Solutionbank M4
Edexcel AS and A Level Modular Mathematics

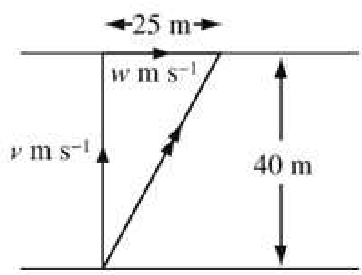
Review Exercise 1
Exercise A, Question 1

Question:

A river of width 40 m flows with uniform and constant speed between straight banks. A swimmer crosses as quickly as possible and takes 30 s to reach the other side. She is carried 25 m downstream.
Find

a the speed of the river,
b the speed of the swimmer relative to the water. **[E]**

Solution:



For the quickest crossing, the swimmer must swim on a course which aims directly across the river; that is the course which is always perpendicular to the river banks.

a Let the speed of the river be w m s^{-1}.

$$R(\rightarrow) \quad \text{speed} = \frac{\text{distance}}{\text{time}}$$

$$w = \frac{25}{30} = \frac{5}{6}$$

As the course of the swimmer is perpendicular to the river bank, her only motion parallel to the bank is due to the flow of the river. She moves 25 m downstream in 30 s.

The speed of the river is $\frac{5}{6}$ m s^{-1}.

b Let the speed of the swimmer relative to the water be v m s^{-1}.

$$R(\uparrow) \quad \text{speed} = \frac{\text{distance}}{\text{time}}$$

$$v = \frac{40}{30} = \frac{4}{3}$$

Across the stream, the swimmer moves 40 m in 30 s.

The speed of the swimmer relative to the water is $\frac{4}{3}$ m s^{-1}.

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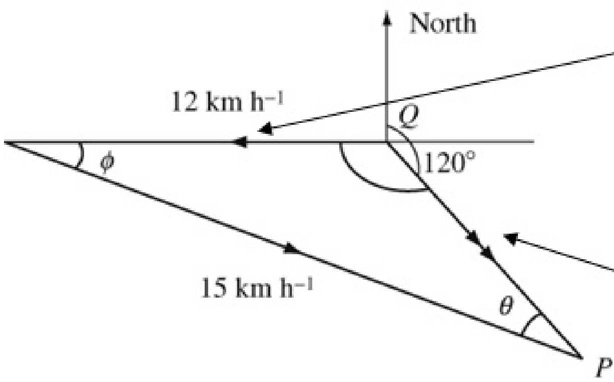
Review Exercise 1
Exercise A, Question 2

Question:

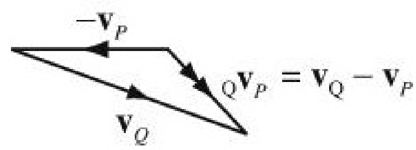
At noon, a boat P is on a bearing of $120°$ from boat Q. Boat P is moving due east at a constant speed of 12 km h^{-1}. Boat Q is moving in a straight line with a constant speed of 15 km h^{-1} on a course to intercept P. Find the direction of motion of Q, giving your answer as a bearing. **[E]**

Solution:



You fix P by introducing the velocity that is 'minus the velocity of P'. This vector represents a velocity equal in magnitude to the velocity of P but in the opposite direction.

This vector represents the velocity of Q relative to P; ${}_Q\mathbf{v}_P = \mathbf{v}_Q - \mathbf{v}_P$. The diagram can be illustrated as



As Q wants to intercept P, ${}_Q\mathbf{v}_P$ is in direction QP.

Using the sine rule

$$\frac{\sin\theta}{12} = \frac{\sin 150°}{15}$$

$$\sin\theta = \frac{12\sin 150°}{15} = 0.4$$

$\theta = 24°$, to the nearest degree

$\phi = 180° - 150° - \theta$

$\quad = 6°$ (nearest degree)

The direction of motion of Q, as a bearing, is $090° + \phi = 096°$ (nearest degree)

Bearings are measured from north, clockwise, and are usually given to the nearest degree or nearest tenth of a degree.

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Review Exercise 1
Exercise A, Question 3

Question:

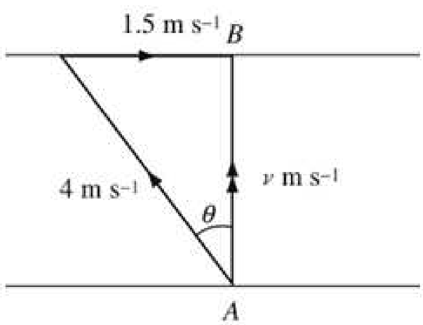
Points A and B are directly opposite each other on the parallel banks of a river. A motorboat, which travels at 4 m s^{-1} relative to the water, crosses from A to B. Given that the distance AB is 400 m and that the river is flowing at 1.5 m s^{-1} parallel to the banks, calculate

a the angle, to the nearest degree, between AB and the direction in which the boat is being steered,

b the speed, in m s^{-1} to 2 significant figures, of the motorboat relative to the bank,

c the time, to the nearest second, taken by the motorboat to cross the river. **[E]**

Solution:



As the motorboat goes directly from A to B, the resultant of the velocity of the motorboat and the velocity of the river must be in the direction AB.

a Let the angle between AB and the direction in which the boat is being steered be θ.

$\tan\theta = \frac{1.5}{4} = 0.375$

$\theta = 21°$ (nearest degree)

b Let $v \text{ m s}^{-1}$ be the speed of the motorboat relative to the bank.
Using a Pythagoras relation

$v^2 = 4^2 - 1.5^2 = 13.75$

$v = \sqrt{13.75} \approx 3.71$

The speed of the motorboat relative to the bank is 3.7 m s^{-1} (2 s.f.)

The triangle of velocities drawn in the diagram is used to answer both parts **a** and **b**. The mathematics involved is elementary trigonometry and Pythagoras' theorem.

c $\text{time} = \frac{\text{distance}}{\text{speed}}$

$= \frac{400}{\sqrt{13.75}} \text{ s}$

$= 108 \text{ s}$ (nearest second)

The speed found in part **b** is the effective speed with which the motorboat progresses from A to B and the time is found using distance = speed × time.

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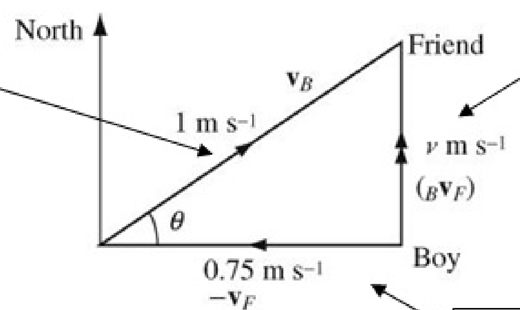
Review Exercise 1
Exercise A, Question 4

Question:

A boy enters a large horizontal field and sees a friend 100 m due north. The friend is walking in an easterly direction at a constant speed of 0.75 m s^{-1}. The boy can walk at a maximum speed of 1 m s^{-1}.
Find the shortest time for the boy to intercept his friend and the bearing on which he must travel to achieve this. **[E]**

Solution:



This is the direction the boy must take.

For interception, the velocity of the boy relative to the friend must be the direction of the line joining the initial position of the boy to the initial position of the friend.

You fix the position of the friend by introducing the velocity that is 'minus the velocity of the friend'. This vector represents a velocity equal in magnitude to the velocity of the friend but in the opposite direction.

Let the speed of the boy relative to the friend be $v \text{ m s}^{-1}$.

$$v^2 = 1^2 - 0.75^2 = 0.4375$$

$$\text{times} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{100}{\sqrt{0.4375}} \text{ s} \approx 151.2 \text{ s}$$

$$\cos\theta = \frac{0.75}{1} \Rightarrow \theta \approx 41.4°$$

The bearing is $090° - \theta \approx 048.6°$

This can be thought of as the 'relative distance', 100 m, divided by the 'relative speed', $\sqrt{0.4375} \text{ m s}^{-1}$.

Bearings are measured from north, clockwise, and are usually given to the nearest degree or nearest tenth of a degree.

The shortest time for the boy to intercept his friend is 151 s (nearest second), and the bearing on which he must travel is $049°$ (nearest degree).

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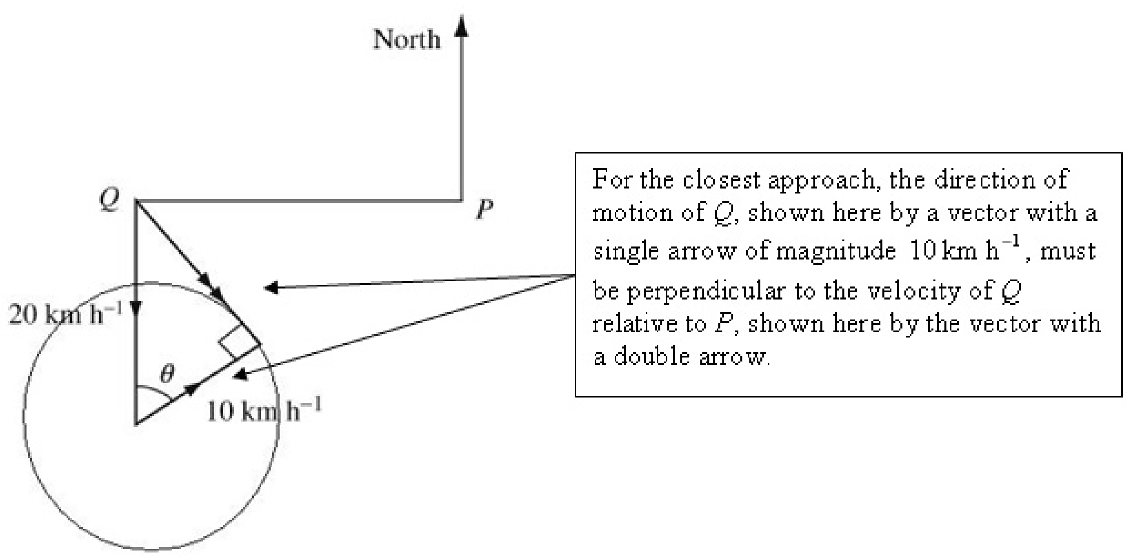
Review Exercise 1
Exercise A, Question 5

Question:

A cyclist P is cycling due north at a constant speed of 20 km h^{-1}. At 12 noon another cyclist Q is due west of P. The speed of Q is constant at 10 km h^{-1}.
Find the course which Q should set in order to pass as close to P as possible, giving your answer as a bearing. **[E]**

Solution:



$$\cos\theta = \frac{10}{20} = \frac{1}{2}$$
$$\theta = 60°$$

The course Q should set is $060°$

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Review Exercise 1
Exercise A, Question 6

Question:

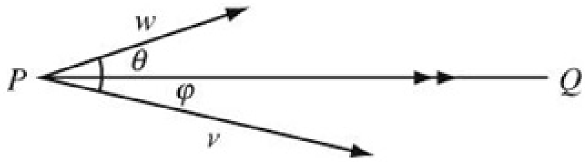
[*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively.*]
An aeroplane makes a journey from a point P to point Q which is due east of P. The wind velocity is $w(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, where w is a positive constant. The velocity of the aeroplane relative to the wind is $v(\cos\phi\mathbf{i} - \sin\phi\mathbf{j})$, where v is a constant and $v > w$.
Given that θ and ϕ are both acute angles,

a show that $v\sin\phi = w\sin\theta$,

b find, in terms of v, w and θ, the speed of the aeroplane relative to the ground. **[E]**

Solution:

a



Let ${}_a\mathbf{v}_w$ be the velocity of the aeroplane relative to the wind,

$\mathbf{v}_w$ be the velocity of the wind and

$\mathbf{v}_a$ be the velocity of the aeroplane relative to the ground.

If x is the speed of the aeroplane relative to the ground

$\mathbf{v}_a = x\mathbf{i}$

> As the aeroplane moves from P to Q, that is due east, the velocity of the aeroplane is in the direction of $\mathbf{i}$. The magnitude of the velocity is the speed x and so $\mathbf{v}_a = x\mathbf{i}$.

${}_a\mathbf{v}_w = \mathbf{v}_a - \mathbf{v}_w$

$v(\cos\phi\mathbf{i} - \sin\phi\mathbf{j}) = x\mathbf{i} - w(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

Equating $\mathbf{j}$ components

$-v\sin\phi = -w\sin\theta$

$v\sin\phi = w\sin\theta$, as required

> By definition, the velocity of A relative to B is given by ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B$. The specification for M4 requires you to know this formula.

b Equating the $\mathbf{i}$ components

$v\cos\phi = x - w\cos\theta$

$x = v\cos\phi + w\cos\theta$

From part **a**

$\sin\phi = \frac{w}{v}\sin\theta$

$\cos^2\phi = 1 - \sin^2\phi = 1 - \frac{w^2}{v^2}\sin^2\theta$

> The question asks you to find the speed in terms of v, w and θ, so you must eliminate ϕ. You do this using the answer to part **a** and the identity $\sin^2\phi + \cos^2\phi = 1$.

Hence

$$x = v\left(1 - \frac{w^2}{v^2}\sin^2\theta\right)^{\frac{1}{2}} + w\cos\theta$$

$$= \left(v^2 - w^2\sin^2\theta\right)^{\frac{1}{2}} + w\cos\theta$$

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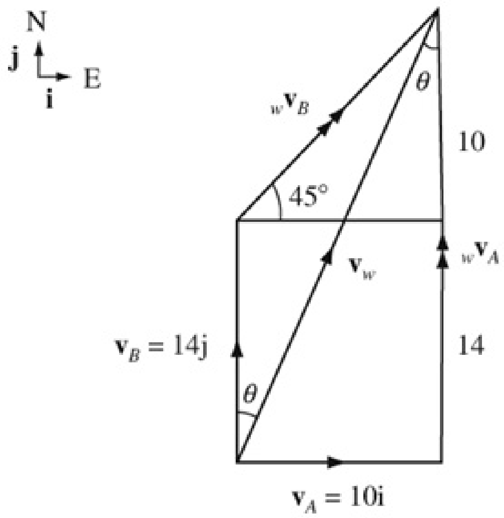
Review Exercise 1
Exercise A, Question 7

Question:

Boat A is sailing due east at a constant speed of 10 km h^{-1}. To an observer on A, the wind appears to be blowing from due south. A second boat B is sailing due north at a constant speed of 14 km h^{-1}. To an observer on B, the wind appears to be blowing from the south west. The velocity of the wind relative to the Earth is constant and is the same for both boats.
Find the velocity of the wind relative to the Earth, stating its magnitude and direction.
[E]

Solution:



If you draw a diagram combining the velocity vector triangles for the velocity of the wind relative to A and the velocity of the wind relative to B, then it is possible just to write down, from the diagram, that the velocity of the wind is $(10\mathbf{i}+24\mathbf{j})\text{m s}^{-1}$. An alternative solution using vectors is given below.

Let $\mathbf{v}_w$ be the velocity of the wind relative to the ground,
$\mathbf{v}_A$ be the velocity of A and
${}_w\mathbf{v}_A$ be the velocity of the wind relative to A.
Taking $\mathbf{i}$ and $\mathbf{j}$ as horizontal unit vectors due east and due north respectively

$${}_w\mathbf{v}_A = \mathbf{v}_w - \mathbf{v}_A = \lambda\mathbf{j}, \text{ say}$$
$$\Rightarrow \mathbf{v}_w - 10\mathbf{i} = \lambda\mathbf{j}$$
$$\mathbf{v}_w = 10\mathbf{i} + \lambda\mathbf{j} \quad ①$$

From A, the wind appears to blow from the south, so the velocity of the wind relative to A is a multiple of $\mathbf{j}$.

Let $\mathbf{v}_B$ be the velocity of B and
${}_w\mathbf{v}_B$ be the velocity of the wind relative to B.

$${}_w\mathbf{v}_B = \mathbf{v}_B - \mathbf{v}_w = \mu\mathbf{i} + \mu\mathbf{j}, \text{ say}$$
$$\Rightarrow \mathbf{v}_w - 14\mathbf{j} = \mu\mathbf{i} + \mu\mathbf{j}$$
$$\mathbf{v}_w = \mu\mathbf{i} + (\mu + 14)\mathbf{j} \quad ②$$

From B, the wind appears to be blowing from the south west, so the velocity of the wind relative to B must be parallel to $\mathbf{i}+\mathbf{j}$.

Equating the $\mathbf{i}$ components of ① and ②
$\mu = 10$
Hence

$$\mathbf{v}_w = 10\mathbf{i} + 24\mathbf{j}$$

Substituting $\mu = 10$ into ②.

$$|\mathbf{v}_w|^2 = 10^2 + 24^2 = 676 \Rightarrow v_w = \sqrt{676} = 26$$
$$\tan\theta = \frac{10}{24} \Rightarrow \theta \approx 22.6^\circ$$

The velocity of the wind relative to the Earth has magnitude 26 m s^{-1} and is on a bearing 023° (nearest degree).

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 8

Question:

Ship A is steaming on a bearing of $060°$ at 30 km h^{-1} and at 9 a.m. it is 20 km due west of a second ship B. Ship B steams in a straight line.

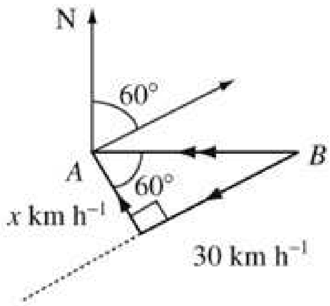
a Find the least speed of B if it is to intercept A.

Given that the speed of B is 24 km h^{-1},

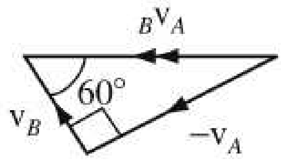
b find the earliest time at which it can intercept A. **[E]**

Solution:

a



The side representing the velocity of B in the triangle of velocity will have the least possible magnitude when it is perpendicular to the side representing the negative of the velocity of A.



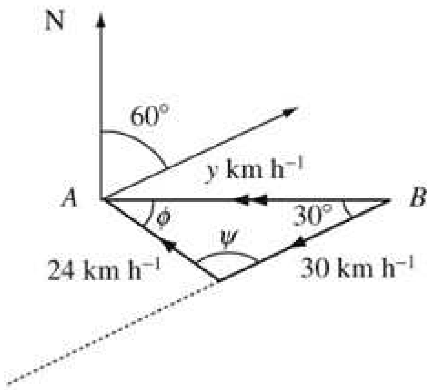
Let the least speed be $x \text{ km h}^{-1}$

$$\frac{30}{x} = \tan 60°$$

$$x = \frac{30}{\tan 60°} = \frac{30}{\sqrt{3}} = 10\sqrt{3}$$

The least speed of B if it is to intercept A is $10\sqrt{3} \text{ km h}^{-1}$.

b



The diagram for part a must be modified as the velocity of B is no longer perpendicular to path of A. In these circumstances, it is advisable to draw a separate diagram.

Using the sine rule

$$\frac{\sin\phi}{30} = \frac{\sin 30°}{24}$$

$$\sin\phi = \frac{30\sin 30°}{24} = \frac{5}{8}$$

Working to 2 decimal places

$$\phi = 38.68°$$

$$\psi = 180° - 30° - 38.68° = 111.32°$$

There is a second solution where $\phi \approx 141.32°$ but this would give a smaller value of y and a later time of interception.

Let $y \text{ km h}^{-1}$ be the magnitude of the velocity of B relative to A.

Using the sine rule

$$\frac{y}{\sin 111.32°} = \frac{24}{\sin 30°}$$

$$y = 44.72$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{20}{44.72}\text{ h} = 0.45\text{ h}$$

$$= 0.45 \times 60 \text{ min} = 27 \text{ min}$$

You can think of this as the 'relative distance', 20 km divided by the 'relative speed', 44.72 km h^{-1}.

The earliest time at which B can intercept A is 9.27 a.m. (nearest minute).

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 9

Question:

A cyclist C is moving with a constant speed of 10 m s^{-1} due south. Cyclist D is moving with a constant speed of 16 m s^{-1} on a bearing of $240°$.

a Show that the magnitude of the velocity of C relative to D is 14 m s^{-1}.

At 2 p.m., D is 4 km due east of C.

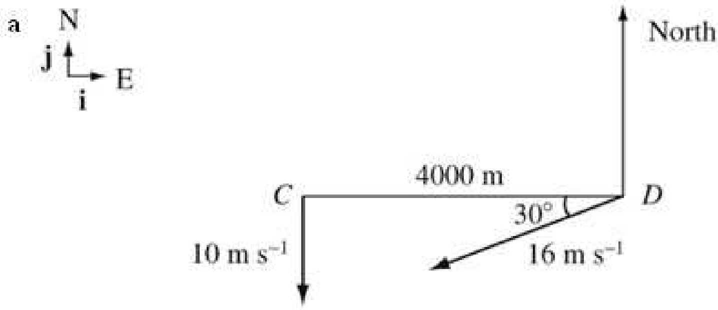
b Find

i the shortest distance between C and D during the subsequent motion,

ii the time, to the nearest minute, at which this shortest distance occurs. **[E]**

Solution:

a



Let **i** and **j** be horizontal unit vectors due east and due north respectively.
Let $\mathbf{v}_C$ m s^{-1} be the velocity of C and $\mathbf{v}_D$ m s^{-1} be the velocity of D.

$\mathbf{v}_C = -10\mathbf{j}$

$\mathbf{v}_D = -16\cos 30°\mathbf{i} - 16\sin 30°\mathbf{j}$

$= -8\sqrt{3}\mathbf{i} - 8\mathbf{j}$

> You resolve the velocity of D along the directions of **i** and **j**.

The velocity of C relative to D is given by

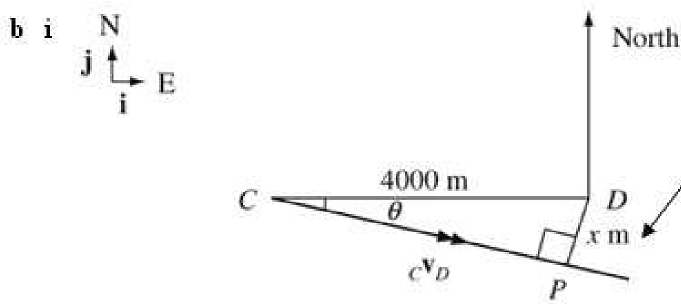
$${}_C\mathbf{v}_D = \mathbf{v}_C - \mathbf{v}_D$$
$$= -10\mathbf{j} - (-8\sqrt{3}\mathbf{i} - 8\mathbf{j})$$
$$= 8\sqrt{3}\mathbf{i} - 2\mathbf{j}$$

$$|{}_C\mathbf{v}_D|^2 = (8\sqrt{3})^2 + (-2)^2 = 192 + 4 = 196$$

$$|{}_C\mathbf{v}_D| = \sqrt{196} = 14$$

The magnitude of velocity of C relative to D is 14 m s^{-1}, as required.

b i



> The shortest distance, is given by the perpendicular from D to the direction of the velocity of C relative to D.

Let the foot of the perpendicular from D to the direction of the velocity of C relative to D be P. Let $DP = x$ m and $CP = y$ m .

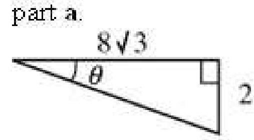
From part **a**

$${}_C\mathbf{v}_D = 8\sqrt{3}\mathbf{i} - 2\mathbf{j}$$

Hence

$$\tan\theta = \frac{2}{8\sqrt{3}} \Rightarrow \theta = 8.213° \text{ (3 d.p)}$$

> You find the angle CP makes with CD using the vector form of the relative velocity you found in part **a**.



