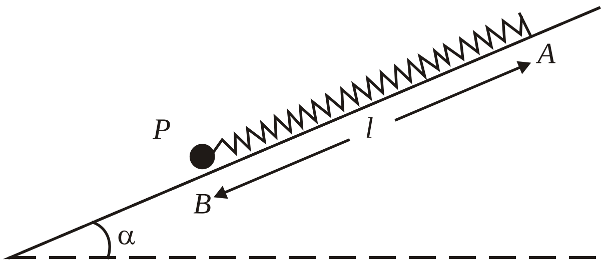
1.



A light elastic spring has natural length l and modulus of elasticity $4mg$. One end of the spring is attached to a point A on a plane that is inclined to the horizontal at an angle α, where $\tan \alpha = \frac{3}{4}$. The other end of the spring is attached to a particle P of mass m. The plane is rough and the coefficient of friction between P and the plane is $\frac{1}{2}$. The particle P is held at a point B on the plane where B is below A and $AB = l$, with the spring lying along a line of the plane, as shown in the figure above. At time $t = 0$, the particle is projected up the plane towards A with speed $\frac{1}{2}\sqrt{(gl)}$. At time t, the compression of the spring is x.

(a) Show that $\frac{d^2x}{dt^2} + 4\omega^2 x = -g$, where $\omega = \sqrt{\left(\frac{g}{l}\right)}$.

(6)

(b) Find x in terms of l, ω and t.

(7)

(c) Find the distance that P travels up the plane before first coming to rest.

(4)
(Total 17 marks)

2. A particle P of mass m is suspended from a fixed point by a light elastic spring. The spring has natural length a and modulus of elasticity $2m\omega^2 a$, where ω is a positive constant. At time $t = 0$ the particle is projected vertically downwards with speed U from its equilibrium position. The motion of the particle is resisted by a force of magnitude $2m\omega v$ where v is the speed of the particle. At time t, the displacement of P downwards from its equilibrium position is x.

(a) Show that $\frac{d^2x}{dt^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0$

(5)

Given that the solution of this differential equation is $x = e^{-\omega t}(A \cos \omega t + B \sin \omega t)$, where A and B are constants,

(b) find A and B.

(4)

(c) Find an expression for the time at which P first comes to rest.

(3)
(Total 12 marks)

3. A light elastic string, of natural length $2a$ and modulus of elasticity mg, has a particle P of mass m attached to its mid-point. One end of the string is attached to a fixed point A and the other end is attached to a fixed point B which is at a distance $4a$ vertically below A.

(a) Show that P hangs in equilibrium at the point E where $AE = \frac{5}{2}a$.

(5)

The particle P is held at a distance $3a$ vertically below A and is released from rest at time $t = 0$. When the speed of the particle is v, there is a resistance to motion of magnitude $2mkv$, where $k = \sqrt{\left(\frac{g}{a}\right)}$. At time t the particle is at a distance $(\frac{5}{2}a + x)$ from A.

(b) Show that

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 2k^2x = 0.$$

(5)

(c) Hence find x in terms of t.

(7)
(Total 17 marks)

4. A particle P moves in a straight line. At time t seconds, its displacement from a fixed point O on the line is x metres. The motion of P is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 12\cos 2t - 6\sin 2t.$$

When $t = 0$, P is at rest at O.

(a) Find, in terms of t, the displacement of P from O.

(11)

(b) Show that P comes to instantaneous rest when $t = \frac{\pi}{4}$.

(2)

(c) Find, in metres to 3 significant figures, the displacement of P from O when $t = \frac{\pi}{4}$.

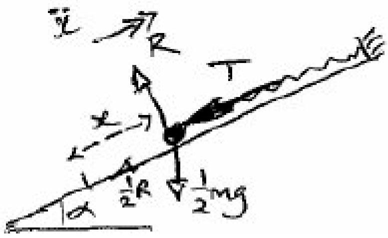
(2)

(d) Find the approximate period of the motion for large values of t.

