1. Two points A and B lie on a smooth horizontal table with $AB = 4a$. One end of a light elastic spring, of natural length a and modulus of elasticity $2mg$, is attached to A. The other end of the spring is attached to a particle P of mass m. Another light elastic spring, of natural length a and modulus of elasticity mg, has one end attached to B and the other end attached to P. The particle P is on the table at rest and in equilibrium.

(a) Show that $AP = \dfrac{5a}{3}$.

(4)

The particle P is now moved along the table from its equilibrium position through a distance $0.5a$ towards B and released from rest at time $t = 0$. At time t, P is moving with speed v and has displacement x from its equilibrium position. There is a resistance to motion of magnitude $4m\omega v$ where $\omega = \sqrt{\left(\dfrac{g}{a}\right)}$.

(b) Show that $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\omega\dfrac{\mathrm{d}x}{\mathrm{d}t} + 3\omega^2 x = 0$.

(5)

(c) Find the velocity, $\dfrac{\mathrm{d}x}{\mathrm{d}t}$, of P in terms of a, ω and t.

(8)
(Total 17 marks)

2. A light elastic spring AB has natural length $2a$ and modulus of elasticity $2mn^2a$, where n is a constant. A particle P of mass m is attached to the end A of the spring. At time $t = 0$, the spring, with P attached, lies at rest and unstretched on a smooth horizontal plane. The other end B of the spring is then pulled along the plane in the direction AB with constant acceleration f. At time t the extension of the spring is x.

(a) Show that $$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + n^2x = f.$$

(6)

(b) Find x in terms of n, f and t.

(8)

Hence find

(c) the maximum extension of the spring,

(3)

(d) the speed of *P* when the spring first reaches its maximum extension.

(2)
(Total 19 marks)

1. (a)

$T_1 = \frac{2mge}{a}$; $T_2 = \frac{mg(2a-e)}{a}$ B1 (either)

$T_1 = T_2$

$2e = (2a - e)$ M1 A1

$e = \frac{2a}{3}$

$AP = a + \frac{2a}{3} = \frac{5a}{3}$ * * A1 4

(b)

$T_2 - T_1 - 4m\omega\dot{x} = m\ddot{x}$

$\frac{mg}{a}\left(\frac{4a}{3} - x\right) - \frac{2mg}{a}\left(\frac{2a}{3} + x\right) - 4m\omega\dot{x} = m\ddot{x}$ M1 A3

$\ddot{x} + 4\omega\dot{x} + \frac{3g}{a}x = 0$

$\ddot{x} + 4\omega\dot{x} + 3\omega^2 x = 0$ * * A1 5

(c)

$\lambda^2 + 4\omega\lambda + 3\omega^2 = 0$

$(\lambda + 3\omega)(\lambda + \omega) = 0$ M1

$\lambda = -3\omega$ or $\lambda = -\omega$

$x = Ae^{-\omega t} + Be^{-3\omega t}$ A1

$\dot{x} = -\omega Ae^{-\omega t} - 3\omega Be^{-3\omega t}$ M1 A1

$t = 0$, $x = \frac{1}{2}a$, $\dot{x} = 0$ M1

$\frac{1}{2}a = A + B$

$0 = -\omega A - 3\omega B$ A1

$A = \frac{3}{4}a$, $B = -\frac{1}{4}a$ A1

$\dot{x} = v = \frac{3}{4}a\omega\,(\mathrm{e}^{-3\omega t} - \mathrm{e}^{-\omega t})$ A1 8

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2. (a)

$(\rightarrow)$, $\mathrm{T} = m\,\ddot{y}$ M1

Hooke's Law:

$T = \frac{2mn^2ax}{2a} = mn^2x$ B1

$\left.\begin{aligned} x + y &= \tfrac{1}{2}ft^2 \\ \dot{x} + \dot{y} &= ft \\ \ddot{x} + \ddot{y} &= f \end{aligned}\right\}$ B2

so, $(\rightarrow)$, $mn^2x = m\ddot{y} = m(f - \ddot{x})$ DM1

$\ddot{x} + n^2x = f$ * * A1 6

(b)	C.F. : $x = A\cos nt + B\sin nt$		B1	
	P.I. : $x = \dfrac{f}{n^2}$		B1	
	Gen solution: $x = A\cos nt + B\sin nt + \dfrac{f}{n^2}$		M1	
	$\dot{x} = -An\sin nt + Bn\cos nt$	follow their PI	M1 A1ft	
	$\left.\begin{array}{l} t = 0, x = 0 \Rightarrow A = -\dfrac{f}{n^2} \\ t = 0, \dot{x} = 0 \Rightarrow B = 0 \end{array}\right\}$		M1 A1	
	$x = \dfrac{f}{n^2}(1 - \cos nt)$		A1	8
(c)	$\dot{x} = 0 \Rightarrow \quad nt = \pi$		M1	
	$x_{\max} = \dfrac{f}{n^2}(1 - -1) = \dfrac{2f}{n^2}$		M1 A1	3
(d)	$\dot{y} = ft - \dot{x}$		M1	
	$= f\dfrac{\pi}{n} - 0\dfrac{f\pi}{n}$		A1	2

[19]

1. Part (a) was answered well, and candidates tended to be successful in applying Hooke's law and equating the tensions in order to find the distance AP at equilibrium.

In part (b) clear diagrams showing the point from which x was measured were an advantage here and were often lacking, resulting in tension being in the wrong directions and/or with the wrong magnitudes. With x defined in the question, candidates often had difficulty in obtaining correct expressions for the extension in each string in terms of x and fudges aimed at obtaining the given differential equation were very common. Some candidates struggled with the mechanics involved and did not include all the relevant terms in setting up their equation of motion.

In part (c) a great many candidates were able to solve the second order differential equation correctly, demonstrating a sound grasp of the pure mathematics involved, although a few were expecting a trigonometric solution and forced their auxiliary equation to produce one. Many candidates were able to use correct boundary conditions to find the unknowns and give a correct final result. The use of a or 0 in place of $0.5a$ for the initial displacement of P was surprisingly common. A few candidates overlooked the request to give the velocity of P as their final answer.

2. In part (a) many candidates claimed to have derived the given equation, but very few actually did so correctly having considered the equation of motion of the particle P. Those candidates who started with a clear diagram were far more likely to realise that they needed to consider both the distance moved by P and the extension in the spring. It was very common for candidates to score only one mark here, for finding the tension in the spring correctly.

Almost all candidates in part (b) were confident in attempting to solve the second order differential equation, although several did not choose a correct form for the complementary function, and a few struggled with the particular integral. It was common to see the correct solution for x.

In part (c) it was reassuring to find many candidates knowing that the maximum value of $(1 - \cos nt)$ is 2, although 1 was a popular alternative answer. Candidates who did not use the basic properties of the trig function were often able to find the maximum extension by using calculus, but here too there were some difficulties in identifying the value(s) of t for which $\sin nt = 0$.

The response to part (d) confirmed that many candidates had little or no idea of the correct derivation of the equation in (a). Most candidates believed that the speed of P and the rate of change of the extension in the spring were equal. Correct responses were seen, but usually only from the stronger candidates.