In ΔCPD

$$\frac{x}{4000} = \sin\theta$$

$x = 4000 \sin 8.213° = 571\text{ m}$ (nearest whole number)

The shortest distance between C and D during the subsequent motion is 571 m (nearest metre).

ii In ΔCPD

$$\frac{y}{4000} = \cos 8.213°$$

$$y = 4000 \cos 8.213° = 3959 \text{ (nearest whole number)}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{3959}{14}\text{ s} = 283\text{ s, to the nearest second.}$$

$$= 5 \text{ minutes (nearest minute)}$$

The time, to the nearest minute, is 2.05 p.m.

$$\text{time} = \frac{\text{relative distance travelled}}{\text{relative speed}}$$

$$283\text{ s} = \frac{283}{60}\text{ minutes} \approx 5\text{ minutes}$$

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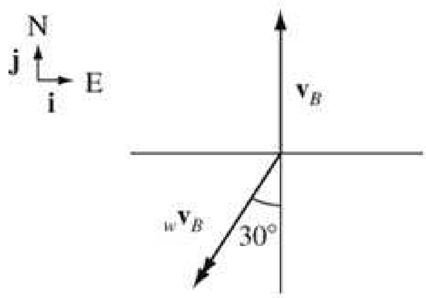
Review Exercise 1
Exercise A, Question 10

Question:

A boat is sailing north at a speed of 15 km h^{-1}. To an observer on the boat the wind appears to blow from a direction $030°$.
The boat turns round and sails due south at the same speed. The velocity of the wind relative to the Earth remains constant, but to an observer on the boat it now appears to blow from $120°$.
Find the velocity of the wind relative to the Earth. **[E]**

Solution:



Let **i** and **j** be horizontal unit vectors due east and due north respectively.
Let $\mathbf{v}_B \text{ km h}^{-1}$ be the velocity of the boat,
$\mathbf{v}_W \text{ km h}^{-1}$ be the velocity of the wind and
$_W\mathbf{v}_B \text{ km h}^{-1}$ be the velocity of the wind relative to the boat.
If the velocity of the wind relative to the boat has magnitude $x \text{ m s}^{-1}$ then

$$_W\mathbf{v}_B = -x\sin 30°\mathbf{i} - x\cos 30°\mathbf{j}$$
$$= -\frac{x}{2}\mathbf{i} - \frac{x\sqrt{3}}{2}\mathbf{j}$$

You resolve the relative velocity along the directions of **i** and **j**.

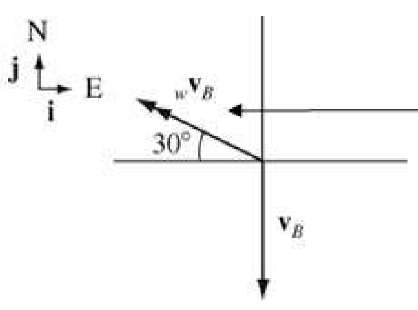
Also

$$_W\mathbf{v}_B = \mathbf{v}_W - \mathbf{v}_B$$

As the boat is moving north, that is in the direction of **j**, at 15 km h^{-1}, $\mathbf{v}_B = 15\mathbf{j}$.

$$-\frac{x}{2}\mathbf{i} - \frac{x\sqrt{3}}{2}\mathbf{j} = \mathbf{v}_W - 15\mathbf{j}$$
$$\mathbf{v}_W = -\frac{x}{2}\mathbf{i} + \left(15 - \frac{x\sqrt{3}}{2}\right)\mathbf{j} \quad ①$$

Solve this equation for $\mathbf{v}_W$.



The boat now reverses its direction and the wind appears to come from a different direction.

Now let the velocity of the wind relative to the boat have magnitude y m s^{-1}.

$${}_W\mathbf{v}_B = -y\cos 30°\mathbf{i} + y\sin 30°\mathbf{j}$$
$$= -\frac{y\sqrt{3}}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$$

You again resolve the relative velocity along the directions of **i** and **j**.

Also

$${}_W\mathbf{v}_B = \mathbf{v}_W - \mathbf{v}_B$$
$$-\frac{y\sqrt{3}}{2}\mathbf{i} + \frac{y}{2}\mathbf{j} = \mathbf{v}_W - (-15\mathbf{j})$$

As the boat is moving south, that is in the direction of $-\mathbf{j}$, at 15 km h^{-1}, $\mathbf{v}_B = -15\mathbf{j}$.

$$\mathbf{v}_W = -\frac{y\sqrt{3}}{2}\mathbf{i} + \left(\frac{y}{2} - 15\right)\mathbf{j} \quad ②$$

You now have two equations for the velocity of the wind and equating the **i** and **j** components will give you a pair of simultaneous equations in x and y.

Equating the **i** components in equations ① and ②

$$-\frac{x}{2} = -\frac{y\sqrt{3}}{2} \Rightarrow x = y\sqrt{3}$$

Equating the **j** components in equations ① and ②

$$\frac{y}{2} - 15 = 15 - \frac{x\sqrt{3}}{2}$$

Substituting $x = y\sqrt{3}$

$$\frac{y}{2} - 15 = 15 - \frac{3y}{2}$$
$$2y = 30 \Rightarrow y = 15$$

Substituting $y = 15$ into ②

$$\mathbf{v}_W = -\frac{15\sqrt{3}}{2}\mathbf{i} - \frac{15}{2}\mathbf{j}$$

The velocity of the wind relative to the Earth is

$$\left(-\frac{15\sqrt{3}}{2}\mathbf{i} - \frac{15}{2}\mathbf{j}\right)\text{km h}^{-1}$$

This is an acceptable vector form for the velocity but the answer can be given in other forms. For example, as a speed of 15 km h^{-1} blowing from the direction 060°. There are also many alternative ways of solving this question.

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Review Exercise 1
Exercise A, Question 11

Question:

A pilot flying an aircraft at a constant speed of 2000 km h^{-1} detects an enemy aircraft 100 km away on a bearing of 045°. The enemy aircraft is flying at a constant velocity of 1500 km h^{-1} due west.

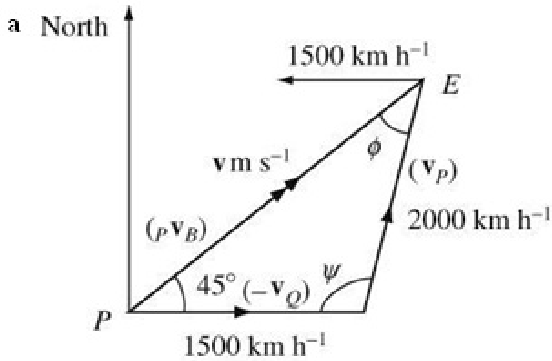
Find

a the course, as a bearing to the nearest degree, that the pilot should set in order to intercept the enemy aircraft,

b the time, to the nearest s, that the pilot will take to reach the enemy aircraft. **[E]**

Solution:

a



For interception, the velocity of the pilot (P) relative to the enemy (E) must be the direction of the line joining the initial position of P to the initial position of E. In this diagram, this relative velocity is shown with a double arrow.

Using the sine rule

$$\frac{\sin\phi}{1500} = \frac{\sin 45^\circ}{2000}$$

$$\sin\phi = \frac{3}{4\sqrt{2}}$$

$$\phi = 32.028^\circ \quad (3\text{ d.p.})$$

As ϕ is opposite the smallest side in the vector triangle, it must be acute.

$$\psi = 180^\circ - 45^\circ - \phi = 102.972^\circ$$

The bearing on which the pilot must fly is

$$\psi - 90^\circ = 013^\circ \quad \text{(nearest degree)}$$

Bearings are measured from north, clockwise. This question requires you to give your answer to the nearest degree.

b Let the magnitude of the velocity of the pilot relative to the enemy be v m s^{-1}

Using the cosine rule

$$v^2 = 1500^2 + 2000^2 - 2\times1500\times2000\cos\psi$$

$$= 7\,596\,849\ldots$$

$$v = 2756.238\ldots$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{100}{v}\text{ m} \approx 0.03628\text{ h}$$

$$= 131\text{ s} \quad \text{(nearest second)}$$

$0.03628\text{ h} = 0.03628\times3600\text{ s} \approx 130.6\text{ s}$

Solutionbank M4

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Review Exercise 1
Exercise A, Question 12

Question:

At noon, two boats A and B are 6 km apart with A due east of B. Boat B is moving due north at a constant speed of 13 km h^{-1}.

Boat A is moving with constant speed 12 km h^{-1} and sets a course so as to pass as close as possible to boat B. Find

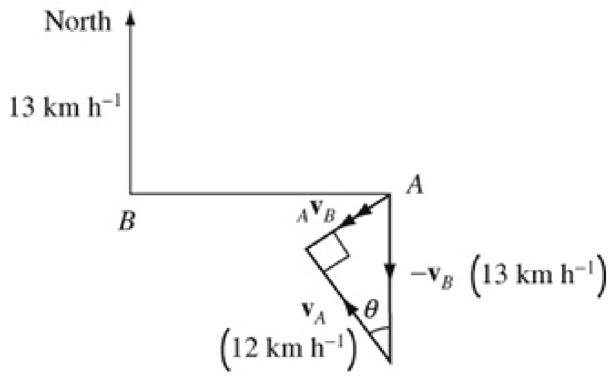
a the direction of motion of A, giving your answer as a bearing,

b the time when the boats are closest,

c the shortest distance between the boats. **[E]**

Solution:

a



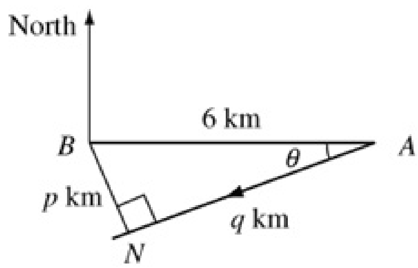
For the closest approach, the direction of motion of A, shown here by a vector with a single arrow of magnitude 12 km h^{-1}, must be perpendicular to the velocity of A relative to B, shown here by the vector with a double arrow.

$\cos\theta = \frac{12}{13} \Rightarrow \theta \approx 22.62°$

The bearing on which A moves is $360° - \theta = 337°$ (nearest degree)

b Let the magnitude of the velocity of A relative to B be $x \text{ m s}^{-1}$

$x^2 = 13^2 - 12^2 = 25 \Rightarrow x = 5$



Let the foot of the perpendicular from B to the direction of travel of A relative to B be N and let $BN = p$ km and $AN = q$ km.

In ΔBNA

$\frac{q}{6} = \cos\theta$

$q = 6\cos\theta = 6\cos 22.62° = 5.538$ (3 d.p.)

The θ in part **b** is the θ you found in part **a**. The two angles are equal.

$\text{time} = \frac{\text{distance}}{\text{speed}}$

$= \frac{5.538}{5} \text{ h} = 1.108 \text{ h}$

$0.108 \text{ h} = 0.108 \times 60 \text{ minutes} \approx 6 \text{ minutes}$

$= 1 \text{h } 6 \text{min}$

The time when the boats are closest is 1306, (nearest minute)

c In ΔBNA

$\frac{p}{6} = \sin\theta$

$p = 6\sin\theta = 6\sin 22.62° = 2.30\ldots$

The shortest distance is 2.3 km (nearest 0.1 km)

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 13

Question:

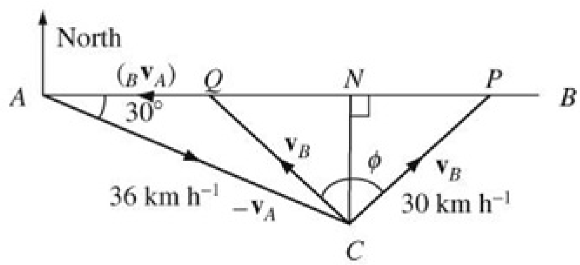
A ship A has maximum speed 30 km h^{-1}. At time $t = 0$, A is 70 km due west of B which is moving at a constant speed of 36 km h^{-1} on a bearing of 300°. Ship A moves on a straight course at constant speed and intercepts B. The course of A makes an angle θ with due north.

a Show that $-\arctan\frac{4}{3} \le \theta \le \arctan\frac{4}{3}$.

b Find the least time for A to intercept B. **[E]**

Solution:

a



In this diagram, the minimum speed for interception is represented by CN and the maximum speed of A, 30 km h^{-1}, by CP and CQ. If $\angle NCP = \phi$, then θ can vary from $-\phi$ to ϕ.

In ΔANC

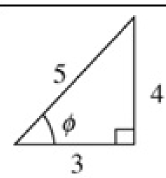
$$\frac{CN}{36} = \sin 30°$$

$$CN = 36\sin 30° = 18$$

In ΔCNP

$$\cos\phi = \frac{CN}{CP} = \frac{18}{30} = \frac{3}{5}$$

$$\tan\phi = \frac{4}{3} \Rightarrow \phi = \arctan\frac{4}{3}$$



From a 3, 4, 5 triangle, you can see that if $\cos\phi = \frac{3}{5} \Rightarrow \tan\phi = \frac{4}{3}$.

As

$$-\phi \le \theta \le \phi$$

$$-\arctan\frac{4}{3} \le \theta \le \arctan\frac{4}{3}, \quad \text{as required.}$$

b

$$AP = AN + NP$$

$$= 36\cos 30° + 30\sin\phi$$

$$= 36 \times \frac{\sqrt{3}}{2} + 30 \times \frac{4}{5} = 18\sqrt{3} + 24$$

The greatest possible velocity of B relative to A is represented on the diagram by AP. The greatest velocity of B relative to A will give you the least time.

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{70}{18\sqrt{3} + 24}\text{ h} = 1.27\text{ h} \quad (3\text{ s.f.})$$

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Review Exercise 1
Exercise A, Question 14

Question:

At 12 noon, ship A is 20 km from ship B, on a bearing of $300°$. Ship A is moving at a constant speed of 15 km h^{-1} on a bearing of $070°$. Ship B moves in a straight line with constant speed $V \text{ km h}^{-1}$ and intercepts A.

a Find, giving your answer to 3 significant figures, the minimum possible value for V.

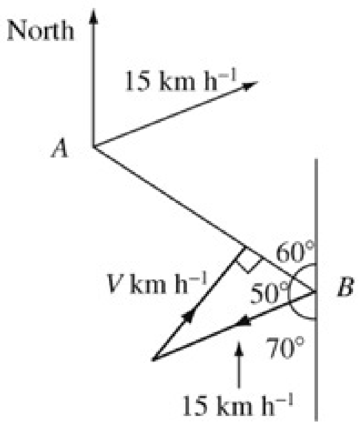
It is now given that $V = 13$.

b Explain why there are two possible times at which ship A can intercept ship B.

c Find, giving your answer to the nearest minute, the earlier time at which ship B can intercept ship A. **[E]**

Solution:

a



You fix A by introducing a vector equal and opposite to the velocity of A to the system.

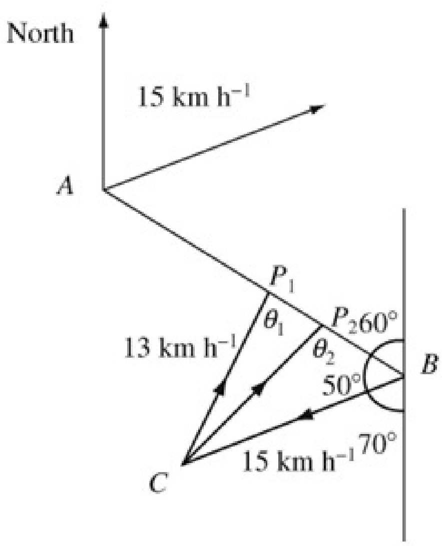
The smallest value of V is given by a vector perpendicular to the direction of the line joining A to B.

The smallest value of V is given by

$$\frac{V}{15} = \sin 50^\circ$$

$$V = 15\sin 50^\circ = 11.5 \quad (3 \text{ s.f.})$$

b



If $V = 13$, there are two possible courses for interception represented by the vectors $\overrightarrow{CP_1}$ and $\overrightarrow{CP_2}$ on the diagram above. When you calculate the value of θ using the sine rule

$$\frac{\sin\theta}{15} = \frac{\sin 50^\circ}{13},$$

the sine rule is ambiguous.

As $\sin\theta = \sin(180^\circ - \theta)$, there are two possible values of the angle θ which satisfy the rule, related by the relation $\theta_1 + \theta_2 = 180^\circ$.

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Review Exercise 1
Exercise A, Question 15

Question:

At time $t = 0$ particles P and Q start simultaneously from points which have position vectors $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$m and $(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$m respectively, relative to a fixed origin O.

The velocities of P and Q are $(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ m s^{-1} and $(2\mathbf{i} + \mathbf{k})$m s^{-1} respectively.

a Show that P and Q collide and find the position vector of the point at which they collide.

A third particle R moves in such a way that its velocity relative to P is parallel to the vector $(-5\mathbf{i} + 4\mathbf{j} - \mathbf{k})$ and its velocity relative to Q is parallel to the vector $(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

Given that all three particles collide simultaneously, find

b **i** the velocity of R,

ii the position vector of R at time $t = 0$. **[E]**

Solution:

a The path of P is given by

$$\mathbf{p} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + t\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

You may use either the **i**, **j**, **k** notation vectors or, as is used here, column vectors. In three dimensions, column vectors can often be written more quickly.

The path of Q is given by

$$\mathbf{q} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + t\begin{pmatrix}2\\0\\1\end{pmatrix}$$

In this question, you need to use the equation of a line using vectors in three dimensions and you need to know how to show that two lines intersect. These are topics in C4. The prerequisites for M4 require the knowledge of books C1 to C4 as well as M1 to M3.

Equating the **i** components

$1+t=-1+2t \Rightarrow t=2$

When $t=2$

$$\mathbf{p} = \begin{pmatrix}1\\-2\\3\end{pmatrix} + 2\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix}$$

and

$$\mathbf{q} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} = \mathbf{p}$$

Hence, when $t=2$, P and Q collide at the point with position vector $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$ m .

b i Let ${}_R\mathbf{v}_P = \lambda\begin{pmatrix}-5\\4\\-1\end{pmatrix}$

As the velocity of R relative to P is in the direction of $(-5\mathbf{i}+4\mathbf{j}-\mathbf{k})$, it must be a multiple of $(-5\mathbf{i}+4\mathbf{j}-\mathbf{k})$.

$${}_R\mathbf{v}_P = \mathbf{v}_R - \mathbf{v}_P$$

$$\lambda\begin{pmatrix}-5\\4\\-1\end{pmatrix} = \mathbf{v}_R - \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\mathbf{v}_R = \begin{pmatrix}1-5\lambda\\2+4\lambda\\-1-\lambda\end{pmatrix} \quad ①$$

Let ${}_R\mathbf{v}_Q = \mu\begin{pmatrix}-2\\2\\-1\end{pmatrix}$

$${}_R\mathbf{v}_Q = \mathbf{v}_R - \mathbf{v}_Q$$

$$\mu\begin{pmatrix}-2\\2\\-1\end{pmatrix} = \mathbf{v}_R - \begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\mathbf{v}_R = \begin{pmatrix}2-2\mu\\2\mu\\1-\mu\end{pmatrix} \quad ②$$

You now have two expressions for $\mathbf{v}_R$. Equating any two of the components will give you a pair of simultaneous equations in λ and μ.

Equating the **i** components of ① and ②

$1-5\lambda = 2-2\mu$

$-5\lambda+2\mu=1 \quad ③$

Equating the **j** components of ① and ②

$2+4\lambda = 2\mu$

$-4\lambda+2\mu=2$

Subtracting ④ − ③

$\lambda=1$

Substituting $\lambda=1$ into ①

$$\mathbf{v}_R = \begin{pmatrix}1-5\\2+4\\-1-1\end{pmatrix} = \begin{pmatrix}-4\\6\\-2\end{pmatrix}$$

The velocity of R is $(-4\mathbf{i}+6\mathbf{j}-2\mathbf{k})\text{ m s}^{-1}$.

ii The path of R is given by

$$\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix} + t\begin{pmatrix}-4\\6\\-2\end{pmatrix}$$

The path of a particle moving with constant velocity can be written as $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where t is the time, $\mathbf{r}_0$ the position vector when $t=0$, and $\mathbf{v}$ the velocity.

where $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$m is the position vector of R at time $t=0$.

From part **a**, when $t=2, \mathbf{r} = \begin{pmatrix}3\\2\\1\end{pmatrix}$

When $t=2$, all three particles are at the point with position vector $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$m.

$$\begin{pmatrix}x\\y\\z\end{pmatrix} + 2\begin{pmatrix}-4\\6\\-2\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3+8\\2-12\\1+4\end{pmatrix} = \begin{pmatrix}11\\-10\\5\end{pmatrix}$$

The position vector of R at time $t=0$ is $(11\mathbf{i}-10\mathbf{j}+5\mathbf{k})$m.

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Review Exercise 1
Exercise A, Question 16

Question:

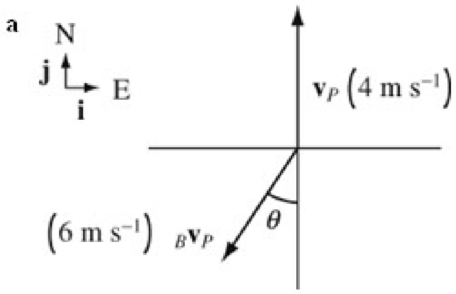
A rugby player is running due north with speed 4 m s^{-1}. He throws the ball horizontally and the ball has an initial velocity relative to the player of 6 m s^{-1} in the direction $\theta°$ west of south, i.e. on a bearing of $(180+\theta)°$, where $\tan\theta° = \frac{4}{3}$.

a Find the magnitude and direction of the initial velocity of the ball relative to a stationary spectator.

b Find also the bearing on which the ball appears to move initially to the referee who is running with speed $2\sqrt{2} \text{ m s}^{-1}$ in a north-westerly direction. **[E]**

Solution:

a



Let **i** and **j** be horizontal unit vectors due east and due north respectively.

$$\mathbf{v}_P = 4\mathbf{j}$$

$$_B\mathbf{v}_P = -6\sin\theta\mathbf{i} - 6\cos\theta\mathbf{j}$$

$$\tan\theta = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$$

$$_B\mathbf{v}_P = \mathbf{v}_B - \mathbf{v}_P$$

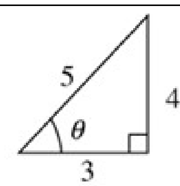
$$-6\times\frac{4}{5}\mathbf{i} - 6\times\frac{3}{5}\mathbf{j} = \mathbf{v}_B - 4\mathbf{j}$$

$$\mathbf{v}_B = -4.8\mathbf{i} + (4-3.6)\mathbf{j} = -4.8\mathbf{i} + 0.4\mathbf{j}$$

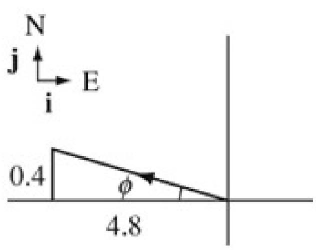
$$|\mathbf{v}_B|^2 = (-4.8)^2 + (0.4)^2 = 23.2$$

$$|\mathbf{v}_B| = 4.82 \quad (3\text{ s.f.})$$

You resolve the velocity of the player, $\mathbf{v}_P$, and the velocity of the ball relative to the player, $_B\mathbf{v}_P$, parallel to the unit vectors **i** and **j**.



You can see from a sketch that if $\tan\theta = \frac{4}{3}$, then $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.