(2)

(Total 17 marks)

1. (a)



$F = \frac{1}{2}R$ M1

$R = mg\cos\alpha$ B1

$T = \frac{4mgx}{l}$ B1

(↗): $-F - mg\sin\alpha - T = m\ddot{x}$ M1 A1

$-\frac{1}{2}.\frac{4}{5}mg - \frac{3}{5}mg - \frac{4mgx}{l} = m\ddot{x}$

$\Rightarrow \frac{d^2x}{dt^2} + 4w^2x = -g$ A1 6

(b) $\left(w = \sqrt{\frac{s}{L}}\right)$

$m^2 + 4w^2 = 0 \quad \Rightarrow m = \pm 2wi$

C.F. $x = A\sin 2wt + B\cos 2wt$ M1

P.I. $x = \frac{-g}{4w^2} = \frac{-l}{4}$ B1

G.S. $x = A\sin 2wt + B\cos 2wt - \frac{l}{4}$ B1

$t = 0, x = 0 \qquad B = \frac{l}{4}$

$\dot{x} = 2wA\cos 2wt - 2wB\sin 2wt$ M1 A1

$t = 0,\ \dot{x} = \frac{1}{2}\sqrt{gl} : \frac{\sqrt{gl}}{2} = 2\sqrt{\frac{g}{l}}A \Rightarrow A = \frac{1}{4}$ M1

$\Rightarrow x = \frac{l}{4}(\sin 2wt + \cos 2wt - 1)$ A1 7

(c) $\dot{x} = 0 \Rightarrow \cancel{2w}A\cos 2wt - \cancel{2w}B\sin 2wt = 0$ M1

$\Rightarrow \tan 2wt = \frac{A}{B} = 1$

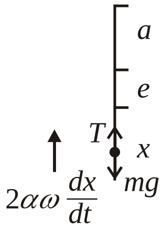
$\Rightarrow 2wt = \frac{\pi}{4}$ (first value) A1

$\Rightarrow x = \frac{1}{4}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1)$ M1

$= \frac{l}{4}(\sqrt{2} - 1)$ A1 4

[17]

2. (a)



R(ϕ) $m\frac{d^2x}{dt^2} = mg - T - 2m\omega \frac{dx}{dt}$ (4 terms) M1 A1

$m\frac{d^2x}{dt^2} = mg - \frac{2m\omega^2 a}{a}(e + x) - 2m\omega\frac{dx}{dt}$ M1

$\rightarrow \underline{\frac{d^2x}{dx^2} + 2\omega\frac{dx}{dt} + 2\omega^2 x = 0}$ M1 A1 5

(b) $x = e^{-\omega t}(A \cos \omega t + b \sin \omega t)$

$t = 0, x = 0 \Rightarrow \underline{A = 0}$ B1

$\frac{dx}{dt} = -\omega e^{-\omega t}.B \sin \omega t + e^{-\omega t}.B\omega \cos \omega t = 0$ (use of product rule) M1

$t = 0, \frac{dx}{dt} = U : U = B\omega \Rightarrow \underline{. B = \frac{U}{\omega}}$ M1 A1 4

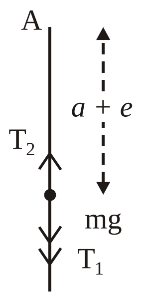
(c) $\frac{dx}{dt} = -Ue^{-\omega t} \sin \omega t + Ue^{-\omega t} \cos \omega t = 0$ M1

$\Rightarrow \tan \omega t = 1$ (solve for tan ωt) M1

$\Rightarrow t = \frac{\pi}{4\omega}$ A1 3

[12]

3. (a)



$T_2 = T_1 + mg$	M1	
$\dfrac{mge}{a} = \dfrac{mg}{a}(2a - e) + mg$	M1 A1	
$e = \dfrac{3a}{2} \Rightarrow AE = \dfrac{5a}{2}$ (*)	A1 A1 cso	5

(b)

$mg + \dfrac{mg}{a}\left(\tfrac{1}{2}a - x\right) - \dfrac{mg}{a}\left(\tfrac{3}{2}a + x\right) - 2m\sqrt{\dfrac{g}{a}}\dot{x} = m\ddot{x}$	M1 A3	
(–1eeoo)		
$\Rightarrow \underline{\ddot{x} + 2k\dot{x} + 2k^2x = 0}$ (*)	A1	5

(c)

AE: $m^2 + 2km + 2k^2 = 0$	M1	
$m = -k \pm k\text{i}$	A1	
GS: $x = \text{e}^{-kt}(A\cos kt + B\sin kt)$	A1 ft	
$t = 0, x = \tfrac{1}{2}a \Rightarrow A = \tfrac{1}{2}a$	B1	
$\dot{x} = k\text{e}^{-kt}(A\cos kt + B\sin kt) + \text{e}^{-kt}(-kA\sin kt + kB\cos kt)$	M1	
$t = 0, \dot{x} = 0 \Rightarrow -kA + kB = 0 \Rightarrow B = A = \tfrac{1}{2}a$	M1	
$x = \tfrac{1}{2}a\,\text{e}^{-kt}(\cos kt + \sin kt)$	A1	7

