\\1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\\1\end{pmatrix} + \begin{pmatrix}-5\\1\\-3\end{pmatrix}$

$= \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}6\\4\\1\end{pmatrix} \times \begin{pmatrix}4\\1\\2\end{pmatrix} + \begin{pmatrix}1\\5\\-2\end{pmatrix} \times \begin{pmatrix}1\\-2\\1\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}-5\\1\\-3\end{pmatrix}$

$= \begin{pmatrix}7\\-8\\-10\end{pmatrix} + \begin{pmatrix}1\\-3\\-7\end{pmatrix} + \begin{pmatrix}-4\\-2\\6\end{pmatrix}$

$= \begin{pmatrix}4\\-13\\-11\end{pmatrix} = \mathbf{G}$

$|\mathbf{G}| = \sqrt{4^2 + (-13)^2 + (-11)^2} = \sqrt{16+169+121} = \sqrt{306}$ Nm

b $\begin{pmatrix}4\\1\\2\end{pmatrix} + \begin{pmatrix}1\\-2\\1\end{pmatrix} + \mathbf{F}_4 = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\mathbf{F}_4 = \begin{pmatrix}-5\\1\\-3\end{pmatrix} \Rightarrow |\mathbf{F}_4| = \sqrt{25+1+9} = \sqrt{35}$ N

c From **a**,

$\sum \mathbf{r}_i \times \mathbf{F}_1 = \begin{pmatrix}7\\-8\\-10\end{pmatrix} + \begin{pmatrix}1\\-3\\-7\end{pmatrix} + \mathbf{r} \times \begin{pmatrix}-5\\1\\-3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} \times \begin{pmatrix}-5\\1\\-3\end{pmatrix} = \begin{pmatrix}-8\\11\\17\end{pmatrix}$

$\begin{pmatrix}-3y-z\\-5z+3x\\x+5y\end{pmatrix} = \begin{pmatrix}-8\\11\\17\end{pmatrix}$

Put $y = 0$: $x = 17, z = 8$

Equation of line of action of $\mathbf{F}_4$ is $\mathbf{r} = \begin{pmatrix}17\\0\\8\end{pmatrix} + \lambda \begin{pmatrix}-5\\1\\-3\end{pmatrix}$

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Exercise E, Question 7

Question:

Three forces of magnitudes 26, $4\sqrt{41}$ and 15 N act respectively along the sides OP, PQ and QO of the triangle POQ where O is the origin. Relative to O, the coordinates of P and Q are (5, 12, 0) and (3, 0, 4) respectively.

a Show that the resultant is $(-3\mathbf{i}-4\mathbf{k})$N.

b Find the magnitude of the moment of the resultant about O.

Solution:

a $$\mathbf{F}_1 = \frac{1}{13}\begin{pmatrix}5\\12\\0\end{pmatrix}\times 26 = \begin{pmatrix}10\\24\\0\end{pmatrix}\text{N}$$

$$\overrightarrow{PQ} = \begin{pmatrix}3\\0\\4\end{pmatrix} - \begin{pmatrix}5\\12\\0\end{pmatrix} = \begin{pmatrix}-2\\-12\\4\end{pmatrix}$$

$$\mathbf{F}_2 = \frac{1}{\sqrt{(-2)^2+(-12)^2+4^2}}\begin{pmatrix}-2\\-12\\4\end{pmatrix}\cdot 4\sqrt{41}$$

$$= \frac{1}{\sqrt{164}}\begin{pmatrix}-2\\-12\\4\end{pmatrix}4\sqrt{41} = \begin{pmatrix}-4\\-24\\8\end{pmatrix}\text{N}$$

$$\overrightarrow{QO} = \begin{pmatrix}-3\\0\\-4\end{pmatrix}$$

$$\mathbf{F}_3 = \frac{1}{5}\begin{pmatrix}-3\\0\\-4\end{pmatrix}\cdot 15 = \begin{pmatrix}-9\\0\\-12\end{pmatrix}\text{N}$$

$$\sum \mathbf{F}_i = \begin{pmatrix}10\\24\\0\end{pmatrix} + \begin{pmatrix}-4\\-24\\8\end{pmatrix} + \begin{pmatrix}-9\\0\\-12\end{pmatrix} = \begin{pmatrix}-3\\0\\-4\end{pmatrix} = (-3\mathbf{i}-4\mathbf{k})\text{N}$$

b $$\sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix}5\\12\\0\end{pmatrix} \times \begin{pmatrix}-4\\-24\\8\end{pmatrix} = \begin{pmatrix}96\\-40\\-72\end{pmatrix} = \mathbf{G}$$

$$|\mathbf{G}| = 8\sqrt{12^2+(-5)^2+(-9)^2}$$
$$= 8\sqrt{144+25+81}$$
$$= 8\sqrt{250}$$
$$= 40\sqrt{10}\text{ Nm}$$

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Exercise E, Question 8

Question:

The point A has coordinates $(2,-5,1)$, the point B has coordinates $(-8,-1,4)$, the point C has coordinates $(0,-13,5)$ and the point D has coordinates $(4,3,-3)$.

a Show that the lines AB and CD intersect at right angles.

b Find the coordinates of the point of intersection.