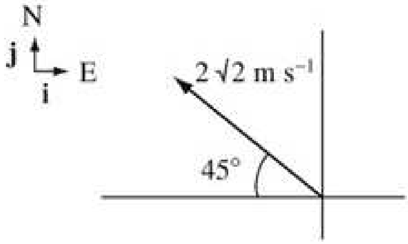
$$\tan\phi = \frac{0.4}{4.8} = 0.083$$

$$\phi = 5^\circ \text{(nearest degree)}$$

The velocity of the ball has magnitude 4.82 m s^{-1} (3 s.f.) and is on a bearing of 275° (nearest degree).

No accuracy is specified in this question and any reasonable accuracy would be accepted. In this context, 2 or 3 significant figures is sensible.

b



The velocity of the referee is given by

$\mathbf{v}_R = -2\sqrt{2}\cos 45°\mathbf{i} + 2\sqrt{2}\sin 45°\mathbf{j}$

$= -2\mathbf{i} + 2\mathbf{j}$

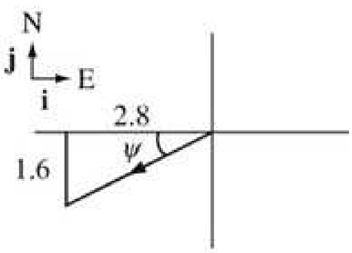
Using $\cos 45° = \sin 45° = \frac{1}{\sqrt{2}}$.

The velocity of the ball relative to the referee is given by

${}_B\mathbf{v}_R = \mathbf{v}_B - \mathbf{v}_R$

$= -4.8\mathbf{i} + 0.4\mathbf{j} - (-2\mathbf{i} + 2\mathbf{j})$

$= -2.8\mathbf{i} - 1.6\mathbf{j}$



There is an interesting interpretation of this question. The ball is actually being passed forward but appears to be backwards to both the player and the referee.

$\tan\psi = \frac{1.6}{2.8} \Rightarrow \psi = 29.7°\ (1\text{ d.p.})$

The bearing on which the ball will appear to move to the referee is $240°$ (nearest degree).

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Review Exercise 1
Exercise A, Question 17

Question:

Two ships A and B are travelling with constant speeds $2u$ m s^{-1} and u m s^{-1} respectively, A on a bearing θ and B on a bearing $90° + \theta$. It is also assumed that a third ship C has a constant, but unknown, velocity which is taken to be a speed v m s^{-1} on a bearing ϕ. To an observer on ship B the velocity of C appears to be due north.

a Show that $\dfrac{u}{\sin\phi} = \dfrac{v}{\cos\theta}$.

To an observer on ship A the velocity of C appears to be on a bearing of $135°$.

b Show that $2u(\cos\theta + \sin\theta) = v(\cos\phi + \sin\phi)$.

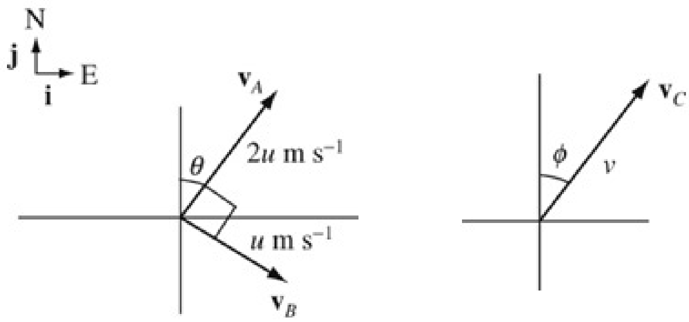
c Hence, find $\tan\phi$ in terms of $\tan\theta$.

Given that $\theta = 30°$ and $u = 10$,

d find the true velocity of C, giving your answer to 3 significant figures. **[E]**

Solution:

a



Let $\mathbf{i}$ and $\mathbf{j}$ be horizontal unit vectors due east and due north respectively.

$\mathbf{v}_A = 2u\sin\theta\mathbf{i} + 2u\cos\theta\mathbf{j}$

$\mathbf{v}_B = u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}$

$\mathbf{v}_C = v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j}$

You resolve the velocities of A, B and C in the directions of $\mathbf{i}$ and $\mathbf{j}$.

${}_C\mathbf{v}_B = \mathbf{v}_C - \mathbf{v}_B = \lambda\mathbf{j}$, say

$v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j} - (u\cos\theta\mathbf{i} - u\sin\theta\mathbf{j}) = \lambda\mathbf{j}$

As the velocity of C relative to B, ${}_C\mathbf{v}_B$, is due north then it must have the form $\lambda\mathbf{j}$, where λ is a constant.

Equating $\mathbf{i}$ components

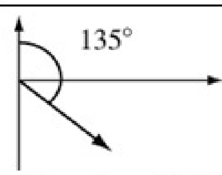
$v\sin\phi - u\cos\theta = 0$

$v\sin\phi = u\cos\theta$

$\dfrac{u}{\sin\phi} = \dfrac{v}{\cos\theta}$, as required

b ${}_C\mathbf{v}_A = \mathbf{v}_C - \mathbf{v}_A = \mu(\mathbf{i} - \mathbf{j})$, say

$v\sin\phi\mathbf{i} + v\cos\phi\mathbf{j} - (2u\sin\theta\mathbf{i} + 2u\cos\theta\mathbf{j}) = \mu(\mathbf{i} - \mathbf{j})$



A bearing of 135° is in the direction $\mathbf{i} - \mathbf{j}$.

Equating $\mathbf{i}$ components

$v\sin\phi - 2u\sin\theta = \mu$ ①

Equating $\mathbf{j}$ components

$v\cos\phi - 2u\cos\theta = -\mu$ ②

Eliminating μ between ① and ②

$v\sin\phi - 2u\sin\theta = -v\cos\phi + 2u\cos\theta$

$2u(\cos\theta + \sin\theta) = v(\cos\phi + \sin\phi)$, as required.

c From the answer to part **a**

$$u = \frac{v\sin\phi}{\cos\theta}$$

Hence

$$2\frac{v\sin\phi}{\cos\theta}(\cos\theta + \sin\theta) = v(\cos\phi + \sin\phi)$$

$$2\left(\frac{\cos\theta + \sin\theta}{\cos\theta}\right) = \frac{\cos\phi + \sin\phi}{\sin\phi}$$

$$2 + 2\tan\theta = \cot\phi + 1$$

$$\cot\phi = 1 + 2\tan\theta$$

$$\tan\phi = \frac{1}{1 + 2\tan\theta}$$

The first step in part **c** is to eliminate u between the answers to part **a** and part **b**. When you do this v also 'cancels' and you obtain a trigonometric relation between θ and ϕ.

d $\tan\phi = \dfrac{1}{1 + 2\tan 30^\circ} = 0.4641\ldots$

$\phi = 24.9^\circ$ (3 s.f.)

$v = \dfrac{u\cos\theta}{\sin\phi} = \dfrac{10\cos 30^\circ}{\sin 24.896\ldots^\circ} = 20.6$ (3 s.f.)

The true velocity of C has magnitude $20.6\ \text{m s}^{-1}$ (3 s.f.) and is on the bearing 024.9° (3 s.f.).

An answer in vector form is also acceptable. This would be $(8.66\mathbf{i} + 18.7\mathbf{j})\ \text{m s}^{-1}$.

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Review Exercise 1
Exercise A, Question 18

Question:

[*In this question* **i** *and* **j** *are horizontal unit vectors due east and due north respectively.*]
The airport B is due north of airport A. On a particular day the velocity of the wind is $(70\mathbf{i} + 25\mathbf{j})\text{km h}^{-1}$. Relative to the air, an aircraft flies with constant speed 250 km h^{-1}.
When the aircraft flies directly from A to B, find

a its speed relative to the ground,

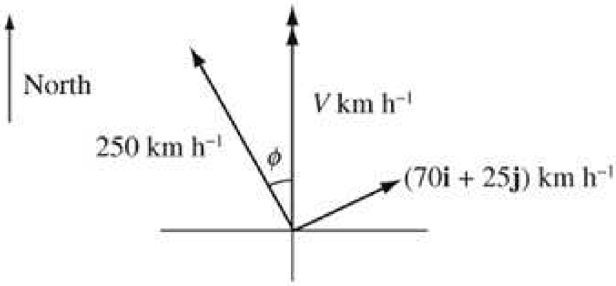
b its direction, as a bearing to the nearest degree, in which it must head.

After flying from A to B, the aircraft returns directly to A.

c Calculate the ratio of the time taken on the outward journey to the time taken on the return flight. **[E]**

Solution:

a



Let the aircraft fly on a bearing of $(360° - \phi)$ and its speed relative to the ground be V km h^{-1}.

$R(\rightarrow)\quad 70 - 250\sin\phi = 0$

$$\sin\phi = \frac{7}{25}$$

The velocity relative to the ground is the resultant of the velocity relative to the air and the velocity of the wind. As this resultant is north, the sum of the component velocities in the easterly direction must be 0.

$R(\uparrow)\quad V = 250\cos\phi + 25$

$$= 250 \times \frac{24}{25} + 25 = 265$$

$\cos^2\phi = 1 - \sin^2\phi = 1 - \frac{7^2}{25^2} = \frac{576}{25^2} = \left(\frac{24}{25}\right)^2$

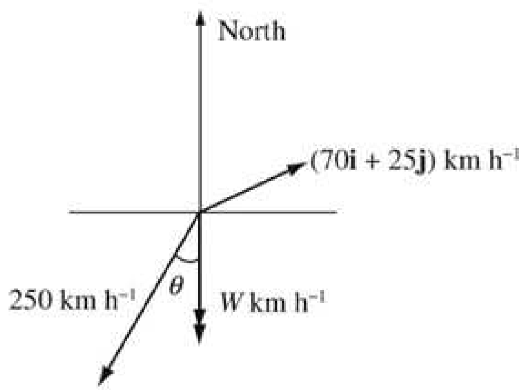
As ϕ is acute, $\cos\phi = \frac{24}{25}$.

The speed of the aircraft relative to the ground is 265 km h^{-1}.

b $\sin\phi = \frac{7}{25} \Rightarrow \phi = 16°$ (nearest degree)

The direction of the aircraft is on the bearing $(360° - \phi) = 344°$ (nearest degree).

c



On the return journey, let the aircraft fly on a bearing of $(180° + \theta)$ and its velocity relative to the ground be W km h^{-1}.

$R(\rightarrow)\quad 70 - 250\sin\theta = 0$

$$\sin\theta = \frac{7}{25}$$

$R(\downarrow)\quad W = 250\cos\theta - 25$

$$= 250 \times \frac{24}{25} - 25 = 215$$

θ has the same value as ϕ in part a

Let T_1 be the time for the outward journey and T_2 the time for the return journey.

$$\frac{T_1}{T_2} = \frac{W}{V} = \frac{215}{265} = \frac{43}{53}$$

As the distances are the same, the time of a journey is inversely proportional to the true speed of the aircraft, $T \propto \frac{1}{V}$. The actual distance travelled is not relevant in part c.

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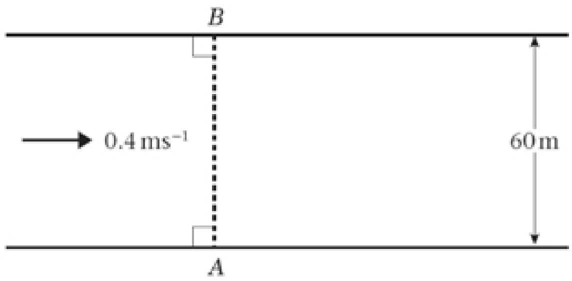
Review Exercise 1
Exercise A, Question 19

Question:

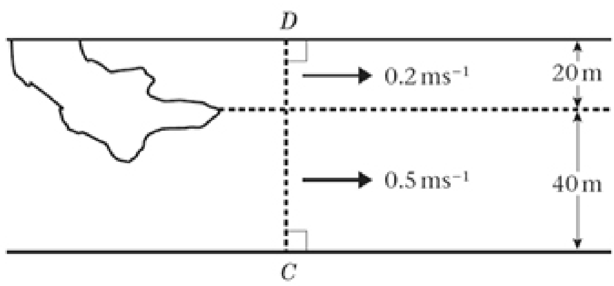


Mary swims in still water at 0.85 m s^{-1}. She swims across a straight river which is 60 m wide and flowing at 0.4 m s^{-1}. She sets off from a point A on the near bank and lands at a point B, which is directly opposite A on the far bank, as shown in the figure above.

Find

a the angle between the near bank and the direction in which Mary swims,

b the time she takes to cross the river.



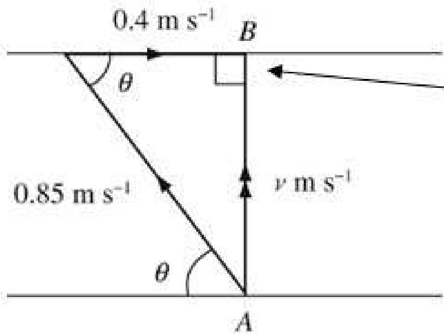
A little further downstream a large tree has fallen from the far bank into the river. The river is modelled as flowing at 0.5 m s^{-1} for a width of 40 m from the near bank, and 0.2 m s^{-1} beyond that. Nassim swims at 0.85 m s^{-1} in still water. He swims across the river from a point C on the near bank. The point D on the far bank is directly opposite C as shown above. Nassim swims at the same angle to the near bank as Mary.

c Find the maximum distance, downstream from CD, of Nassim during the crossing.

d Show that he will land at the point D. **[E]**

Solution:

a



As Mary crosses directly from A to B, the resultant of the velocity of her swimming and the velocity of the water must be perpendicular to the bank. You then use trigonometry and Pythagoras' theorem to solve parts **a** and **b**.

Let the angle between the near bank and the direction in which Mary swims be θ.

$$\cos\theta = \frac{0.4}{0.85} = \frac{8}{17}$$

$$\theta = 61.9^\circ \quad \text{(nearest } 0.1^\circ\text{)}$$

b Let Mary's speed relative to the bank be v m s^{-1}.

$$v^2 = 0.85^2 - 0.4^2 = 0.5625$$

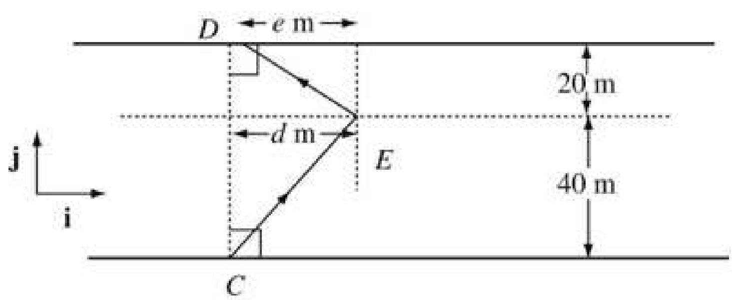
$$v = \sqrt{0.5625} = 0.75$$

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{60}{0.75}\text{ s} = 80\text{ s}$$

Mary takes 80 s to cross the river.

c



Let E be the point where Nassim is furthest downstream.

Let **i** and **j** be unit vectors parallel and perpendicular to the banks in the directions shown in the diagram.

It is possible to solve parts **c** and **d** using vector triangles. However, to get full marks you must show that Nassim lands exactly at D and this is easier to show using a vector or component method.

As Nassim moves from C to E

$${}_N\mathbf{v}_W = -0.4\mathbf{i} + 0.75\mathbf{j}$$

The velocity of Nassim relative to the water, ${}_N\mathbf{v}_W$ m s^{-1}, is the same as Mary's in parts **a** and **b**, that is $(-0.4\mathbf{i} + v\mathbf{j})$ m s^{-1} and in part **b** you showed that $v = 0.75$.

$${}_N\mathbf{v}_W = \mathbf{v}_N - \mathbf{v}_W$$
$$-0.4\mathbf{i} + 0.75\mathbf{j} = \mathbf{v}_N - 0.5\mathbf{i}$$
$$\mathbf{v}_N = 0.1\mathbf{i} + 0.75\mathbf{j}$$

From C to E, the velocity of the water, $\mathbf{v}_W$ m s^{-1} has magnitude 0.5 m s^{-1}.

Considering Nassim's motion parallel to **j** as he moves from C to E

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$
$$= \frac{40}{0.75}\text{ s} = \frac{160}{3}\text{ s}$$

Parallel to **j**, Nassim moves a distance of 40 m with speed 0.75 m s^{-1}.

Let the distance moved downstream be d m.
Considering Nassim's motion parallel to **i** as he moves from C to E

$$\text{distance} = \text{speed} \times \text{time}$$
$$d = 0.1 \times \frac{160}{3} = \frac{16}{3}$$

The resultant velocity $(0.1\mathbf{i} + 0.75\mathbf{j})$ m s^{-1} shows that Nassim is pushed downstream at a rate of 0.1 m s^{-1}.

The maximum distance downstream of Nassim during the crossing is $\frac{16}{3}$ m.

d From E to the far bank the velocity of the water has magnitude 0.2 m s^{-1} and equation * becomes

$$-0.4\mathbf{i} + 0.75\mathbf{j} = \mathbf{v}_N - 0.2\mathbf{i} \quad *$$

$$\mathbf{v}_N = -0.2\mathbf{i} + 0.75\mathbf{j}$$

You repeat the method you used in part **c** to find the distance that Nassim moves upstream as he swims from E to the far bank.

Considering Nassim's motion parallel to $\mathbf{j}$ as he moves from E to the far bank

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{20}{0.75} \text{ s} = \frac{80}{3} \text{ s}$$

Parallel to $\mathbf{j}$, Nassim moves a distance of 20 m with speed 0.75 m s^{-1}. In this direction, his speed has not changed.

Let the distance moved upstream be e m.
Considering Nassim's motion parallel to $\mathbf{i}$ as he moves from C to E

$$\text{distance} = \text{speed} \times \text{time}$$

$$e = 0.2 \times \frac{80}{3} = \frac{16}{3}$$

As $e = d$, Nassim lands at the point D.

The resultant velocity $(-0.2\mathbf{i} + 0.75\mathbf{j}) \text{ m s}^{-1}$ shows that Nassim moves upstream at a rate of 0.2 m s^{-1}.

Solutionbank M4
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Review Exercise 1
Exercise A, Question 20

Question:

A girl wishes to swim across a river from a fixed point O on the bank, to a point B on the opposite bank. The position vector of B relative to O is $20\mathbf{j}$m. In a simple model the water is assumed to be flowing with uniform velocity $u\mathbf{i}$ m s^{-1} and the girl intends to swim in such a way that she moves along the line OB.

a Given that $u = 0.6$ and that the speed of the girl relative to the water is 1 m s^{-1}, show that the time taken to swim across the river is 25 s.

A geographer points out that the flow of the river will be faster nearer the middle than closer to the banks and the model for the flow of the river is refined. When the girl is at a point R on the river, with position vector $(x\mathbf{i} + y\mathbf{j})$m , the velocity of the river at that point is $v\mathbf{i}$ m s^{-1}, where

$$v = \frac{y}{25}(20 - y), \quad 0 \le y \le 20.$$

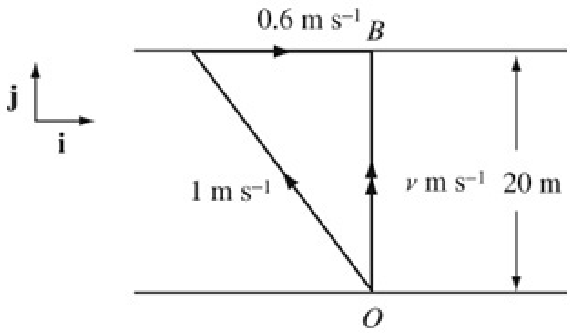
The girl swims with velocity $(-p\mathbf{j} + q\mathbf{j})$m s^{-1} relative to the water, where p and q are positive constants. The girl starts to swim from O at time $t = 0$ and the time taken to cross from O to B is now 50 s.

b Find the value of q and hence show that, at time t seconds, $y = 0.4t$.

c By considering the motion of the girl in the $\mathbf{i}$ direction, find the value of p. **[E]**

Solution:

a



Let the speed of the swimmer relative to the bank be v m s^{-1}.

$v^2 = 1^2 - 0.6^2 = 0.64$

$v = \sqrt{0.64} = 0.8$

$\text{time} = \dfrac{\text{distance}}{\text{speed}}$

$= \dfrac{20}{0.8}\text{ s} = 25\text{ s},$ as required

b Let $\mathbf{v}_W$ be the velocity of the water, $\mathbf{v}_G$ m s^{-1} $= (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$ m s^{-1} be the velocity of the girl, and ${}_G\mathbf{v}_W$ m s^{-1} be the velocity of the girl relative to the water.

You interpret the conditions of the question as

$\mathbf{v}_W = \dfrac{y}{25}(20 - y)\mathbf{i}$ and

${}_G\mathbf{v}_W = -p\mathbf{i} + q\mathbf{j}$.

$${}_G\mathbf{v}_W = \mathbf{v}_G - \mathbf{v}_W$$

$$-p\mathbf{i} + q\mathbf{j} = \mathbf{v}_G - \frac{y}{25}(20 - y)\mathbf{i}$$

$$\mathbf{v}_G = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \left(\frac{y}{25}(20 - y) - p\right)\mathbf{i} + q\mathbf{j} \quad *$$

Equation * shows that, in the $\mathbf{j}$ direction, the girl is moving with the constant speed q m s^{-1}.

Considering the motion in the $\mathbf{j}$ direction

distance = speed × time

$20 = q \times 50 \Rightarrow q = 0.4$

At time t seconds

distance = speed × time

$y = qt = 0.4t,$ as required

c Taking the **i** components of equation *

$$\dot{x} = \frac{dx}{dt} = \frac{y}{25}(20 - y) - p$$

$$= \frac{4y}{5} - \frac{y^2}{25} - p$$

$$= \frac{1.6t}{5} - \frac{0.16t^2}{25} - p$$

Using the result of part a by substituting $y = 0.4t$.

Integrating

$$x = \frac{1.6t^2}{10} - \frac{0.16t^3}{75} - pt + A$$

A is a constant of integration.

When $t = 0, x = 0 \Rightarrow A = 0$

$$x = \frac{1.6t^2}{10} - \frac{0.16t^3}{75} - pt$$

When $t = 50$, $x = 0$

When the girl reaches B after 50 s, there has been no displacement upstream or downstream.

$$0 = \frac{1.6 \times 50^2}{10} - \frac{0.16 \times 50^3}{75} - 50p$$

$$50p = 400 - \frac{800}{3}$$

$$p = 8 - \frac{16}{3} = \frac{8}{3}$$

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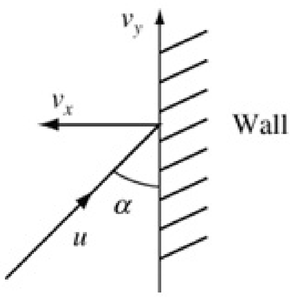
Review Exercise 1
Exercise A, Question 21

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal plane when it collides with a fixed smooth vertical wall. Immediately before the collision, the speed of S is U and its direction of motion makes an angle α with the wall. The coefficient of restitution between S and the wall is e.
Find the kinetic energy of S immediately after the collision. **[E]**

Solution:



Let the components of the velocity perpendicular and parallel to the wall immediately after the collision be v_x and v_y respectively.

Parallel to the wall

$v_y = u\cos\alpha$

> The impulse is perpendicular to the wall and so the component of the velocity parallel to the wall is unchanged.

Perpendicular to the wall

Newton's law of restitution

$v_x = eu\sin\alpha$

> Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, the component after collision $= e \times$ the component before collision
> The component of the velocity perpendicular to the wall before collision is $u\sin\alpha$.