[17]

4.	(a)	Auxiliary Equation.: $m^2 + 2m + 2 = 0, \Rightarrow m = -1 \pm i$	M1, A1	
		$\therefore$ Complementary. Function is: $x = e^{-t}(A\cos t + B\sin t)$	M1 ft	
		Let $x = p\cos 2t + q\sin 2t$, $\dot{x} = -2p\sin 2t + 2q\cos 2t$, $\ddot{x} = -4x$	M1	
		Sub. in D.E.		
		$-2p\cos 2t - 2q\sin 2t - 4p\sin 2t + 4q\cos 2t = 12\cos 2t - 6\sin 2t$	M1	
		$-2p + 4q = 12, -4p - 2q = -6$	A1	
		$-10p = 0 \Rightarrow p = 0, q = 3$	M1	
		$\therefore$ $x = 3\sin 2t + e^{-t}(A\cos t + B\sin t)$	A1	
		$t = 0, x = 0 \Rightarrow 0 = A$	B1	
		$\dot{x} = 6\cos 2t - e^{-t}B\sin t + e^{-t}B\cos t$	M1	
		$t = 0, x = 0 \Rightarrow 0 = 6 + B \therefore B = -6$		
		$\therefore$ $x = 3\sin 2t - 6e^{-t}\sin t$	A1	11
	(b)	$\dot{x} = 6[\cos 2t + e^{-t}\sin t - e^{-t}\cos t]$		
		Sub $t = \frac{\pi}{4}$ $\dot{x} = 6[\cos 2t + e^{-t} - 6e^{-t}\cos t]$		
		$\dot{x} = 6\left[0 + e^{-\frac{x}{4}} \times \frac{1}{\sqrt{2}} - e^{-\frac{x}{4}} \times \frac{1}{\sqrt{2}}\right] = 0$	M1	
		$\therefore$ P comes to instantaneous rest when $t = \frac{\pi}{4}$	A1	2
	(c)	sub $t = \frac{\pi}{4}$ in x $= 3\sin\frac{\pi}{2} - 6e^{-\frac{\pi}{4}}\frac{1}{\sqrt{2}}$, $= 1.07$	M1, A1	2
	(d)	$t \to \infty$ $x \approx 3\sin 2t$, approximate period is π	M1, A1	2

[17]

1. (a) Most candidates were able to make a reasonable attempt although there were some sign errors in Newton's second law. Some weaker candidates missed out the component of the weight.

(b) The auxiliary equation was usually solved correctly. Subsequently a common error was either not to find a particular integral or to attempt to find it having already used the initial conditions on the complementary function – this was heavily penalised.

(c) Although the straightforward method of equating the velocity to zero was usually known, those candidates who had not simplified their answer to part (b) were often unable to complete this part. There were a very small number who attempted to use an energy method, occasionally correctly.

2. In part (a), many candidates failed to realise that they needed to consider the equilibrium position and obtained the printed answer by ignoring the mg term in F = ma. In (b), a few wasted time by solving the differential equation when the solution was given but full marks were often obtained in this part. A mark was sometimes lost in (c) because candidates gave the general solution of $\tan\omega t = 1$ instead of the time at which P first comes to rest.

3. Part (a) was reasonably well done, though with also a fair number of fudges to produce the given answer. A clearly drawn diagram here would have helped. The derivation of the differential equation in part (b) was not well done: many failed to see that there were four forces acting on the particle and the correct extensions in the two parts of the string were rarely correct. Again a clear diagram would have helped many here. Part (c) was familiar to many though by no means always producing full marks. Several made slips in solving the auxiliary equation, and many gave the wrong initial value of x to find one of the constants of integration.

4. This was another question that was answered well. The method for solving the differential equation was well known and, apart from a few candidates who made errors in finding the particular integral, the majority of solutions were correct. Most candidates knew what was required in parts (b) and (c) and $t = \frac{\pi}{4}$ was substituted in the appropriate formula. In the final part only the best candidates realised that for large values of t the displacement is approximately $3 \sin 2t$ and therefore the period is π. Some tried to argue that the period was 4 times the value $\frac{\pi}{4}$ used in parts (b) and (c).