A force of magnitude F acting in the direction AD moves a particle from B to D.

c Find the work done by the force.

Solution:

a $\overrightarrow{AB} = \begin{pmatrix} -8-2 \\ -1+5 \\ 4-1 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix}$

$\overrightarrow{CD} = \begin{pmatrix} 4-0 \\ 3+13 \\ -3-5 \end{pmatrix} = \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix} = -40+64-24 = 0$

$\therefore$ perpendicular

b $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -13 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 16 \\ -8 \end{pmatrix}$

$\Rightarrow 2-10\lambda = 4\mu$ ①

$-5+4\lambda = -13+16\mu$ ②

$1+3\lambda = 5-8\mu$ ③

$2+6\lambda = 10-16\mu$ ③×3

$-3+10\lambda = -3$

$\lambda = 0 \Rightarrow \mu = \frac{1}{2}$

Point of intersection is $(2,-5,1)$

c $\mathbf{F} = F \times$unit vector along $\overrightarrow{AD}$

$\overrightarrow{AD} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} |\overrightarrow{AD}| = \sqrt{2^2+8^2+(-4)^2}$

$= \sqrt{4+64+16}$

$= \sqrt{84}$

So, $\mathbf{F} = \frac{F}{\sqrt{84}} \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix}$; $\overrightarrow{BD} = \begin{pmatrix} 4+8 \\ 3+1 \\ -3-4 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -7 \end{pmatrix}$

work done $= \frac{F}{\sqrt{84}} \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 4 \\ -7 \end{pmatrix}$

$= \frac{F}{\sqrt{84}}(24+32+28) = F\sqrt{84} = 2F\sqrt{21}$

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Exercise E, Question 9

Question:

The force $\mathbf{F} = (X\mathbf{i} + Y\mathbf{j})\text{N}$ acts through the point with position vector $\mathbf{r} = (x\mathbf{i} - y\mathbf{j})\text{ m}$.

a Prove that this force is equivalent to an equal force at O together with a couple.

Three variable forces $\mathbf{F_1} = 2\cos t\mathbf{i}, \mathbf{F_2} = \cos t\mathbf{i} + 2\sin t\mathbf{j}, \mathbf{F_3} = 3\sin t\mathbf{i} + \cos t\mathbf{j}$ act at the points with position vectors $\mathbf{O}$, $\mathbf{i} + \mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.

If the system is reduced to a single force $\mathbf{R}$ acting at O with a couple $\mathbf{G}$,

b find the values of $\mathbf{R}$ and $\mathbf{G}$.

c reduce an equation of the line of action of the resultant.

d Show that this line passes through a fixed point which is independent of t.

Solution:

a $\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} \times \begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ xY + yX \end{pmatrix} = \mathbf{G}$

Couple is $(xY + yX)\mathbf{k}$ Nm .

b $$\mathbf{R} = \sum \mathbf{F}_i = \begin{pmatrix} 2\cos t \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos t \\ 2\sin t \\ 0 \end{pmatrix} + \begin{pmatrix} 3\sin t \\ \cos t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix} \text{N}$$

$$\mathbf{G} = \sum \mathbf{r}_i \times \mathbf{F}_i = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos t \\ 2\sin t \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3\sin t \\ \cos t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 2\sin t - \cos t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -3\cos t - 6\sin t \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$

$$\begin{pmatrix} -z(2\sin t + \cos t) \\ z(3\cos t + 3\sin t) \\ x(2\sin t + \cos t) - y(3\cos t + 3\sin t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -4\sin t - 4\cos t \end{pmatrix}$$

$\Rightarrow 2x - 3y = -4;\ x - 3y = -4;\ z = 0$

If $x = 0,\ y = \dfrac{4}{3},\ z = 0$ ($\forall$ (for all) t)

Hence line of action is $\mathbf{r} = \begin{pmatrix} 0 \\ \frac{4}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3\cos t + 3\sin t \\ 2\sin t + \cos t \\ 0 \end{pmatrix}$

d This line passes through the fixed point $\begin{pmatrix} 0 \\ \frac{4}{3} \\ 0 \end{pmatrix}$ which is independent of t.

Solutionbank M5
Edexcel AS and A Level Modular Mathematics

Exercise E, Question 10

Question:

A force of unit magnitude has equal vector moments about points with position vectors $\mathbf{j}$ m and $(\mathbf{i}+2\mathbf{j}-\mathbf{k})$ m . Find the possible forces.

Solution:

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}-1\\-1\\1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{0}$$

$$\Rightarrow\begin{pmatrix}-z-y\\x+z\\-y+x\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow y=-z$$

$$x=-z$$

$$\Rightarrow x=y=-z$$

and $x^2+y^2+z^2=1$

$$3x^2=1$$

$$x=\pm\frac{1}{\sqrt{3}}$$

$$x=\frac{1}{\sqrt{3}};y=\frac{1}{\sqrt{3}};z=-\frac{1}{\sqrt{3}}$$

$$x=\frac{-1}{\sqrt{3}};y=\frac{-1}{\sqrt{3}};z=\frac{1}{\sqrt{3}}$$

Possible forces $\pm\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})\text{N}$