The kinetic energy of S after the collision is given by

$$\frac{1}{2}m\left(v_x^2 + v_y^2\right)$$

$$= \frac{1}{2}m\left(e^2u^2\sin^2\alpha + u^2\cos^2\alpha\right)$$

$$= \frac{1}{2}mu^2\left(e^2\sin^2\alpha + \cos^2\alpha\right)$$

> If v is the velocity after collision, the kinetic energy of S after the collision is $\frac{1}{2}mv^2$ and $v^2 = v_x^2 + v_y^2$.

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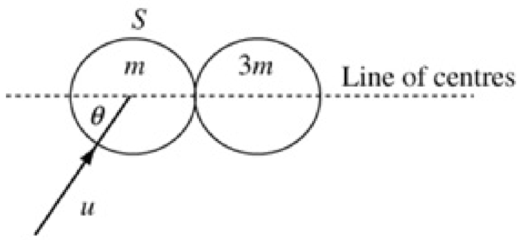
Review Exercise 1
Exercise A, Question 22

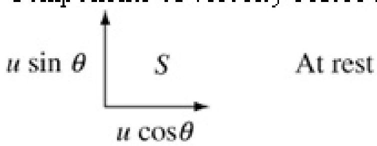
Question:

A smooth sphere S, of mass m, is moving with speed u on a horizontal plane when it collides with another smooth sphere, of mass $3m$ and having the same radius as S, which is at rest on the horizontal plane. The direction of motion of S before impact makes an angle $\theta, 0 < \theta < \frac{\pi}{2}$, with the line of centres of the two spheres. The coefficient of restitution between the spheres is e. After impact the spheres are moving in directions which are perpendicular to each other.
Find the value of e. **[E]**

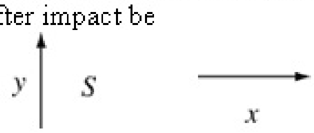
Solution:



Components of velocity before impact



Let the components of the velocity after impact be



Before the impact, the second sphere is at rest and the impulse on this sphere acts along the line of centres. So the second sphere must move along the line of centres. The question gives you that, after the impact, the spheres are moving in perpendicular directions, so S is moving perpendicular to the line of centres.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos\theta = 3mx$

$x = \frac{1}{3}u\cos\theta$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$x = eu\cos\theta$ ②

The velocity of S has no component along the line of centres and so the velocity of separation in this direction is just the velocity of the second sphere, x.

From ① and ②

$\frac{1}{3}u\cos\theta = eu\cos\theta$

Hence

$e = \frac{1}{3}$

You could find y. As the component of the velocity of S perpendicular to the line of centres is unchanged, $y = u\sin\theta$. However, in this question, this is not required.

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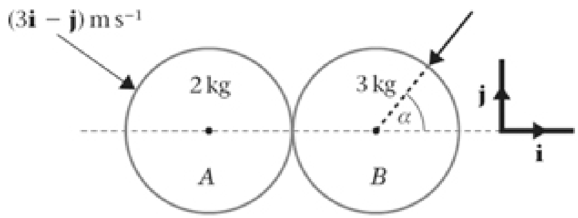
Review Exercise 1
Exercise A, Question 23

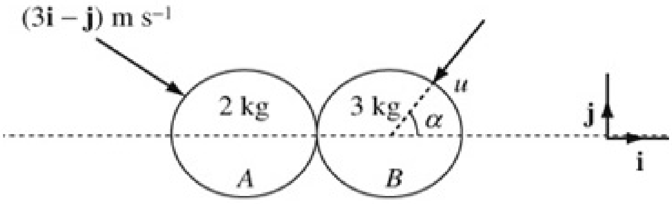
Question:



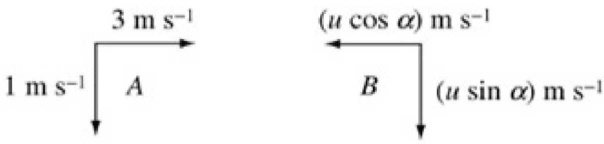
Two smooth uniform spheres A and B, of equal radius, are moving on a smooth horizontal plane. Sphere A has mass 2 kg and sphere B has mass 3 kg. The spheres collide and at the instant of collision the line joining their centres is parallel to **i**. Before the collision A has velocity $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$ and after the collision it has velocity $(-2\mathbf{i}-\mathbf{j})\text{m s}^{-1}$. Before the collision the velocity of B makes an angle α with the line of centres, as shown in the figure, where $\tan\alpha = 2$. The coefficient of restitution between the spheres is $\frac{1}{2}$. Find, in terms of **i** and **j**, the velocity of B before the collision. **[E]**

Solution:

Let the speed of B before the collision be u m s^{-1}



Components of velocity before collision



Let the components of velocity after collision be

Parallel to $\mathbf{j}$, the components of the velocity are unchanged and you can just write them down.

Parallel to $\mathbf{i}$

Conservation of linear momentum

$$2\times 3-3\times u\cos\alpha = 2\times(-2)+3z$$

$$6-3u\cos\alpha = -4+3z$$

$$3u\cos\alpha+3z=10 \quad ①$$

Equations ① and ② are a pair of simultaneous equations in $u\cos\alpha$ and z. The question asks you to find the velocity of B before the collision. You do not need to know z, so eliminate it.

Newton's law of restitution

velocity of separation $= e\times$velocity of approach

$$2+z = \frac{1}{2}(3+u\cos\alpha)$$

$$u\cos\alpha-2z=1 \quad ②$$

①×2

$$6u\cos\alpha+6z=20 \quad ③$$

②×3

$$3u\cos\alpha-6z=3 \quad ④$$

③ + ④

$$9u\cos\alpha=23$$

$$u\cos\alpha=\frac{23}{9}$$

$$\tan\alpha=\frac{u\sin\alpha}{u\cos\alpha}=2$$

The question gives you that $\tan\alpha=2$ and, as you have found $u\cos\alpha$, you can use this result to find $u\sin\alpha$.

$$u\sin\alpha=2u\cos\alpha=2\times\frac{23}{9}=\frac{46}{9}$$

The velocity of B before the collision is

$$(-u\cos\alpha\mathbf{i}-u\sin\alpha\mathbf{j})\text{m s}^{-1}=\left(-\frac{23}{9}\mathbf{i}-\frac{46}{9}\mathbf{j}\right)\text{m s}^{-1}$$

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Review Exercise 1
Exercise A, Question 24

Question:

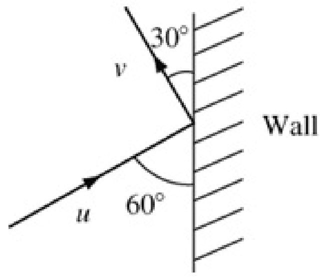
A small ball is moving on a horizontal plane when it strikes a smooth vertical wall. The coefficient of restitution between the ball and the wall is e. Immediately before the impact the direction of motion of the ball makes an angle of $60°$ with the wall. Immediately after the impact the direction of motion of the ball makes an angle of $30°$ with the wall.

a Find the fraction of the kinetic energy of the ball which is lost in the impact.

b Find the value of e. **[E]**

Solution:

a



Let the speed of the ball before impact be u m s^{-1} and the speed of the ball after impact be v m s^{-1}.

Parallel to the wall

$u\cos 60° = v\cos 30°$

$$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow v = \frac{u}{\sqrt{3}}$$

As the impulse of the wall on the ball is perpendicular to the wall, parallel to the wall the component of the velocity of the ball is unchanged.

The kinetic energy lost is

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}m\left(\frac{u}{\sqrt{3}}\right)^2$$

$$= \frac{1}{2}mu^2 - \frac{1}{6}mu^2 = \frac{1}{3}mu^2$$

Substituting $v = \frac{u}{\sqrt{3}}$.

The fraction of the kinetic energy lost is

$$\frac{\frac{1}{3}mu^2}{\frac{1}{2}mu^2} = \frac{2}{3}$$

You find the $\frac{\text{loss in kinetic energy}}{\text{original kinetic energy}}$.

b *Perpendicular to the wall*

Newton's law of restitution

$v\sin 30° = eu\cos 60°$

$$\frac{1}{2}v = \frac{\sqrt{3}}{2}eu$$

Perpendicular to the wall, Newton's law of restitution gives that, for the velocity, component after collision $= e \times$ component before collision.

As $v = \frac{u}{\sqrt{3}}$

$$\frac{1}{2} \times \frac{u}{\sqrt{3}} = \frac{\sqrt{3}}{2}eu \Rightarrow e = \frac{1}{\sqrt{3}\times\sqrt{3}} = \frac{1}{3}$$

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Edexcel AS and A Level Modular Mathematics

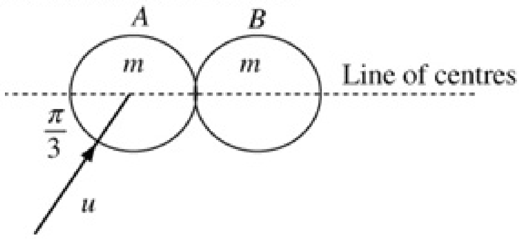
Review Exercise 1
Exercise A, Question 25

Question:

A smooth sphere A moving with speed u collides with an identical sphere B which is at rest. The directions of motion of A before and after impact makes angles $\frac{\pi}{3}$ and β respectively with the line of centres at the moment of impact. The coefficient of restitution between the spheres is 0.8.
Show that $\tan B = 10\sqrt{3}$. **[E]**

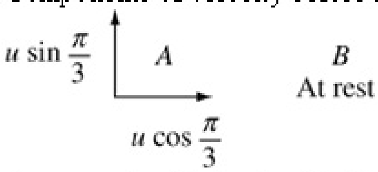
Solution:

Let the mass of the spheres be m, the speed of A after the collision be v and the speed of B after the collision be w.

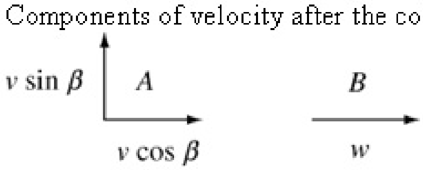


> In questions about identical spheres, the mass of the spheres will usually cancel out of any equations but it is sensible to introduce a variable for the mass, m, so that you can write down the equation for conservation of linear momentum.

Components of velocity before the collision



Components of velocity after the collision



> Before the collision, B is at rest and the impulse on B acts along the line of centres. Hence, after the collision, B must move along the line of centres.

Perpendicular to the line of centres

$$v\sin\beta = u\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}u \quad ①$$

> Perpendicular to the line of centres the component of the velocity is unchanged.

Parallel to line of centres

Conservation of linear momentum

$$mu\cos\frac{\pi}{3} = mv\cos\beta + mw$$

$$w = \frac{1}{2}u - v\cos\beta \quad ②$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$w - v\cos\beta = 0.8u\cos\frac{\pi}{3}$$

$$w = 0.4u + v\cos\beta \ldots \quad ③$$

> The value of w is not required, so you eliminate it between equations ② and ③.

Eliminating w from ② and ③

$$\frac{1}{2}u - v\cos\beta = 0.4u + v\cos\beta$$

$$v\cos\beta = 0.05u \quad ④$$

Dividing ① by ④

$$\frac{\not{v}}{\not{v}}\frac{\sin\beta}{\cos\beta} = \frac{\frac{\sqrt{3}}{2}}{0.05}\frac{\not{u}}{\not{u}}$$

$$\tan\beta = 10\sqrt{3}, \quad \text{as required}$$

> You eliminate u and v between equations ① and ④ by dividing the equations.

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Review Exercise 1
Exercise A, Question 26

Question:

A smooth uniform sphere P of mass m is falling vertically and strikes a fixed smooth inclined plane with speed u. The plane is inclined at an angle $\theta, \theta < 45°$, to the horizontal.
The coefficient of restitution between P and the inclined plane is e. Immediately after P strikes the plane, P moves horizontally.

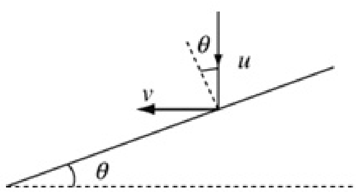
a Show that $e = \tan^2 \theta$.

b Show that the magnitude of the impulse exerted by P on the plane is $mu \sec \theta$.

[E]

Solution:

a



Let the speed of P immediately after the impact be v.

Parallel to the plane
$u \sin \theta = v \cos \theta$ ①

Parallel to the plane, the component of the velocity is unchanged by the impact.

Perpendicular to the plane

Newton's law of restitution
$v \sin \theta = eu \cos \theta$
$eu \cos \theta = v \sin \theta$ ②

Dividing ② by ①

$$\frac{e\not{u} \cos \theta}{\not{u} \sin \theta} = \frac{\not{v} \sin \theta}{\not{v} \cos \theta}$$

$$e = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta, \quad \text{as required}$$

b Resolving perpendicular to the plane, the impulse is given by

$$I = mv \sin \theta - m(-u \cos \theta)$$
$$= m(v \sin \theta + u \cos \theta)$$
$$= m\left(\frac{u \sin \theta}{\cos \theta} \sin \theta + u \cos \theta\right)$$
$$= mu\left(\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta}\right) = mu\left(\frac{1}{\cos \theta}\right)$$
$$= mu \sec \theta, \quad \text{as required}$$

The impulse is perpendicular to the plane and its magnitude is the difference in the linear momenta resolved perpendicular to the plane.

From equation ①, in part a, $v = \dfrac{u \sin \theta}{\cos \theta}$

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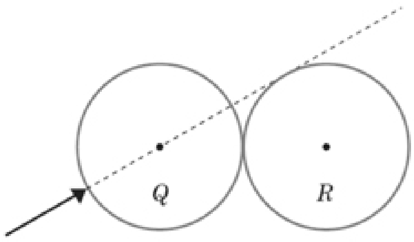
Edexcel AS and A Level Modular Mathematics

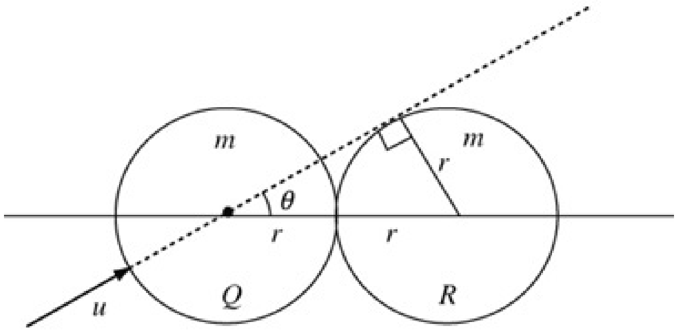
Review Exercise 1
Exercise A, Question 27

Question:



A smooth uniform sphere P is at rest on a smooth horizontal plane, when it is struck by an identical sphere Q moving on the plane. Immediately before the impact, the line of motion of the centre of Q is tangential to the sphere P, as shown the figure. The direction of motion of Q is turned through $30°$ by the impact. Find the coefficient of restitution between the spheres. **[E]**

Solution:



Let the mass of each of the spheres be m and the radius of each of the spheres be r.
Let the angle the direction of motion of Q makes with the line of centres before the impact be θ.

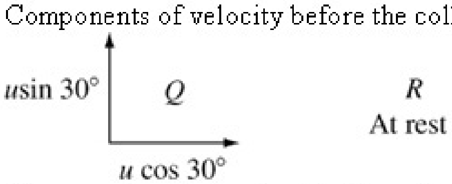
$$\sin\theta = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = 30^\circ$$

Hence, the angle the direction of motion of Q makes with the line of centres, after the impact, is 60°.

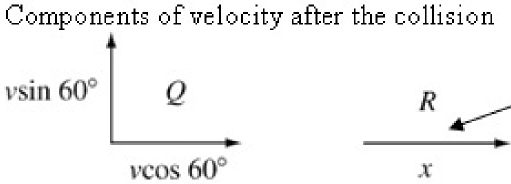
> Q is turned through 30° so, after the collision, it makes an angle of $30^\circ + 30^\circ = 60^\circ$ with the line of centres.

Let the speed of Q immediately after the impact be v and the speed of R immediately after the impact be x.

Components of velocity before the collision



Components of velocity after the collision



> Initially, R is at rest and the impulse of Q on R acts along the line of centres. So, after the impact, R moves along the line of centres.

Perpendicular to the line of centres

For Q

$$u\sin 30^\circ = v\sin 60^\circ$$

$$\frac{1}{2}u = \frac{\sqrt{3}}{2}v \Rightarrow u = v\sqrt{3}. \quad ①$$

> As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

Parallel to the line of centres

Conservation of linear momentum

$$mu\cos 30^\circ = mv\cos 60^\circ + mx$$

$$x = \frac{\sqrt{3}}{2}u - \frac{1}{2}v \quad ②$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x - v\cos 60^\circ = eu\cos 30^\circ$$

$$x = \frac{\sqrt{3}}{2}eu + \frac{1}{2}v \qquad ③$$

Eliminating x from ② and ③

$$\frac{\sqrt{3}}{2}eu + \frac{1}{2}v = \frac{\sqrt{3}}{2}u - \frac{1}{2}v$$

$$v = \frac{\sqrt{3}}{2}u(1-e)$$

You now use equation ① to eliminate u and v.

Using ①

$$v = \frac{\sqrt{3}}{2}v\sqrt{3}(1-e)$$

Dividing both sides by v and using $\sqrt{3}\times\sqrt{3}=3$.

$$1 = \frac{3}{2}(1-e) \Rightarrow \frac{2}{3} = 1-e$$

$$e = \frac{1}{3}$$

The coefficient of restitution between the spheres is $\frac{1}{3}$.

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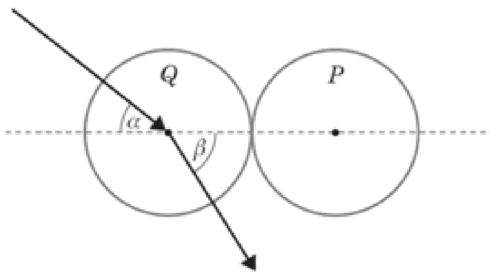
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 28

Question:

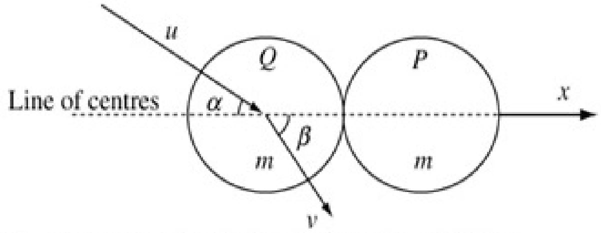


A smooth sphere P lies at rest on a smooth horizontal plane. A second identical sphere Q, moving on the plane, collides with the sphere P. Immediately before the collision the direction of motion of Q makes an angle α with the line joining the centres of the spheres. Immediately after the collision the direction of motion of Q makes an angle β with the line joining the centres of spheres, as shown in the figure. The coefficient of restitution between the spheres is e.
Show that $(1-e)\tan\beta = 2\tan\alpha$. **[E]**

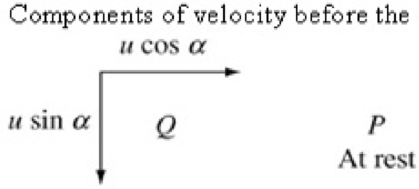
Solution:

Let the mass of each of the spheres be m.
Let the speed of Q immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of P immediately after the collision be x.

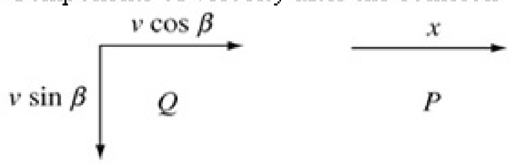
You need to introduce a number of variables to solve this question and you need to make clear to an examiner what the variables stand for. You can do this with a clearly labelled diagram.



Components of velocity before the collision



Components of velocity after the collision



Perpendicular to the line of centres
For Q
$u\sin\alpha = v\sin\beta$ ①

As the impulse is along the line of centres, the component of the velocity of Q perpendicular to the line of centres is unchanged.

Along line of centres

Conservation of linear momentum
$\not{m}u\cos\alpha = \not{m}v\sin\beta + \not{m}x$
$x = u\cos\alpha - v\cos\beta$ ②

Use these two equations to eliminate x, the speed of P.

Newton's law of restitution
velocity of separation $= e \times$ velocity of approach
$$x - v\cos\beta = eu\cos\alpha$$
$$x = eu\cos\alpha + v\cos\beta \quad ③$$

Eliminating x between ② and ③
$u\cos\alpha - v\cos\beta = eu\cos\alpha + v\cos\beta$
$(1-e)u\cos\alpha = 2v\cos\beta$ ④

Dividing ① by ④
$$\frac{u\sin\alpha}{(1-e)u\cos\alpha} = \frac{v\sin\beta}{2v\cos\beta}$$
$$\frac{\tan\alpha}{1-e} = \frac{\tan\beta}{2}$$
$(1-e)\tan\beta = 2\tan\alpha$, as required

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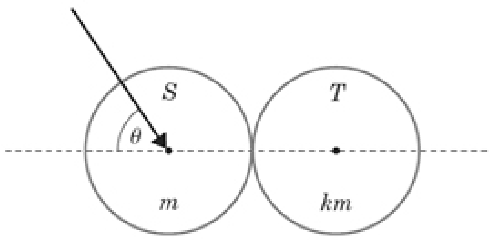
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 29

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal table. The sphere S collides with another smooth uniform sphere T, of the same radius as S but of mass $km, k > 1$, which is at rest on the table. The coefficient of restitution between the spheres is e. Immediately before the spheres collide the direction of motion of S makes an angle θ with the line joining their centres, as shown in the figure.



Immediately after the collision the directions of motion of S and T are perpendicular.

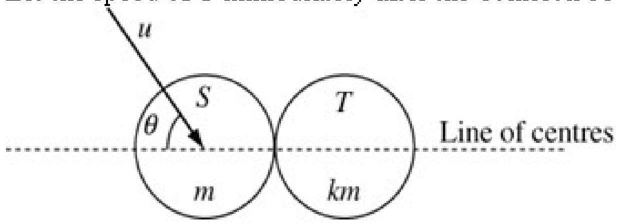
a Show that $e = \frac{1}{k}$.

Given that $k = 2$ and that the kinetic energy lost in the collision is one quarter of the initial kinetic energy,

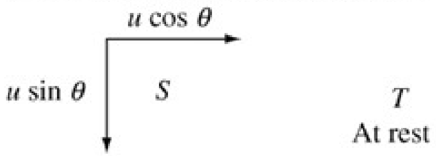
b find the value of θ. **[E]**

Solution:

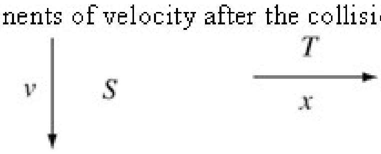
a Let the speed of S immediately before the collision be u and its speed immediately after the collision be v.
Let the speed of T immediately after the collision be x.



Components of velocity before the collision



Components of velocity after the collision



Before the collision, T is at rest and the impulse on T acts along the line of centres. So, after the collision, T moves along the line of centres. After the collision, the spheres are moving in perpendicular directions, so S is moving perpendicular to the line of centres.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos\theta = kmx$

$kx = u\cos\theta$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$x = eu\cos\theta$ ②

Parallel to the line of centres, the velocity of separation of the spheres is just x.

Dividing ② by ①

$$\frac{\cancel{x}}{k\cancel{x}} = \frac{e\cancel{u}\cancel{\cos\theta}}{\cancel{u}\cancel{\cos\theta}}$$

$e = \frac{1}{k}$, as required

b If $k = 2$, then $e = \frac{1}{2}$.

Using the printed answer to part **a**.

From equation ② in part **a**

$x = \frac{1}{2}u\cos\theta$

Perpendicular to the line of centres

For S

$v = u\sin\theta$

The initial kinetic energy is $\frac{1}{2}mu^2$

The final kinetic energy is

The mass of T is km.

$$\frac{1}{2}mv^2 + \frac{1}{2}kmx^2 = \frac{1}{2}m(u\sin\theta)^2 + \frac{1}{2}km\left(\frac{1}{2}u\cos\theta\right)^2$$

$$= \frac{1}{2}mu^2\sin^2\theta + \frac{1}{4}mu^2\cos^2\theta$$

As $k = 2$.

The total loss in kinetic energy is

$$\frac{1}{2}mu^2 - \left(\frac{1}{2}mu^2\sin^2\theta + \frac{1}{4}mu^2\cos^2\theta\right)$$

$$= \frac{1}{2}mu^2(1-\sin^2\theta) - \frac{1}{4}mu^2\cos^2\theta$$

$$= \frac{1}{2}mu^2\cos^2\theta - \frac{1}{4}mu^2\cos^2\theta = \frac{1}{4}mu^2\cos^2\theta$$

Hence

$$\frac{1}{4}mu^2\cos^2\theta = \frac{1}{4}\times\frac{1}{2}mu^2$$

The loss of energy is one quarter of the initial kinetic energy; that is one quarter of $\frac{1}{2}mu^2$.

$$\cos^2\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = 45°$$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 30

Question:

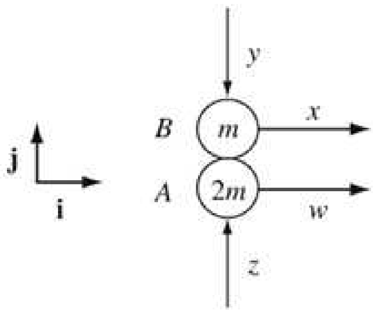
A smooth uniform sphere A has mass $2m$ kg and another smooth uniform sphere B, with the same radius as A, has mass m kg. The spheres are moving on a smooth horizontal plane when they collide. At the instant of collision the line joining the centres of the spheres is parallel to $\mathbf{j}$. Immediately **after** the collision, the velocity of A is $(3\mathbf{i}-\mathbf{j})\text{m s}^{-1}$ and the velocity of B is $(2\mathbf{i}+\mathbf{j})\text{m s}^{-1}$. The coefficient of restitution between the spheres is $\frac{1}{2}$.

a Find the velocities of the two spheres immediately before the collision.
b Find the magnitude of the impulse in the collision.
c Find, to the nearest degree, the angle through which the direction of motion of A is deflected by the collision. **[E]**

Solution:

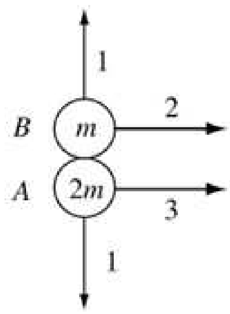
a Let the velocity of B before the collision be $(x\mathbf{i} - y\mathbf{j})\text{m s}^{-1}$ and the velocity of A before the collision be $(w\mathbf{i} + z\mathbf{j})\text{m s}^{-1}$.

Before the collision



The components of the velocities are in m s^{-1}.

After the collision



Parallel to $\mathbf{i}$

$x = 2, w = 3$

As the impulse is in the direction of $\mathbf{j}$, the components of the velocities of both A and B in the direction of $\mathbf{i}$ are unchanged.

Parallel to $\mathbf{j}$

Conservation of linear momentum

$-my + 2mz = m \times 1 - 2m \times 1$

$-y + 2z = -1$ ①

Newton's law of restitution

velocity of separation $= e \times$velocity of approach

$$1 - (-1) = \frac{1}{2}(y + z)$$

$$y + z = 4 \quad ②$$

① + ②

$3z = 3 \Rightarrow z = 1$

Substituting $z = 1$ into ②

$y + 1 = 4 \Rightarrow y = 3$

The velocity of A is $(3\mathbf{i} + \mathbf{j})\text{m s}^{-1}$.

The velocity of B is $(2\mathbf{i} - 3\mathbf{j})\text{m s}^{-1}$.

b Considering the change in momentum of A in the direction of $\mathbf{j}$.

As the impulse is in the direction of $\mathbf{j}$, you can consider the change of momentum of either A or B in the direction of $\mathbf{j}$.

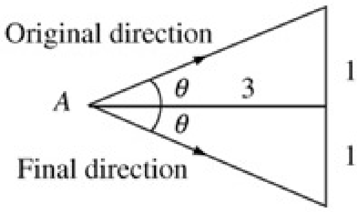
$I = 2mv - 2mu$

$= 2m \times 1 - 2m \times (-1) = 4m$

The mass of A is $2m$.

The magnitude of the impulse is $4m$ N s.

c



A is deflected from the direction of $(3\mathbf{i} + \mathbf{j})$ to the direction of $(3\mathbf{i} - \mathbf{j})$.

The angle of deflection is given by

$2\theta = 2\arctan\frac{1}{3} = 37°$ (nearest degree)

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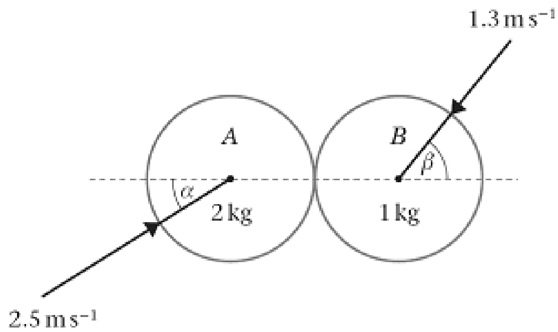
Review Exercise 1
Exercise A, Question 31

Question:



Two smooth uniform spheres A and B of equal radius have masses 2 kg and 1 kg respectively. They are moving on a smooth horizontal plane when they collide. Immediately before the collision the speed of A is 2.5 m s^{-1} and the speed of B is 1.3 m s^{-1}. When they collide the line joining their centres makes an angle α with the direction of motion of A and an angle β with the direction of motion of B, where $\tan \alpha = \frac{4}{3}$ and $\tan \beta = \frac{12}{5}$, as shown in the figure.

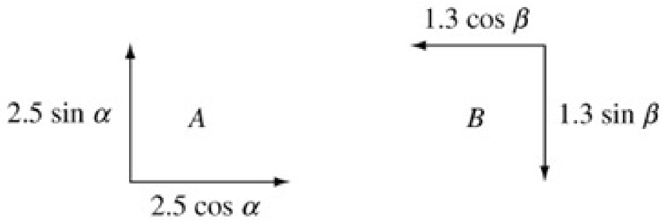
a Find the components of the velocities of A and B perpendicular and parallel to the line of centres immediately before the collision.

The coefficient of restitution between A and B is $\frac{1}{2}$.

b Find, to one decimal place, the speed of each sphere after the collision. **[E]**

Solution:

a Components of the velocity before the collision.
All velocities are in m s^{-1}

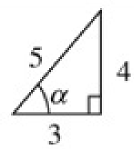


The component of the velocity of A perpendicular to the line of centres immediately before the collision is

$2.5 \sin \alpha \text{ m s}^{-1} = 2.5 \times \frac{4}{5} \text{ m s}^{-1} = 2 \text{ m s}^{-1}$

The component of the velocity of A parallel to the line of centres immediately before the collision is

$2.5 \cos \alpha \text{ m s}^{-1} = 2.5 \times \frac{3}{5} \text{ m s}^{-1} = 1.5 \text{ m s}^{-1}$



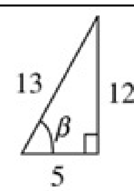
This sketch illustrates that, as $3^2 + 4^2 = 5^2$, if $\tan \beta = \frac{4}{3}$, then $\sin \beta = \frac{4}{5}$ and $\cos \beta = \frac{3}{5}$.

The component of the velocity of B perpendicular to the line of centres immediately before the collision is

$1.3 \sin \beta \text{ m s}^{-1} = 1.3 \times \frac{12}{13} \text{ m s}^{-1} = 1.2 \text{ m s}^{-1}$

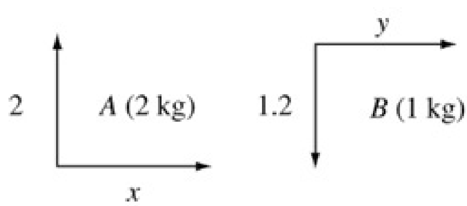
The component of the velocity of B parallel to the line of centres immediately before the collision is

$1.3 \cos \beta \text{ m s}^{-1} = 1.3 \times \frac{5}{13} \text{ m s}^{-1} = 0.5 \text{ m s}^{-1}$



This sketch illustrates that, as $5^2 + 12^2 = 13^2$, if $\tan \beta = \frac{12}{5}$, then $\sin \beta = \frac{12}{13}$ and $\cos \beta = \frac{5}{13}$.

b Let the components of the velocity after the collision be, with all velocities in m s^{-1},



The components perpendicular to the line of centres are unchanged.

Parallel to the line of centres

Conservation of linear momentum

$2x + y = 2 \times 1.5 - 1 \times 0.5$

$2x + y = 2.5$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$y - x = \frac{1}{2}(1.5 + 0.5)$$

$$y - x = 1 \qquad ②$$

A approaches the point of the collision with speed 1.5 m s^{-1} along the line of centres. B approaches the point of the collision with speed 0.5 m s^{-1} along the line of centres. So the two spheres are approaching one another at the speed of $(1.5 + 0.5) \text{ m s}^{-1} = 2 \text{ m s}^{-1}$

① − ②

$3x = 1.5 \Rightarrow x = 0.5$

From ②
$y = 1 + x = 1.5$

The speed of A is
$\sqrt{(0.5^2 + 2^2)} \text{ m s}^{-1} = \sqrt{4.25} \text{ m s}^{-1} = 2.1 \text{ m s}^{-1}$ (1 d.p.)

The speed of B is
$\sqrt{(1.2^2 + 1.5^2)} \text{ m s}^{-1} = \sqrt{3.69} \text{ m s}^{-1} = 1.9 \text{ m s}^{-1}$ (1 d.p.)

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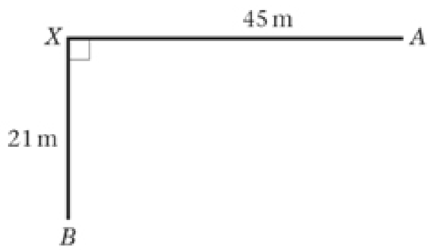
Review Exercise 1
Exercise A, Question 32

Question:



The figure represents the scene of a road accident. A car of mass 600 kg collided at the point X with a stationary van of mass 800 kg. After the collision the van came to rest at the point A having travelled a horizontal distance of 45 m, and the car came to rest at the point B having travelled a horizontal distance of 21 m. The angle AXB is $90°$.
The accident investigators are trying to establish the speed of the car before the collision and they model both vehicle as small spheres.

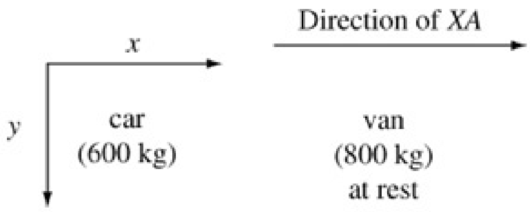
a Find the coefficient of restitution between the car and the van.

The investigators assume that after the collision, and until the vehicles came to rest, the van was subject to a constant horizontal force of 500 N acting along AX and the car to a constant horizontal force of 300 N along BX.

b Find the speed of the car immediately before the collision. **[E]**

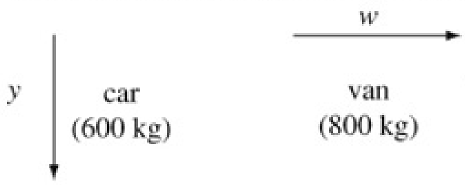
Solution:

a Let the components of the velocity of the car before the collision, with all components in m s^{-1}, be



As the van is at rest, after the collision it must travel along the line of centres of the car and the van. In the diagram in the question, *XA* must be the line of centres, so you consider the components of velocity perpendicular and parallel to *XA*.

Let the components of the velocity of the car and van after the collision, with all components in m s^{-1}, be



After the collision, the van is moving along *XA* and the car is moving perpendicular to *XA*.

Parallel to XA

Conservation of linear momentum

$$600x = 800w \Rightarrow w = \frac{3}{4}x \quad *$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$w = ex$$

Hence

$$\frac{3}{4}x = ex$$

$$e = \frac{3}{4}$$

b For the van

$$\mathbf{F} = m\mathbf{a}$$

$$-500 = 800a \Rightarrow a = -0.625$$

$$v^2 = u^2 + 2as$$

$$0^2 = w^2 - 2 \times 0.625 \times 45$$

$$w^2 = 56.25 \Rightarrow w = 7.5$$

To find w (and hence x) and y, you need to use both Newton's second law and the kinematic equation for constant acceleration, $v^2 = u^2 + 2as$.

For the car

$$\mathbf{F} = m\mathbf{a}$$

$$-300 = 600a \Rightarrow a = -0.5$$

$$v^2 = u^2 + 2as$$

$$0^2 = y^2 - 2 \times 0.5 \times 21$$

$$y^2 = 21 \Rightarrow y = \sqrt{21}$$

From equation * in part **a**

$$w = \frac{3}{4}x$$

$$7.5 = \frac{3}{4}x \Rightarrow x = \frac{7.5}{\frac{3}{4}} = 10$$

Let the speed of the car immediately before the collision be U m s^{-1}

$$U^2 = x^2 + y^2 = 10^2 + 21 = 121$$

$$U = \sqrt{121} = 11$$

The speed of the car immediately before the collision is 11 m s^{-1}.

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Review Exercise 1
Exercise A, Question 33

Question:

A smooth sphere T is at rest on a smooth horizontal table. An identical sphere S moving on the table with speed U collides with T. The directions of motion of S before and after impact make angles of $30°$ and $\beta°$ $(0 < \beta < 90)$ respectively with L, the line of centres at the moment of impact. The coefficient of restitution between S and T is e.

a Show that V, the speed of T immediately after impact, is given by $V = \frac{U\sqrt{3}}{4}(1+e)$.

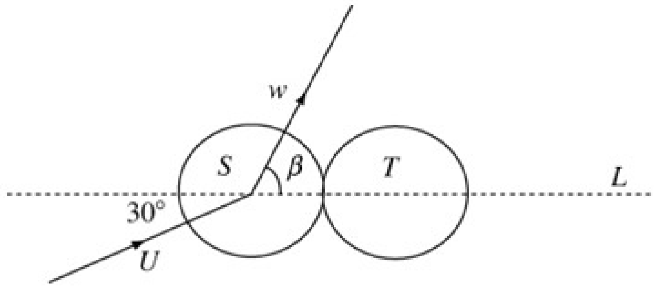
b Find the components of the velocity of S, parallel and perpendicular to L, immediately after impact.

Given that $e = \frac{2}{3}$,

c find, to 1 decimal place, the value of β. **[E]**

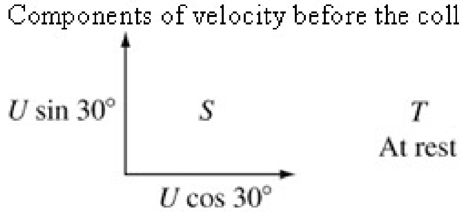
Solution:

a

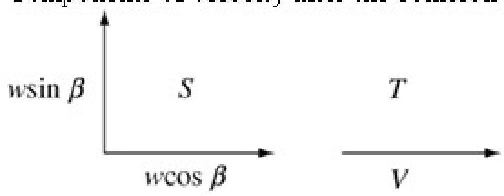


Let the speed of S after the collision be w and the mass of both S and T be m.

Components of velocity before the collision



Components of velocity after the collision



As the impact is along L, sphere T must move along L.

Parallel to the L

Conservation of linear momentum

$mU\cos 30° = mw\cos\beta + mV$

$w\cos\beta + V = \frac{\sqrt{3}}{2}U$ ①

Using $\cos 30° = \frac{\sqrt{3}}{2}$.

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$V - w\cos\beta = eU\cos 30°$$

$$V - w\cos\beta = \frac{\sqrt{3}}{2}eU \quad ②$$

① + ②

$$2V = \frac{\sqrt{3}}{2}U + \frac{\sqrt{3}}{2}eU = \frac{U\sqrt{3}}{2}(1+e)$$

$V = \frac{U\sqrt{3}}{4}(1+e)$, as required

b Subtracting ② from ①

$2w\cos\beta = \frac{\sqrt{3}}{2}U - \frac{\sqrt{3}}{2}eU = \frac{\sqrt{3}}{2}(1-e)U$

$w\cos\beta = \frac{\sqrt{3}}{4}(1-e)U$ ③

> The components of the velocity of S, parallel and perpendicular to L, immediately after impact, are $w\cos\beta$ and $w\sin\beta$ respectively. You find $w\cos\beta$ using the equations ① and ② in part **a**.

Perpendicular to L
For S

$w\sin\beta = U\sin 30° = \frac{1}{2}U$ ④

> The component of the velocity of S perpendicular to the impulse is unchanged.

The components of the velocity of S, parallel and perpendicular to L, immediately after impact are $\frac{\sqrt{3}}{4}(1-e)U$ and $\frac{1}{2}U$, respectively.

c If $e = \frac{2}{3}$, from ③

$w\cos\beta = \frac{\sqrt{3}}{4}\left(1-\frac{2}{3}\right)U = \frac{\sqrt{3}}{12}U$

Also $w\sin\beta = \frac{1}{2}U$

Dividing

$\frac{w\sin\beta}{w\cos\beta} = \frac{\frac{1}{2}U}{\frac{\sqrt{3}}{12}U}$

$\tan\beta = \frac{6}{\sqrt{3}} = 2\sqrt{3}$

$\beta = 73.9°$ (1 d.p.)

> Use your calculator to complete the question. As no mode is specified, $\beta = 1.3$ radians is also acceptable.

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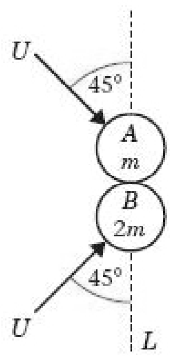
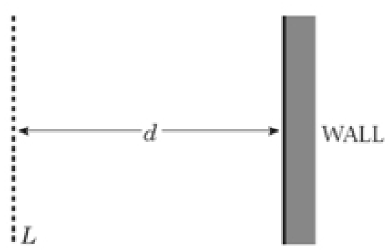
Review Exercise 1
Exercise A, Question 34

Question:

Two small spheres A and B, of equal size and of mass m and $2m$ respectively, are moving initially with the same speed U on a smooth horizontal floor. The spheres collide when their centres are on a line L. Before the collision the spheres are moving towards each other, with their directions of motion perpendicular to each other and each inclined at an angle $45°$ to the line L, as shown in the figure below. The coefficient of restitution between the spheres is $\frac{1}{2}$.



a Find the magnitude of the impulse which acts on A in the collision.

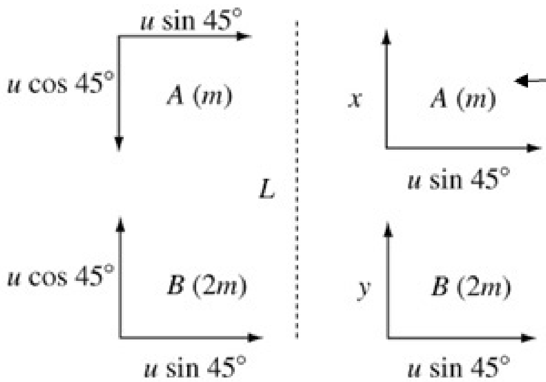


The line L is parallel to and a distance d from a smooth vertical wall, as shown in the second figure.

b Find, in terms of d, the distance between the points at which the spheres first strike the wall. **[E]**

Solution:

a Components before collision Components after collision



The components perpendicular to L are unchanged by the collision. In this case, all of the components perpendicular to L are $u \sin 45° = \frac{u}{\sqrt{2}}$.

Parallel to L

Conservation of linear momentum (↑)

$$2mu\cos 45° - mu\cos 45° = mx + 2my$$

$$x + 2y = \frac{u}{\sqrt{2}} \quad ①$$

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x - y = \frac{1}{2}\left(u\cos 45° + u\cos 45°\right)$$

$$x - y = \frac{u}{\sqrt{2}} \quad ②$$

Use $\cos 45° = \frac{1}{\sqrt{2}}$.

① − ②

$3y = 0 \Rightarrow y = 0$

Hence $x = \frac{u}{\sqrt{2}}$

As $y = 0$, after the collision B is travelling perpendicular to L. You will need this to solve part **b**.

The impulse on A is given by

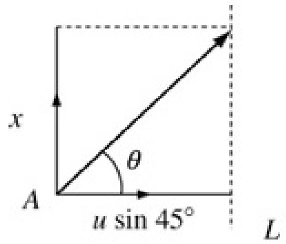
(↑) $I =$ final momentum of $A -$ initial momentum of A

$m\left(-u \sin 45°\right)$

$$= \frac{mu}{\sqrt{2}} + \frac{mu}{\sqrt{2}} = \frac{2mu}{\sqrt{2}} = \sqrt{2}mu$$

The magnitude of the impulse which acts on A in the collision is $\sqrt{2}mu$.

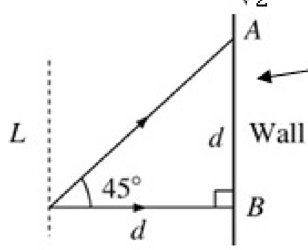
b



This diagram shows the components of the velocity of A parallel and perpendicular to L. As both x and $u \sin 45^\circ$ are $\frac{u}{\sqrt{2}}$, A moves at an angle of 45° to the direction of motion of B.

The direction of motion of A is given by

$$\tan\theta = \frac{x}{u\sin 45^\circ} = \frac{\frac{u}{\sqrt{2}}}{\frac{u}{\sqrt{2}}} = 1 \Rightarrow \theta = 45^\circ$$



This diagram shows the paths of A and B from the point of collision to the points where they hit the wall. The triangle is isosceles and the required distance is d.

The distance between the points at which the spheres first strike the wall is d.

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Review Exercise 1
Exercise A, Question 35

Question:

Two smooth uniform spheres A and B have equal radii. Sphere A has mass m and sphere B has mass km. The spheres are at rest on a smooth horizontal table. Sphere A is then projected along the table with speed u and collides with B. Immediately before the collision, the direction of motion of A makes an angle of $60°$ with the line joining the centres of the two spheres. The coefficient of restitution between the spheres is $\frac{1}{2}$.

a Show that the speed of B immediately after the collision is $\frac{3u}{4(k+1)}$.

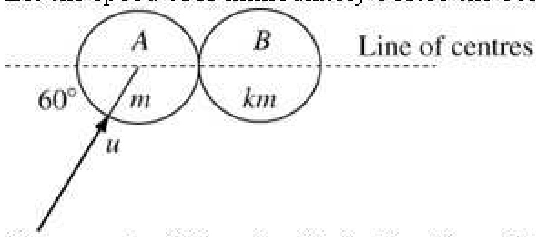
Immediately after the collision the direction of motion of A makes an angle $\arctan(2\sqrt{3})$ with the direction of motion of B.

b Show that $k = \frac{1}{2}$.

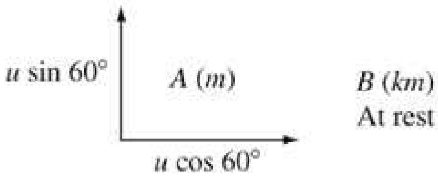
c Find the loss of kinetic energy due to the collision. **[E]**

Solution:

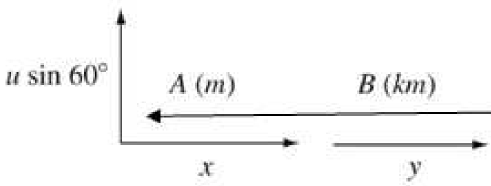
a Let the speed of A immediately before the collision be u.



Components of the velocities before the collision



Let the components of the velocities after collision be



As the impulse is along the line of centres, the component of the velocity of A perpendicular to the line of centres is unchanged.

Parallel to the line of centres

Conservation of linear momentum

$mu\cos 60^\circ = mx + kmy$

$$x + ky = \frac{1}{2}u \quad ①$$

As B moves along the line of centres, the component, y, of the velocity of B along the line of centres is the velocity of B. So to solve part **a**, you must find y from this pair of simultaneous equations.

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$y - x = \frac{1}{2}u\cos 60^\circ = \frac{1}{4}u \quad ②$$

$$ky + y = \frac{3u}{4}$$

$$y = \frac{3u}{4(k+1)}, \text{ as required}$$

b From ②

$$x = y - \frac{u}{4} = \frac{3u}{4(k+1)} - \frac{u}{4} = \frac{3u - u(k+1)}{4(k+1)}$$

$$= \frac{(2-k)u}{4(k+1)}$$

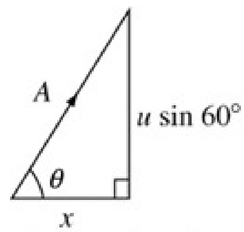
The direction of motion of A is given by

$$\tan\theta = \frac{u\sin 60^\circ}{x} = \frac{\frac{\sqrt{3}u}{2}}{\frac{(2-k)u}{4(k+1)}}$$

$$= \frac{4(k+1)\sqrt{3}}{2(2-k)} = 2\sqrt{3}$$

$$k+1 = 2-k$$

$$2k = 1 \Rightarrow k = \frac{1}{2}\text{, as required}$$



The question gives you that the direction of motion of A makes an angle $\arctan(2\sqrt{3})$ with the line of centres, so $\tan\theta = 2\sqrt{3}$. This gives an equation that you can solve for k.

c If $k = \frac{1}{2}$,

$$y = \frac{3u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{2}u$$

$$x = \frac{\left(2-\frac{1}{2}\right)u}{4\left(\frac{1}{2}+1\right)} = \frac{1}{4}u$$

The kinetic energy of the system after the collision is

$$\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right) + \frac{1}{2}kmy^2$$

$$= \frac{1}{2}m\left(\frac{u^2}{16} + \frac{3u^2}{4}\right) + \frac{1}{4}m \times \frac{1}{4}u^2$$

$$= \frac{1}{2}mu^2\left(\frac{1}{16} + \frac{3}{4} + \frac{1}{8}\right) = \frac{15}{32}mu^2$$

After the collision the velocity of A has components x and $u\sin 60^\circ$. So the kinetic energy of A after the collision is $\frac{1}{2}m\left(x^2 + (u\sin 60^\circ)^2\right)$.

The loss in kinetic energy is

$$\frac{1}{2}mu^2 - \frac{15}{32}mu^2 = \frac{1}{32}mu^2$$

Before the collision only A is moving and it has speed u, so the initial kinetic energy of the system is $\frac{1}{2}mu^2$.

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Review Exercise 1
Exercise A, Question 36

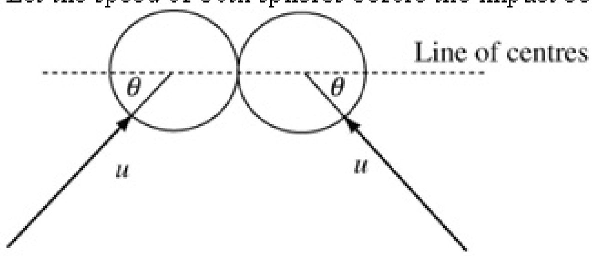
Question:

Two equal smooth spheres approach each other from opposite directions with equal speeds. The coefficient of restitution between the spheres is e. At the moment of impact, their common normal is inclined at an angle θ to the original direction of motion. After impact, each sphere moves at right angles to its original direction of motion.

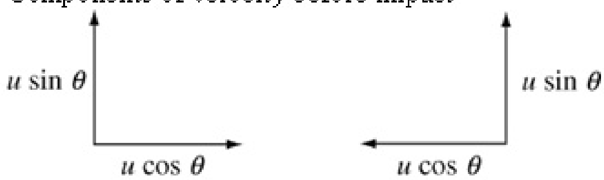
Show that $\tan\theta = \sqrt{e}$. **[E]**

Solution:

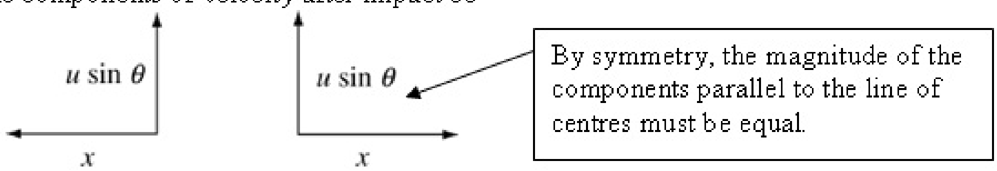
Let the speed of both spheres before the impact be u and the mass of each sphere m.



Components of velocity before impact



Let the components of velocity after impact be



Parallel to the line of centres

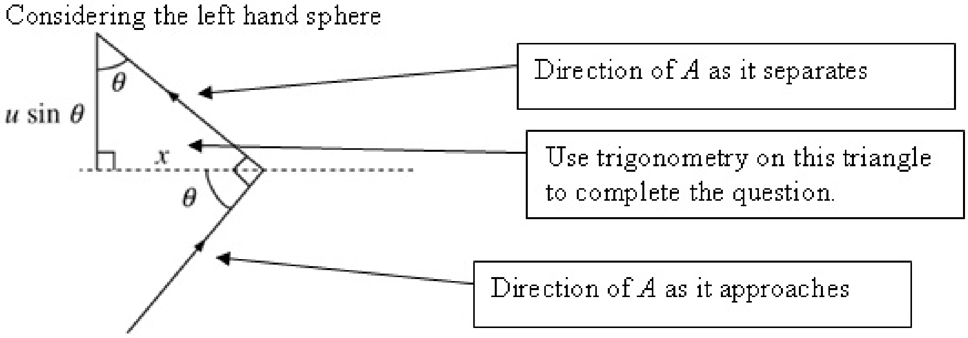
Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$2x = e2u\cos\theta$$

$$x = eu\cos\theta$$

Considering the left hand sphere



$$\tan\theta = \frac{x}{u\sin\theta} = \frac{e\cancel{u}\cos\theta}{\cancel{u}\sin\theta}$$

$$e = \tan\theta \times \frac{\sin\theta}{\cos\theta} = \tan^2\theta$$

$\tan\theta = \sqrt{e}$, as required

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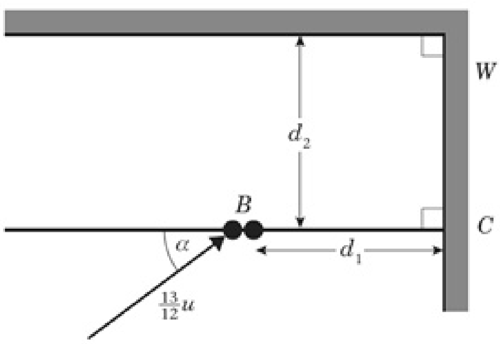
Review Exercise 1
Exercise A, Question 37

Question:



A small ball Q of mass $2m$ is at rest at the point B on a smooth horizontal plane. A second small ball P of mass m is moving on the plane with speed $\frac{13}{12}u$ and collides with Q. Both the balls are smooth, uniform and of the same radius. The point C is on a smooth vertical wall W which is at a distance d_1 from B, and BC is perpendicular to W. A second smooth vertical wall is perpendicular to W and at a distance d_2 from B. Immediately before the collision occurs, the direction of motion of P makes an angle α with BC, as shown in the figure, where $\tan\alpha = \frac{5}{12}$.
The line of centres of P and Q is parallel to BC. After the collision Q moves towards C with speed $\frac{3}{5}u$.

a Show that, after the collision, the velocity components of P parallel and perpendicular to CB are $\frac{1}{5}u$ and $\frac{5}{12}u$ respectively.

b Find the coefficient of restitution between P and Q.

c Show that when Q reaches C, P is at a distance $\frac{4}{3}d_1$ from W.

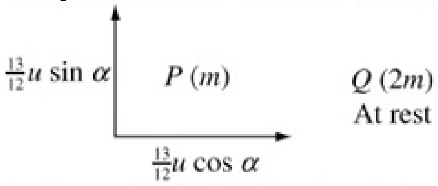
For each collision between a ball and a wall the coefficient of restitution is $\frac{1}{2}$. Given that the balls collide with each other again,

d show that the time between the two collisions of the balls is $\frac{15d_1}{u}$,

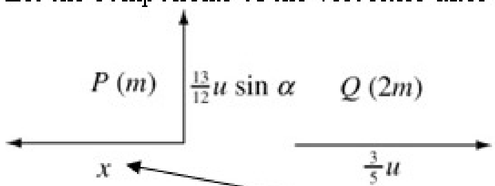
e find the ratio $d_1 : d_2$. **[E]**

Solution:

a Components of the velocities before the collision

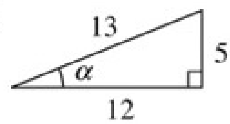


Let the components of the velocities after the collision be



The direction of the component of the velocity of P along the line of centres, here called x, is not obvious. If you put it in the opposite direction to that shown here, you would get a negative value of x and your solution would still be valid.

$$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}, \cos\alpha = \frac{12}{13}$$



This sketch illustrates that, as $5^2 + 12^2 = 13^2$, if $\tan\alpha = \frac{5}{12}$, then $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$.

Perpendicular to the line of centres CB

In this direction, the component of the velocity of P is unchanged and is

$\frac{13}{12}u \sin\alpha = \frac{13}{12}u \times \frac{5}{13} = \frac{5}{12}u$, as required

Perpendicular to the line of centres CB

Conservation of linear momentum

$$m \times \frac{13}{12}u \cos\alpha = -mx + 2m \times \frac{3}{5}u$$

$x = \frac{6}{5}u - \frac{13}{12}u \times \frac{12}{13} = \frac{6}{5}u - u = \frac{1}{5}u$, as required

b Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x + \frac{3}{5}u = e\frac{13}{12}u \cos\alpha$$

$$\frac{13}{12}u \cos\alpha = \frac{13}{12}u \times \frac{12}{13} = u$$

$$\frac{1}{5}u + \frac{3}{5}u = eu$$

$$e = \frac{4}{5}$$

c Let the time after the collision for Q to reach C be t_1.

distance = speed × time

$$d_1 = \frac{3}{5}ut_1 \Rightarrow t_1 = \frac{5d_1}{3u}$$

Perpendicular to W, in time t_1, P travels a distance s given by

distance = speed × time

$$s = \frac{1}{5}u \times t_1 = \frac{1}{5}u \times \frac{5d_1}{3u} = \frac{1}{3}d_1$$

Perpendicular to W, the component of the velocity of P after the collision is $\frac{1}{5}u$. To find the distance of P from W, you need consider only this component.

The distance of P from W is

$d_1 + s = d_1 + \frac{1}{3}d_1 = \frac{4}{3}d_1$, as required

d Before hitting W, Q has speed $\frac{3}{5}u$

After hitting W, Q has speed $e\frac{3}{5}u = \frac{1}{2} \times \frac{3}{5}u = \frac{3}{10}u$

In the direction CB, the velocity of Q relative to P is

$\frac{3}{10}u - \frac{1}{5}u = \frac{1}{10}u$

The time, t_2, for Q to travel from C to the point of the second collision is given by

$$t_2 = \frac{\frac{4}{3}d_1}{\frac{1}{10}u} = \frac{40d_1}{3u}$$

In the direction CB, the time is given by the distance of Q relative to $P\left(\frac{4}{3}d_1\right)$ divided by the velocity of Q relative to $P\left(\frac{1}{10}u\right)$.

The time between the two collisions is

$t_1 + t_2 = \frac{5d_1}{3u} + \frac{40d_1}{3u} = \frac{45d_1}{3u} = \frac{15d_1}{u}$, as required

e Before hitting the perpendicular wall, P has a component velocity $\frac{5}{12}u$ perpendicular to CB.

After hitting the wall, this component becomes

$$e\frac{5}{12}u = \frac{1}{2}\times\frac{5}{12}u = \frac{5}{24}u$$

If t_3 is the time for P to move from B to the wall and t_4 is the time for P to move from the wall back to CB, then

> As Q moves along CB, the second collision must occur on CB. So you need to find the time it takes for P to move to the wall and return to CB.

$$t_3 = \frac{d_2}{\frac{5}{12}u} = \frac{12d_2}{5u}$$

and

$$t_4 = \frac{d_2}{\frac{5}{24}u} = \frac{24d_2}{5u}$$

$$t_3 + t_4 = \frac{15d_1}{u}$$

> Q is moving along CB. So, for the second collision, P must travel from CB to the wall, which is perpendicular to W, and back to the line CB, in time $\frac{15d_1}{u}$.

$$\frac{12d_2}{5u} + \frac{24d_2}{5u} = \frac{15d_1}{u}$$

$$36d_2 = 75d_1$$

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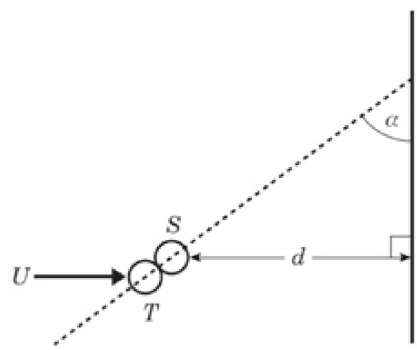
Review Exercise 1
Exercise A, Question 38

Question:



A small smooth uniform sphere S is at rest on a smooth horizontal floor at a distance d from a straight vertical wall. An identical sphere T is projected along the floor with speed U towards S and in a direction which is perpendicular to the wall. At the instant when T strikes S the line joining their centres makes an angle α with the wall, as shown in the figure.

Each sphere is modelled as having negligible diameter in comparison with d. The coefficient of restitution between the spheres is e.

a Show that the components of the velocity of T after the impact, parallel and perpendicular to the line of centres, are $\frac{1}{2}U(1-e)\sin\alpha$ and $U\cos\alpha$ respectively.

b Show that the components of the velocity of T after the impact, parallel and perpendicular to the wall are $\frac{1}{2}U(1+e)\cos\alpha\sin\alpha$ and $\frac{1}{2}U[2-(1+e)\sin^2\alpha]$ respectively.

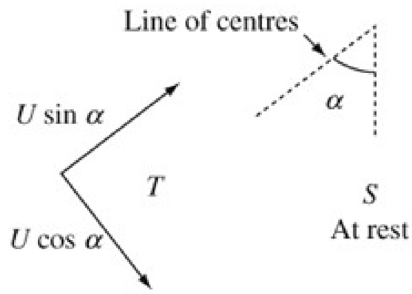
The spheres S and T strike the wall at the points A and B respectively.

Given that $e=\frac{2}{3}$ and $\tan\alpha=\frac{3}{4}$,

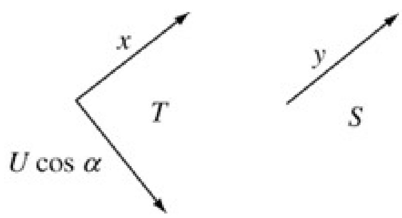
c find, in terms of d, the distance AB. **[E]**

Solution:

a The components of velocity before the collision perpendicular and parallel to the line of centres are



Let the components of velocities after the collision be



Let the mass of each sphere be m

Perpendicular to the line of centres
The component of the velocity is unchanged, so the component of the velocity of T after the impact perpendicular to the line of centres is $U\cos\alpha$, as required.

Parallel to the line of centres
Conservation of linear momentum

$mU\sin\alpha = mx + my$

$x + y = u\sin\alpha$ ①

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

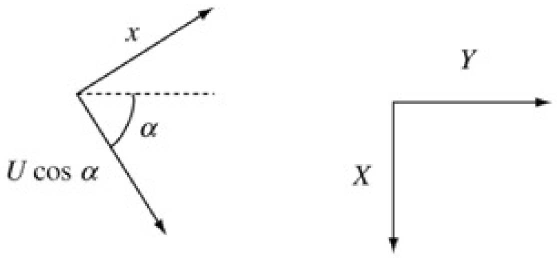
$y - x = eU\sin\alpha$ ②

① − ②

$2x = U\sin\alpha - eU\sin\alpha = U(1-e)\sin\alpha$

$x = \frac{1}{2}U(1-e)\sin\alpha$, as required

b Let the components of the velocity of T after the impact, parallel and perpendicular to the wall be X and Y respectively



$$\text{R}(\downarrow) X = U\cos\alpha\sin\alpha - x\cos\alpha$$
$$= U\cos\alpha\sin\alpha - \frac{1}{2}U(1-e)\sin\alpha\cos\alpha$$
$$= U\cos\alpha\sin\alpha\left(1-\frac{1}{2}+\frac{1}{2}e\right) = U\cos\alpha\sin\alpha\left(\frac{1}{2}+\frac{1}{2}e\right)$$
$$= \frac{1}{2}U(1+e)\cos\alpha\sin\alpha, \text{ as required}$$

$$\text{R}(\rightarrow) Y = U\cos\alpha\cos\alpha + x\sin\alpha$$
$$= U\cos^2\alpha + \frac{1}{2}U(1-e)\sin\alpha\sin\alpha$$
$$= U(1-\sin^2\alpha) + \frac{1}{2}U(1-e)\sin^2\alpha$$
$$= \frac{1}{2}U(2-2\sin^2\alpha+\sin^2\alpha-e\sin^2\alpha)$$
$$= \frac{1}{2}U(2-\sin^2\alpha-e\sin^2\alpha)$$
$$= \frac{1}{2}U\left[2-(1+e)\sin^2\alpha\right], \text{ as required}$$

c $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}, \cos\alpha = \frac{4}{5}$

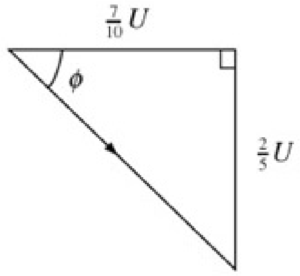
You can just write these down but, if you can't remember these relations, you can find the sine and cosine by sketching a 3, 4, 5 triangle.

With $e = \frac{2}{3}$, the components in part **b** become

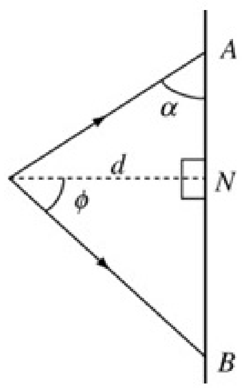
$$X = \frac{1}{2}U\left(1+\frac{2}{3}\right)\times\frac{3}{5}\times\frac{4}{5} = \frac{2}{5}U$$
$$Y = \frac{1}{2}U\left[2-\left(1+\frac{2}{3}\right)\times\frac{9}{25}\right] = \frac{1}{2}U\left(2-\frac{3}{5}\right) = \frac{7}{10}U$$

The direction of motion of S is along the line of centres
The direction of motion of T is given by

To find the points where S and T strike the wall, you need to know the direction of motion after the collision of both S and T.



$$\tan\phi = \frac{\frac{2}{5}U}{\frac{7}{10}U} = \frac{4}{7}$$



N is the foot of the perpendicular from the point of collision to the wall.

$$\frac{d}{AN} = \tan\alpha = \frac{3}{4} \Rightarrow AN = \frac{4}{3}d$$

$$\frac{NB}{d} = \tan\phi = \frac{4}{7} \Rightarrow NB = \frac{4}{7}d$$

$$AB = AN + NB = \frac{4}{3}d + \frac{4}{7}d = \frac{40}{21}d$$

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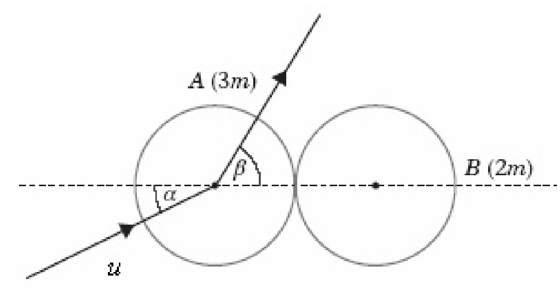
Review Exercise 1
Exercise A, Question 39

Question:



A uniform small smooth sphere of mass $3m$ moving with speed u on a smooth horizontal table collides with a stationary small sphere B of the same size as A and of mass $2m$. The direction of motion of A before impact makes an angle α with the line of centres of A and B, and the direction of motion of A after the impact makes an angle β with the same line, as shown in the figure. The coefficient of restitution between the spheres is $\frac{2}{3}$.

a Show that $\tan\beta = 3\tan\alpha$.

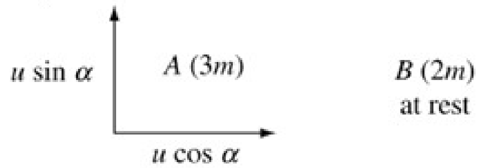
b Express $\tan(\beta - \alpha)$ in terms of t, where $t = \tan\alpha$.

c Hence find, as α varies, the maximum angle of deflection of A caused by the impact. **[E]**

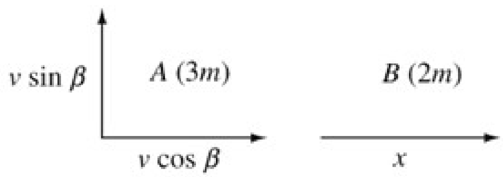
Solution:

a Let the speed of A before the collision be u,
the speed of A after the collision be v and the speed of B after the collision be x

Components before the collision



Components after the collision



Perpendicular to the line of centres

$v\sin\beta = u\sin\alpha$ ①

Parallel to the line of centres

Conservation of linear momentum

$3mu\cos\alpha = 3mv\cos\beta + 2mx$

$2x + 3v\cos\beta = 3u\cos\alpha$ ②

Newton's law of restitution

velocity of separation $= e \times$ velocity of approach

$$x - v\cos\beta = \frac{2}{3}u\cos\alpha \quad ③$$

The relation you are asked to prove contains only angles, so you must eliminate the three velocities, u, v and x, from these 3 equations.

② $-2\times$③

$$5v\cos\beta = 3u\cos\alpha - \frac{4}{3}u\cos\alpha = \frac{5}{3}u\cos\alpha$$

$$v\cos\beta = \frac{1}{3}u\cos\alpha \quad ④$$

Divide ① by ④

$$\frac{v\sin\beta}{v\cos\beta} = \frac{u\sin\alpha}{\frac{1}{3}u\cos\alpha}$$

$\tan\beta = 3\tan\alpha$, as required

b $\tan(\beta-\alpha)=\dfrac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}$

$=\dfrac{3\tan\alpha-\tan\alpha}{1+\tan\alpha\times 3\tan\alpha}=\dfrac{2\tan\alpha}{1+3\tan^2\alpha}$

$=\dfrac{2t}{1+3t^2}$

c Let $\mathrm{f}(t)=\dfrac{2t}{1+3t^2}$

The angle of deflection is the change in the angle due to the impact and that is $(\beta-\alpha)$. The maximum of $(\beta-\alpha)$ will correspond to the maximum of f(t), which can be found using calculus.

$\mathrm{f}'(t)=\dfrac{2(1+3t^2)-2t(6t)}{(1+3t^2)^2}=\dfrac{2-6t^2}{(1+3t^2)^2}$

Using the quotient rule for differentiating.

For a maximum value $\mathrm{f}'(t)=0$

$2-6t^2=0$

$t=\dfrac{1}{\sqrt{3}}$

For a collision, $0\le\alpha<90^\circ$, so the negative solution can be ignored.

At $t=\dfrac{1}{\sqrt{3}}$

$\mathrm{f}(t)=\dfrac{2\times\frac{1}{\sqrt{3}}}{1+3\left(\frac{1}{\sqrt{3}}\right)^2}=\dfrac{2\times\frac{1}{\sqrt{3}}}{1+1}=\dfrac{1}{\sqrt{3}}$

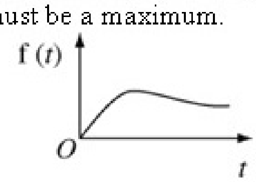
Hence

$\tan(\beta-\alpha)=\dfrac{1}{\sqrt{3}}$

$\beta-\alpha=30^\circ$

The maximum angle of deflection of A caused by the impact is 30°.

The least deflection is clearly when $\alpha=0$ (there is no deflection then) and the deflection initially increases as α increases so, as the function is continuous, the following stationary value must be a maximum.



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Review Exercise 1
Exercise A, Question 40

Question:

Two identical spheres A and B lie at rest on a smooth horizontal table. Sphere B is projected along the table towards sphere A with velocity $u\mathbf{i}+v\mathbf{j}$, where $\mathbf{i}$ is the unit vector along the line of centres at the time of impact and $\mathbf{j}$ is a unit vector perpendicular to $\mathbf{i}$ and in the plane of the table. Given that the coefficient of restitution between the spheres is e,

a find the velocities of the spheres after impact.

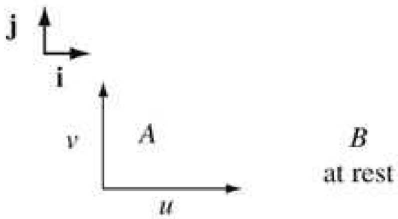
Given further that the velocity of B before impact makes an angle θ with the direction of $\mathbf{i}$ and that the velocity of B after impact makes an angle ϕ with the direction of $\mathbf{i}$,

b show that $\tan(\phi-\theta)=\dfrac{\tan\theta(1+e)}{1-e+2\tan^2\theta}$.

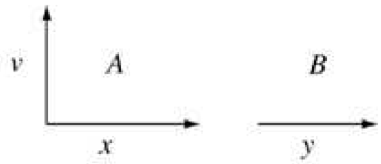
c Hence show that, as θ varies, the maximum value of the angle of deviation, $\phi-\theta$, occurs when $\tan^2\theta=\dfrac{1-e}{2}$. **[E]**

Solution:

Let the mass of both spheres be m
Components of velocity before impact



Let the components of the velocities after impact be



Parallel to **i**
Conservation of linear momentum

$mu = mx + my$

$x + y = u$ ①

Newton's law of restitution
velocity of separation $= e \times$ velocity of approach

$y - x = eu$ ②

① + ②

$2y = u + eu \Rightarrow y = \frac{(1+e)u}{2}$

① − ②

$2x = u - eu \Rightarrow x = \frac{(1-e)u}{2}$

After the impact, the velocity of A is

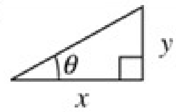
$x\mathbf{i} + v\mathbf{j} = \frac{(1-e)u}{2}\mathbf{i} + v\mathbf{j}$

The component in the **j** direction is unchanged by the impulse as it is perpendicular to the impulse.

and the velocity of B is

$y\mathbf{i} = \frac{(1+e)u}{2}\mathbf{i}$

b $\tan\theta = \dfrac{v}{u}$



If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, then $\tan\theta = \dfrac{y}{x}$.

$$\tan\phi = \frac{v}{x} = \frac{2v}{(1-e)u}$$

$$= \frac{2}{1-e}\times\frac{v}{u} = \frac{2}{1-e}\tan\theta$$

As $\dfrac{v}{u} = \tan\theta$

$$\tan(\phi-\theta) = \frac{\tan\phi - \tan\theta}{1+\tan\phi\tan\theta}$$

The formula for the difference of two tangents is in the C3 specification.

$$= \frac{\frac{2}{1-e}\tan\theta - \tan\theta}{1+\frac{2}{1-e}\tan\theta\times\tan\theta}$$

Multiply the numerator and the denominator by $(1-e)$.

$$= \frac{2\tan\theta - (1-e)\tan\theta}{1-e+2\tan^2\theta}$$

$$= \frac{\tan\theta(1+e)}{1-e+2\tan^2\theta}, \text{ as required}$$

c Let $\tan\theta = t$ and $f(t) = \dfrac{(1+e)t}{1-e+2t^2}$

The maximum of $(\phi-\theta)$ will correspond to the maximum of f(t), which can be found using calculus.

$$f'(t) = \frac{(1-e+2t^2)(1+e) - (1+e)t(4t)}{(1-e+2t^2)^2}$$

For $f'(t) = 0$

$$(1-e+2t^2)\cancel{(1+e)} - \cancel{(1+e)}4t^2 = 0$$

$$1-e+2t^2-4t^2 = 0$$

$$2t^2 = 1-e$$

$$t^2 = \frac{1-e}{2}$$

$$\tan^2\theta = \frac{1-e}{2}, \text{ as required}$$

If you are very familiar with the pure mathematics involved, you may recognise that $f'(t) = 0$ has a solution at $t = \infty$, which corresponds to $\theta = \dfrac{\pi}{2}$. Examination of the situation shows this is not actually a collision (the spheres 'graze' each other) so this case can be ignored.

The least deflection is clearly when $\theta = 0$ (there is no deflection then) and the deflection initially increases as θ increases so, as the function is continuous, the following stationary value must be a maximum.

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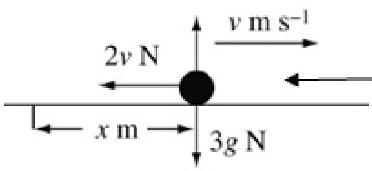
Review Exercise 1
Exercise A, Question 41

Question:

A particle P of mass 3 kg moves in a straight line on a smooth horizontal plane. When the speed of P is v m s^{-1}, the resultant force acting on P is a resistance to motion of magnitude $2v$ N.
Find the distance moved by P while slowing down from 5 m s^{-1} to 2 m s^{-1}. **[E]**

Solution:



The displacement, x m, of P must be measured from a fixed point. Here you can choose to measure the displacement from the point where the velocity of P is 5 m s^{-1}.

R($\rightarrow$) $\mathbf{F} = ma$

$$-2v = 3a = 3v\frac{dv}{dx}$$

$$\frac{dv}{dx} = -\frac{2}{3}$$

When resistance is a function of velocity and the question asks about a relation between distance and velocity, the formula $a = v\frac{dv}{dx}$ is normally used.

The resistance is in the direction of x decreasing and so has a negative sign in this equation.

Integrating with respect to x

$$v = -\frac{2}{3}x + C$$

When $x = 0, v = 5$

$5 = 0 + C \Rightarrow C = 5$

As you are measuring the displacement from the point where the velocity of P is 5 m s^{-1}, you use $x = 0$ when $v = 5$ to evaluate the constant of integration.

Hence

$$v = 5 - \frac{2}{3}x$$

When $v = 2$

$$2 = 5 - \frac{2}{3}x$$

$$\frac{2}{3}x = 5 - 2 = 3 \Rightarrow x = \frac{3}{2} \times 3 = 4.5$$

The distance moved by P while slowing down from 5 m s^{-1} to 2 m s^{-1} is 4.5 m.

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Review Exercise 1
Exercise A, Question 42

Question:

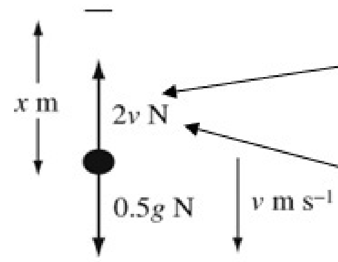
A particle P of mass 0.5 kg is released from rest at time $t = 0$ and falls vertically through a liquid. The motion P is resisted by a force of magnitude $2v$ N, where v m s^{-1} is the speed of P at time t seconds.

a Show that $5\frac{dv}{dt} = 49 - 20v$.

b Find the speed of P when $t = 1$. **[E]**

Solution:

a



The displacement, x m, must be measured from a fixed point. Here you can choose the point from which P is released.

As P is falling, the resistance opposes motion and acts upwards.

R(↓) $\mathbf{F} = ma$

$0.5g - 2v = 0.5a$

The weight acts in the direction of x increasing, so, in this equation, the term $0.5g$ is positive. The resistance acts in the direction of x decreasing, so, in this equation, the term $2v$ is negative.

Using $g = 9.8$ and $a = \frac{dv}{dt}$

$4.9 - 2v = 0.5\frac{dv}{dt}$

$5\frac{dv}{dt} = 49 - 20v$, as required

You multiply this equation by 10 and rearrange the terms to obtain the differential equation printed in the question.

b $\int \frac{5}{49-20v}\,dv = \int 1\,dt$

The answer to part **a** is a separable differential equation and the first step in solving it is to separate the variables.

$-\frac{1}{4}\int \frac{-20}{49-20v}\,dv = \int 1\,dt$

$-\frac{1}{4}\ln(49-20v) = t + A$

$\ln(49-20v) = B - 4t$, where $B = -4A$

When $t = 0, v = 0$

$\ln 49 = B$

$\int \frac{f'(v)}{f(v)}\,dv = \ln f(v) + A$ and $\frac{d}{dv}(49-20v) = -20$. Using $5 = -\frac{1}{4} \times -20$, you adjust the constants so that the integral can just be written down.

Hence

$\ln(49-20v) = \ln 49 - 4t$

$\ln 49 - \ln(49-20v) = \ln\left(\frac{49}{49-20v}\right) = 4t$

You use the log rule $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$ to simplify the expression.

$\frac{49}{49-20v} = e^{4t}$

$49 - 20v = 49e^{-4t}$

$20v = 49 - 49e^{-4t} = 49\left(1 - e^{-4t}\right)$

$v = \frac{49}{20}\left(1 - e^{-4t}\right)$

Taking exponentials of both sides to the equation and using the rule $e^{\ln f(x)} = f(x)$.

When $t = 1$

$v = \frac{49}{20}\left(1 - e^{-4}\right) \approx 2.4$

The speed of P when $t = 1$ is

$\frac{49}{20}\left(1 - e^{-4}\right)\text{ m s}^{-1} = 2.4\text{ m s}^{-1}$ (2 s.f.)

As a numerical value of g has been used, the final answer should be given to 2 or 3 significant figures.

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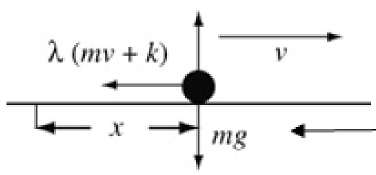
Review Exercise 1
Exercise A, Question 43

Question:

A particle of mass m moves in a straight line on a horizontal table against a resistance of magnitude $\lambda(mv+k)$, where λ and k are constants. Given that the particle starts with speed u at time $t=0$, show that the speed v of the particle at time t is
$v=\frac{k}{m}(e^{-\lambda t}-1)+ue^{-\lambda t}$. **[E]**

Solution:



The displacement x must be measured from a fixed point. Here you measure x from the point where the particle starts. Later, you will use $v=u$ when $t=0$ to evaluate the constant of integration.

$\text{R}(\rightarrow)\quad \mathbf{F}=m\mathbf{a}$

$$-\lambda(mv+k)=ma=m\frac{\mathrm{d}v}{\mathrm{d}t}$$

The resistance is in the direction of x decreasing and so has a negative sign in this equation.

Separating the variables

$$\int\frac{m}{mv+k}\,\mathrm{d}v=-\int\lambda\,\mathrm{d}t$$

$$\ln(mv+k)=-\lambda t+A$$

$$mv+k=\mathrm{e}^{-\lambda t+A}=\mathrm{e}^{A}\mathrm{e}^{-\lambda t}$$

$$=B\,\mathrm{e}^{-\lambda t}$$

e^{A}, where A is an arbitrary constant, is another arbitrary constant, B.

When $t=0, v=u$

$$mu+k=B$$

Hence

$$mv+k=(mu+k)\mathrm{e}^{-\lambda t}$$

You make v the subject of this formula to complete the question.

$$mv=k\mathrm{e}^{-\lambda t}-k+mu\,\mathrm{e}^{-\lambda t}=k(\mathrm{e}^{-\lambda t}-1)+mu\,\mathrm{e}^{-\lambda t}$$

$$v=\frac{k}{m}(\mathrm{e}^{-\lambda t}-1)+u\mathrm{e}^{-\lambda t}\text{, as required}$$

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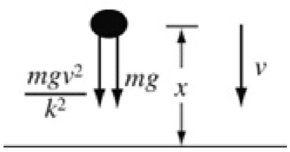
Review Exercise 1
Exercise A, Question 44

Question:

A particle P, of mass m, is projected upwards from horizontal ground with speed U.
The motion takes place in a medium in which the resistance is of magnitude $\frac{mgv^2}{k^2}$,
where v is the speed of P and k is a positive constant.
Show that P reaches its maximum height above ground after a time T given by
$T = \frac{k}{g}\arctan\left(\frac{U}{k}\right)$. **[E]**

Solution:



$$\mathbf{F} = m\mathbf{a}$$

$$-mg - \frac{mgv^2}{k^2} = ma = m\frac{dv}{dt}$$

The displacement x is measured from the point of projection. In this question, both the weight of the particle and the resistance act in the direction of x decreasing and so the both terms representing these forces are negative in the equation of motion.

$$-\cancel{m}g - \frac{\cancel{m}gv^2}{k^2} = \cancel{m}\frac{dv}{dt}$$

$$-g\left(\frac{k^2+v^2}{k^2}\right) = \frac{dv}{dt}$$

Separating the variables

$$\int \frac{1}{k^2+v^2}\,dv = -\int \frac{g}{k^2}\,dt$$

$$\frac{1}{k}\arctan\left(\frac{v}{k}\right) = -\frac{g}{k^2}t + A$$

The prerequisites given in the specification for M4 require you to know that $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$.

When $t = 0, v = U$

$$\frac{1}{k}\arctan\left(\frac{U}{k}\right) = A$$

Hence

$$\frac{1}{k}\arctan\left(\frac{v}{k}\right) = -\frac{g}{k^2}t + \frac{1}{k}\arctan\left(\frac{U}{k}\right)$$

At the maximum height $v = 0$ and $t = T$

When P reaches its maximum height above the ground its velocity is 0.

$$0 = -\frac{g}{k^2}T + \frac{1}{k}\arctan\left(\frac{U}{k}\right)$$

$$T = \frac{k}{g}\arctan\left(\frac{U}{k}\right), \text{ as required}$$

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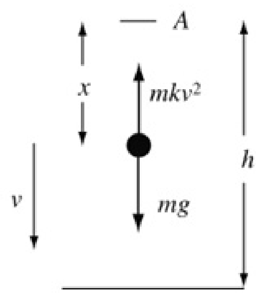
Review Exercise 1
Exercise A, Question 45

Question:

At time $t = 0$ a particle of mass m falls from rest at the point A which is at a height h above a horizontal plane. The particle is subject to a resistance of magnitude mkv^2, where v is the speed of the particle at time t and k is a positive constant. The particle strikes the plane with speed V.
Show that $kV^2 = g(1 - e^{-2kh})$. **[E]**

Solution:



R(↓) $\mathbf{F} = ma$

$mg - mkv^2 = ma = mv\frac{dv}{dx}$

$\not{m}g - \not{m}kv^2 = \not{m}v\frac{dv}{dx}$

The displacement x is measured from A. In this question, the weight of the particle is in the direction of x increasing. In the equation of motion, the term representing the weight, mg, is positive.
The resistance acts in the direction of x decreasing and, in the equation of motion, the term representing the resistance is negative.

Separating the variables

$\int \frac{v}{g - kv^2} dv = \int 1 \, dx$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator, $g - kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$.

Multiply throughout by $-2k$

$\int \frac{-2kv}{g - kv^2} dv = \int -2k \, dx$

$\ln\left(g - kv^2\right) = -2kx + A$

$g - kv^2 = e^{-2kx + A} = e^A e^{-2kx}$

$= B e^{-2kx}$

e^A, where A is an arbitrary constant, is another arbitrary constant, B.

At $x = 0, v = 0$

$g = B e^0 = B$

Hence

$g - kv^2 = g e^{-2kx}$

$kv^2 = g\left(1 - e^{-2kx}\right)$

At $x = h, v = V$

$kV^2 = g\left(1 - e^{-2kh}\right)$, as required

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Review Exercise 1
Exercise A, Question 46

Question:

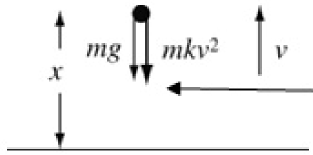
A particle of mass m is projected vertically upwards with speed U. It is subject to air resistance of magnitude mkv^2, where v is its speed and k is a positive constant.

a Show that the greatest height of the particle above its point of projection is $\frac{1}{2k}\ln\left(1+\frac{kU^2}{g}\right)$.

b Find an expression for the total work done against air resistance during the upward motion. **[E]**

Solution:

a



As the particle is moving upwards, the resistance, mkv^2, opposes the motion and acts downwards.

R(↑) $\mathbf{F} = m\mathbf{a}$

$$-mg - mkv^2 = ma = mv\frac{dv}{dx}$$

$$-\not{m}g - \not{m}kv^2 = \not{m}v\frac{dv}{dx}$$

When resistance is a function of velocity and the question asks about a relation between distance and velocity, you usually use the formula $a = v\frac{dv}{dx}$.

Separating the variables

$$\int \frac{v}{g + kv^2}\,dv = -\int 1\,dx$$

If you multiply both sides of this equation by $2k$, on the left hand side, the numerator of the fraction, $2kv$, is the differential of the denominator, $g + kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$.

Multiply throughout by $2k$

$$\int \frac{2kv}{g + kv^2}\,dv = -\int 2k\,dx$$

$$\ln\left(g + kv^2\right) = -2kx + A$$

At $x = 0, v = U$

$$\ln\left(g + kU^2\right) = A$$

Hence

$$\ln\left(g + kv^2\right) = -2kx + \ln\left(g + kU^2\right)$$

The particle reaches its maximum height when $v = 0$.

$$\ln g = -2kx + \ln\left(g + kU^2\right)$$

$$2kx = \ln\left(g + kU^2\right) - \ln g = \ln\left(\frac{g + kU^2}{g}\right)$$

Using the law of logarithms $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$$x = \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)\text{, as required}$$

b Work done = loss in energy

$$= \frac{1}{2}mU^2 - mgx$$

The particle starts with kinetic energy $\frac{1}{2}mU^2$ and, at its maximum height, has, relative to the ground, potential energy $mg \times$ height above the ground. The difference between these energies is the work done by the resistance.

$$= \frac{1}{2}mU^2 - mg \times \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

$$= \frac{1}{2}m\left(U^2 - \frac{g}{k}\ln\left(1 + \frac{kU^2}{g}\right)\right)$$

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Review Exercise 1
Exercise A, Question 47

Question:

A lorry of mass M is moving along a straight horizontal road. The engine produces a constant driving force of magnitude F. The total resistance to motion is modelled as having magnitude kv^2, where k is a constant, and v is the speed of the lorry.

Given that the lorry moves with constant speed V,

a show that $V = \sqrt{\left(\frac{F}{k}\right)}$.

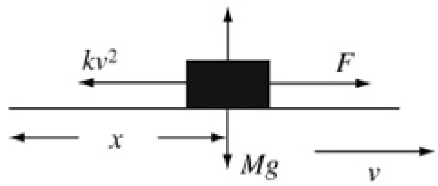
Given instead that the lorry starts from rest,

b show that the distance travelled by the lorry in attaining a speed $\frac{1}{2}V$ is $\frac{M}{2k}\ln\left(\frac{4}{3}\right)$.

[E]

Solution:

a



$\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$

$F - kV^2 = 0$

$V^2 = \dfrac{F}{k}$

$V = \sqrt{\left(\dfrac{F}{k}\right)}$, as required

When the lorry is moving at a constant speed V, the acceleration of the lorry is 0.

The velocity of the lorry is taken in the direction of x increasing and you can ignore the possibility of a negative square root.

b $\text{R}(\rightarrow)\quad \mathbf{F} = m\mathbf{a}$

$F - kv^2 = Ma = Mv\dfrac{\text{d}v}{\text{d}x}$

Separating the variables

$\displaystyle\int \frac{v}{F - kv^2}\,\text{d}v = \int \frac{1}{M}\,\text{d}x$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator $F - kv^2$, and you can integrate using the formula $\displaystyle\int \frac{\text{f}'(x)}{\text{f}(x)}\,\text{d}x = \ln \text{f}(x)$.

Multiply throughout by $-2k$

$\displaystyle\int \frac{-2kv}{F - kv^2}\,\text{d}v = -\int \frac{2k}{M}\,\text{d}x$

$\ln\left(F - kv^2\right) = -\dfrac{2k}{M}x + A$

$\dfrac{2k}{M}x = A - \ln\left(F - kv^2\right)$

At $x = 0, v = 0$

$0 = A - \ln F \Rightarrow A = \ln F$

Hence

$\dfrac{2k}{M}x = \ln F - \ln\left(F - kv^2\right) = \ln\left(\dfrac{F}{F - kv^2}\right)$

Using the law of logarithms $\ln a - \ln b = \ln\left(\dfrac{a}{b}\right)$.

$x = \dfrac{M}{2k}\ln\left(\dfrac{F}{F - kv^2}\right)$

When $v = \dfrac{1}{2}V,\ v^2 = \dfrac{1}{4}V^2 = \dfrac{F}{4k}$

You use the expression for V from part **a**.

$x = \dfrac{M}{2k}\ln\left(\dfrac{F}{F - k \times \dfrac{F}{4k}}\right) = \dfrac{M}{2k}\ln\left(\dfrac{F}{\frac{3}{4}F}\right)$

$= \dfrac{M}{2k}\ln\left(\dfrac{4}{3}\right)$, as required

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 48

Question:

A train of mass m is moving along a straight horizontal railway line. At time t, the train is moving with speed v and the resistance to motion has magnitude kv, where k is a constant. The engine of the train is working at a constant rate P.

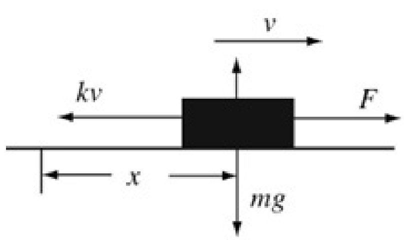
a Show that, when $v > 0$, $mv\frac{dv}{dt} + kv^2 = P$.

When $t = 0$, the speed of the train is $\frac{1}{3}\sqrt{\left(\frac{P}{k}\right)}$.

b Find, in terms of m and k, the time taken for the train to double its initial speed. **[E]**

Solution:

a



power = force × velocity

$$P = Fv \Rightarrow F = \frac{P}{v}$$

As the velocity increases, the tractive force F decreases and, hence, the acceleration will decrease.

$R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$F - kv = ma$$

$$\frac{P}{v} - kv = m\frac{dv}{dt}$$

Multiply this equation throughout by v and rearrange the result to obtain the printed answer.

$$P - kv^2 = mv\frac{dv}{dt}$$

$mv\frac{dv}{dt} + kv^2 = P$, as required

b $mv\frac{dv}{dt} = P - kv^2$

Separating the variables

$$\int \frac{v}{P - kv^2}\,dv = \int \frac{1}{m}\,dt$$

If you multiply both sides of this equation by $-2k$, on the left hand side, the numerator of the fraction, $-2kv$, is the differential of the denominator, $P - kv^2$, and you can integrate using the formula $\int \frac{f'(x)}{f(x)}\,dx = \ln f(x)$.

Multiply throughout by $-2k$

$$\int \frac{-2kv}{P - kv^2}\,dv = -\int \frac{2k}{m}\,dt$$

$$\ln\left(P - kv^2\right) = -\frac{2k}{m}t + A$$

When $t = 0, v = \frac{1}{3}\sqrt{\left(\frac{P}{k}\right)} \Rightarrow v^2 = \frac{P}{9k}$

$$\ln\left(P-\frac{P}{9}\right)=A\Rightarrow A=\ln\left(\frac{8P}{9}\right)$$

$$\ln\left(P-kv^2\right)=-\frac{2k}{m}t+\ln\left(\frac{8P}{9}\right)$$

$$t=\frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right)-\ln\left(P-kv^2\right)\right]$$

You are asked to find an expression for the time, so it is sensible to rearrange this formula, making t the subject of the formula. This will save you time later.

When $v=\frac{2}{3}\sqrt{\left(\frac{P}{k}\right)}\Rightarrow v^2=\frac{4P}{9k}$

The initial speed is $\frac{1}{3}\sqrt{\left(\frac{P}{k}\right)}$, so double the initial speed is $\frac{2}{3}\sqrt{\left(\frac{P}{k}\right)}$.

$$t=\frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right)-\ln\left(P-\frac{4P}{9}\right)\right]$$

$$=\frac{m}{2k}\left[\ln\left(\frac{8P}{9}\right)-\ln\left(\frac{5P}{9}\right)\right]$$

$$=\frac{m}{2k}\ln\left(\frac{\frac{8P}{9}}{\frac{5P}{9}}\right)=\frac{m}{2k}\ln\left(\frac{8}{5}\right)$$

Using the law of logarithms $\ln a-\ln b=\ln\left(\frac{a}{b}\right)$.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 49

Question:

The engine of a car of mass 800 kg works at a constant rate of 32 kW. The car travels along a straight horizontal road and the resistance to motion of the car is proportional to the speed of the car. At time t seconds, $t \geq 0$, the car has a speed v m s^{-1} and when $t = 0$, its speed is 10 m s^{-1}.

a Show that $800v\dfrac{dv}{dt} = 32\,000 - kv^2$, where k is a positive constant.

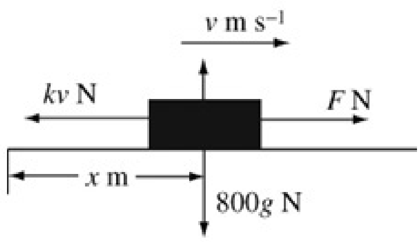
Given that the limiting speed of the car is 40 m s^{-1}, find

b the value of k,

c v^2 in terms of t. **[E]**

Solution:

a



power = force × velocity

$32\,000 = Fv \Rightarrow F = \dfrac{32\,000}{v}$

To use the relation power = force × velocity, you must convert 32 kW to 32 000 W.

Let the resistance be kv, where k is a constant

As the resistance is proportional to the velocity, you can let the resistance = kv, where k is the constant of proportionality.

R(→) $\mathbf{F} = m\mathbf{a}$

$F - kv = 800a$

$\dfrac{32\,000}{v} - kv = 800\dfrac{dv}{dt}$

You multiply this equation throughout by v.

$32\,000 - kv^2 = 800v\dfrac{dv}{dt}$

$800v\dfrac{dv}{dt} = 32\,000 - kv^2$, as required

b When $\dfrac{dv}{dt} = 0, v = 40$

At the limiting, or terminal, speed the speed of the car does not increase, so the acceleration is zero. This enables you to evaluate the constant of proportionality.

$0 = 32\,000 - k \times 1600$

$k = \dfrac{32\,000}{1600} = 20$

c $800v\dfrac{dv}{dt} = 32\,000 - 20v^2$

Separating the variables

$\displaystyle\int \frac{800v}{32\,000 - 20v^2}\,dv = \int 1\,dt$

$\displaystyle -20\int \frac{-40v}{32\,000 - 20v^2}\,dv = \int 1\,dt$

$\displaystyle\int \frac{f'(v)}{f(v)}\,dv = \ln f(v) + A$ and $\dfrac{d}{dv}(32\,000 - 20v^2) = -40v$. Using $800 = -20 \times -40$, you adjust the constants so that the integral can just be written down.

$-20\ln(32\,000 - 20v^2) = t + A$

$\ln(32\,000 - 20v^2) = -\dfrac{t}{20} + B$, where $B = -\dfrac{1}{20}A$

$32\,000 - 20v^2 = e^{-\frac{t}{20}}e^B = C\,e^{-\frac{t}{20}}$, where $C = e^B$

$20v^2 = 32\,000 - Ce^{-\frac{t}{20}}$

When $t = 0, v = 10$

$2000 = 32\,000 - C \Rightarrow C = 30\,000$

You use the initial conditions to find the arbitrary constant. You can choose at which stage you do this. In this solution, you could have found the constant earlier and evaluated A or B instead of C.

$20v^2 = 32\,000 - 30\,000e^{-\frac{t}{20}}$

$v^2 = 1600 - 1500e^{-\frac{t}{20}}$

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Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 50

Question:

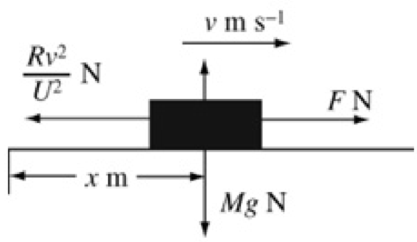
A car of mass M kg is driven by an engine working at a constant power RU watts, where R and U are positive constants. When the speed of the car is v m s^{-1}, the resistance to motion is $\frac{Rv^2}{U^2}$ newtons.

a Show that the acceleration of the car, a m s^{-2}, when its speed is v m s^{-1}, is given by $R(U^3 - v^3) = MU^2va$.

b Hence show that the distance, in m, travelled by the car as it increases its speed from u_1 m s^{-1} to u_2 m s^{-1} $(u_1 < u_2 < U)$ is $\frac{MU^2}{3R}\ln\left(\frac{U^3 - u_1^3}{U^3 - u_2^3}\right)$. **[E]**

Solution:

a



$$\text{power} = \text{force} \times \text{velocity}$$

$$RU = Fv \Rightarrow F = \frac{RU}{v}$$

$R(\rightarrow)$ $\quad \mathbf{F} = ma$

$$F - \frac{Rv^2}{U^2} = Ma$$

$$\frac{RU}{v} - \frac{Rv^2}{U^2} = Ma$$

Multiply throughout by $U^2 v$

$$RU^3 - Rv^3 = MU^2 va$$

$R(U^3 - v^3) = MU^2 va$, as required

The question asks for the distance travelled as the speed increases. Time does not come into the question directly so you must use the relation $a = v\frac{dv}{dx}$.

b $R(U^3 - v^3) = MU^2 v\left(v\frac{dv}{dx}\right) = MU^2 v^2 \frac{dv}{dx}$

Separating the variables

$$\int \frac{v^2}{U^3 - v^3}\,dv = \int \frac{R}{MU^2}\,dx$$

Multiply throughout by -3

$$\int \frac{-3v^2}{U^3 - v^3}\,dv = -3\int \frac{R}{MU^2}\,dx$$

$\frac{d}{dv}(U^3 - v^3) = -3v^2$, so multiplying both sides of the equation by -3 gives, on the left hand side, a fraction in which the numerator is the differential of the denominator which gives a log integral.

Integrating both sides using the limits $v = u_1$ and $v = u_2$, and the limits $x = 0$ and $x = s$

$$\left[\ln(U^3 - v^3)\right]_{u_1}^{u_2} = -\frac{3R}{MU^2}[x]_0^s$$

$$\ln(U^3 - u_2^3) - \ln(U^3 - u_1^3) = -\frac{3R}{MU^2}s$$

$$s = \frac{MU^2}{3R}\left[\ln(U^3 - u_1^3) - \ln(U^3 - u_2^3)\right]$$

$$= \frac{MU^2}{3R}\ln\left(\frac{U^3 - u_1^3}{U^3 - u_2^3}\right), \text{ as required}$$

Using limits is sometimes a convenient way of avoiding evaluating the constant of integration. On the left hand side, v increases from u_1 to u_2. On the right hand side, you take x as increasing from 0 to s. s will then represent the distance travelled by the car as the speed increases from u_1 to u_2.

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 51

Question:

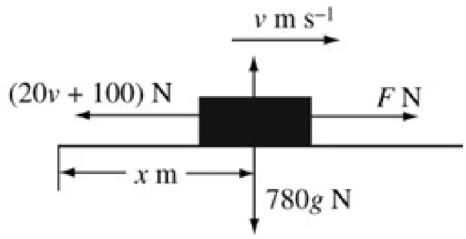
A car of mass 780 kg is moving along a straight horizontal road with the engine of the car working at 21 kW. The total resistance to the motion of the car is $(20v+100)$ N, where v m s^{-1} is the speed of the car at time t seconds.

a Show that $39v\dfrac{dv}{dt}=(30-v)(35+v)$.

b Find an expression for the time taken for the car to accelerate from 15 m s^{-1} to V m s^{-1}. **[E]**

Solution:

a



$$\text{power} = \text{force} \times \text{velocity}$$

$$21\,000 = Fv \Rightarrow F = \frac{21\,000}{v}$$

The relation power = force × velocity, which is part of the M2 specification, lets you express the tractive force in terms of the velocity. To use this relation you must convert 21 kW to 21 000 W.

$$\text{R}(\rightarrow) \quad \mathbf{F} = ma$$

$$F - (20v + 100) = 780\frac{dv}{dt}$$

$$\frac{21\,000}{v} - 20v - 100 = 780\frac{dv}{dt}$$

Divide throughout by 20

$$\frac{1050}{v} - v - 5 = 39\frac{dv}{dt}$$

Multiply throughout by v

$$1050 - v^2 - 5v = 39v\frac{dv}{dt}$$

$$39v\frac{dv}{dt} = 1050 - 5v - v^2$$

$$39v\frac{dv}{dt} = (30 - v)(35 + v), \text{ as required}$$

b Separating the variables

$$\int \frac{39v}{(30-v)(35+v)}\,dv = \int 1\,dt$$

Let $\frac{39v}{(30-v)(35+v)} = \frac{A}{30-v} + \frac{B}{35+v}$

$$39v = A(35+v) + B(30-v)$$

To integrate $\frac{39v}{(30-v)(35+v)}$ you must break the expression up into partial fractions. You may use any method, including the cover-up rule, to find the partial fractions.

Let $v \rightarrow 30$

$$39 \times 30 = A \times 65$$

$$A = \frac{39 \times 30}{65} = 18$$

Let $v \rightarrow -35$

$$39 \times -35 = B \times 65$$

$$B = \frac{39 \times -35}{65} = -21$$

Hence

$$\int\left(\frac{18}{30-v}-\frac{21}{35+v}\right)\mathrm{d}v=\int 1\,\mathrm{d}t$$

$$-18\ln(30-v)-21\ln(35+v)=t+A$$

When $t=0, v=15$

You can take $v=15$ as the starting value of the velocity.

$$-18\ln 15-21\ln 50=A$$

Hence

$$-18\ln(30-v)-21\ln(35+v)=t-18\ln 15-21\ln 50$$

$$t=18\ln 15-18\ln(30-v)+21\ln 50-21\ln(35+v)$$

$$=18\ln\left(\frac{15}{30-v}\right)+21\ln\left(\frac{50}{35+v}\right)$$

When $v=V$

$$t=18\ln\left(\frac{15}{30-V}\right)+21\ln\left(\frac{50}{35+V}\right)$$

The question specifies no form of the answer and there are many possible alternative forms.

Solutionbank M4

Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 52

Question:

A railway truck of mass 3000 kg moves along a straight, horizontal railway line. When its speed is v m s^{-1}, it experiences a total resistance to motion of $(2000+5v^2)$N . A cable is attached to the truck, and the tension in the cable exerts a constant tractive force of 6500 N on the truck.

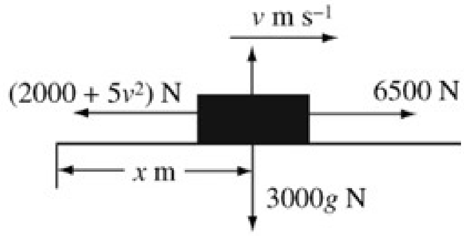
a Find the time taken for the truck to accelerate from rest to a speed of 20 m s^{-1}, giving your answer in seconds to 3 significant figures.

When the speed of the truck is 20 m s^{-1}, the cable breaks.

b Find the time taken after the cable breaks for the truck to come to rest, giving your answer in seconds to 3 significant figures. **[E]**

Solution:

a



$$\text{R}(\rightarrow) \quad \mathbf{F} = ma$$

$$6500 - (2000 + 5v^2) = 3000a$$

> The tension in the cable is in the direction of x increasing and the resistance to motion acts in the direction of x decreasing.

$$4500 - 5v^2 = 3000\frac{dv}{dt}$$

$\div 5$

$$900 - v^2 = 600\frac{dv}{dt}$$

Separating the variables

$$\int 1 dt = 600\int \frac{1}{900 - v^2} dv$$

> To integrate $\frac{1}{900 - v^2} = \frac{1}{(30 - v)(30 + v)}$ you must break the expression up into partial fractions. It is a common error to write $\int \frac{1}{900 - v^2} dv = \ln(900 - v^2)$.

Let $\frac{1}{900 - v^2} = \frac{1}{(30 - v)(30 + v)} = \frac{A}{30 - v} + \frac{B}{30 + v}$

$\times (30 - v)(30 + v)$

$$1 = A(30 + v) + B(30 - v)$$

Let $v \rightarrow 30$

$$1 = 60A \Rightarrow A = \frac{1}{60}$$

Let $v \rightarrow -30$

$$1 = 60B \Rightarrow B = \frac{1}{60}$$

Hence

$$\int 1\, dt = 600\int \left(\frac{1}{60(30 + v)} + \frac{1}{60(30 - v)}\right) dv$$

$$t = 10(\ln(30 + v) - \ln(30 - v)) + A$$

$$= 10\ln\left(\frac{30 + v}{30 - v}\right) + A$$

> $\int \frac{1}{30 + v} dv = \ln(30 + v) + A$ and $\int \frac{1}{30 - v} dv = -\ln(30 - v) + B$. However, after integrating both, you need only add one constant of integration.

When $t = 0, v = 0$

$$0 = 10\ln\left(\frac{30}{30}\right) + A \Rightarrow A = 0$$

Hence

$$t = 10\ln\left(\frac{30+v}{30-v}\right)$$

When $v = 20$

$$t = 10\ln\left(\frac{30+20}{30-20}\right) = 10\ln 5 \approx 16.1$$

There is an exact answer here, 10 ln 5, but the conditions of the question require you to give your answers to 3 significant figures.

The time taken for the truck to accelerate from rest to a speed of 20 m s^{-1} is 16.1 s (3 s.f.).

b After the rope breaks

R($\rightarrow$) $\mathbf{F} = m\mathbf{a}$

$$-2000 - 5v^2 = 3000\frac{dv}{dt}$$

After the cable breaks, the only force acting horizontally on the truck is the total resistance acting in the direction of x decreasing.

$\div 5$

$$-400 - v^2 = 600\frac{dv}{dt}$$

Separating the variables

$$\int \frac{1}{20^2 + v^2}\,dv = -\int \frac{1}{600}\,dt$$

$$\frac{1}{20}\arctan\left(\frac{v}{20}\right) = -\frac{1}{600}t + B$$

The prerequisites given in the specification for M4 require you to know that $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$.

When $t = 0, v = 20$

$$\frac{1}{20}\arctan 1 = B \Rightarrow B = \frac{1}{20} \times \frac{\pi}{4} = \frac{\pi}{80}$$

Using $\arctan 1 = \frac{\pi}{4}$.

Hence

$$\frac{1}{20}\arctan\left(\frac{v}{20}\right) = -\frac{1}{600}t + \frac{\pi}{80}$$

When $v = 0$

$$0 = -\frac{1}{600}t + \frac{\pi}{80}$$

Using $\arctan 0 = 0$.

$$t = \frac{\pi}{80} \times 600 = \frac{15\pi}{2} \approx 23.6$$

The time taken after the cable breaks for the truck to come to rest is 23.6 s (3 s.f.).

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 53

Question:

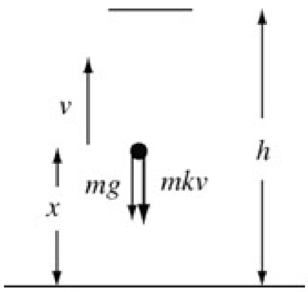
A particle P of mass m moves in a medium which produces a resistance of magnitude mkv, where v is the speed of P and k is a constant. The particle P is projected vertically upwards in this medium with speed $\frac{g}{k}$.

a Show that P comes instantaneously to rest after time $\frac{\ln 2}{k}$.

b Find, in terms of k and g, the greatest height above the point of projection reached by P. **[E]**

Solution:

a



$R(\uparrow)\quad \mathbf{F} = ma$

$-mg - mkv = ma$

$-mg - mkv = m\frac{dv}{dt}$

Both the weight of the particle (mg) and the resistance (mkv) act in the direction of x decreasing and so the both these terms are negative in the equation of motion.

Separating the variables

$$\int 1\,dt = -\int \frac{1}{g+kv}\,dv$$

$$t = -\frac{1}{k}\ln(g+kv) + A$$

As $\int \frac{f'(x)}{f(x)}dx = \ln f(x)$, ignoring the arbitrary constant, $\int \frac{k}{g+kv}dv = \ln(g+kv)$. So $\int \frac{1}{g+kv}dv = \frac{1}{k}\ln(g+kv)$.

When $t = 0, v = \frac{g}{k}$

$$0 = -\frac{1}{k}\ln\left(g + k\times\frac{g}{k}\right) + A \Rightarrow A = \frac{1}{k}\ln 2g$$

Hence

$$t = \frac{1}{k}\ln 2g - \frac{1}{k}\ln(g+kv) = \frac{1}{k}(\ln 2g - \ln(g+kv))$$

$$= \frac{1}{k}\ln\left(\frac{2g}{g+kv}\right)$$

Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

When $v = 0$

$$t = \frac{1}{k}\ln\left(\frac{2g}{g}\right) = \frac{\ln 2}{k}$$, as required

b R($\uparrow$) $\mathbf{F} = m\mathbf{a}$

$-mg - mkv = ma$

$-\cancel{m}g - \cancel{m}kv = \cancel{m}v\frac{dv}{dx}$

> In part **a**, where you are asked for a time, you use $a = \frac{dv}{dt}$.
> In part **b**, where you are asked for a distance, you use $a = v\frac{dv}{dx}$.

Separating the variables

$-\int 1\,dx = \int \frac{v}{g+kv}\,dv$

Let $\frac{v}{g+kv} = A + \frac{B}{g+kv}$

> In the fraction $\frac{v}{g+kv}$, the degree of the numerator is equal to the degree of the denominator, so, before integrating, you must reduce the improper fraction to a proper one. You can use any method to do this.

Multiply throughout by $g+kv$

$v = A(g+kv) + B$

Equating coefficients of v

$1 = Ak \Rightarrow A = \frac{1}{k}$

Equating constant coefficients

$0 = Ag + B \Rightarrow B = -gA = -\frac{g}{k}$

Hence

$-\int 1\,dx = \int\left(\frac{1}{k} - \frac{g}{k(g+kv)}\right)dv$

$-x = \frac{1}{k}v - \frac{g}{k^2}\ln(g+kv) + C$

> As in part **a**, $\int \frac{1}{g+kv}\,dv = \frac{1}{k}\ln(g+kv)$.

When $x = 0, v = \frac{g}{k}$

$0 = \frac{g}{k^2} - \frac{g}{k^2}\ln 2g + C \Rightarrow C = \frac{g}{k^2}\ln 2g - \frac{g}{k^2}$

Hence

$x = -\frac{1}{k}v + \frac{g}{k^2}\ln(g+kv) + \frac{g}{k^2} - \frac{g}{k^2}\ln 2g$

Let the greatest height above the point of projection reached by P be h.

When $x = h, v = 0$

> At the greatest height the velocity of P is 0.

$h = \frac{g}{k^2}\ln g + \frac{g}{k^2} - \frac{g}{k^2}\ln 2g$

$= \frac{g}{k^2} - \frac{g}{k^2}(\ln 2g - \ln g) = \frac{g}{k^2} - \frac{g}{k^2}\ln\left(\frac{2g}{g}\right)$

> Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$= \frac{g}{k^2}(1 - \ln 2)$

Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Review Exercise 1
Exercise A, Question 54

Question:

A particle of mass m moves under gravity down a line of greatest slope of a smooth plane inclined at an angle α to the horizontal. When the speed of the particle is v the resistance to motion of the particle is mkv, where k is a positive constant.

a Show that the limiting speed c of the particle is given by $c = \frac{g \sin \alpha}{k}$.

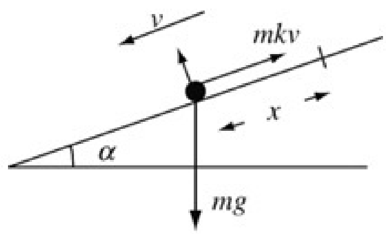
The particle starts from rest.

b Show that the time T taken to reach the speed of $\frac{1}{2}c$ is given by $T = \frac{1}{k} \ln 2$.

c Find, in terms of c and k, the distance travelled by the particle in attaining the speed of $\frac{1}{2}c$. **[E]**

Solution:

a



R(↙) $\mathbf{F} = m\mathbf{a}$

$mg\sin\alpha - mkv = ma$

Dividing throughout by m

$g\sin\alpha - kv = a$ ①

> The component of the weight acts down the plane and the resistance acts up the plane.

At the limiting speed c, $a = 0$

$g\sin\alpha - kc = 0$

Hence

$c = \dfrac{g\sin\alpha}{k}$, as required

> The limiting speed cannot be exceeded so, at the limiting speed, the acceleration is 0.

b From part a, $g\sin\alpha = kc$

Equation ① in part a can be written as

$kc - kv = a$ ②

> Replacing $g\sin\alpha$ by kc simplifies the algebra considerably.

$k(c - v) = \dfrac{dv}{dt}$

> To find a time, you use $a = \dfrac{dv}{dt}$. In part c, you are asked for distance and there you will use $a = v\dfrac{dv}{dx}$.

Separating the variables

$$\int k\,dt = \int \frac{1}{c-v}\,dt$$

$$kt = -\ln(c - v) + A$$

When $t = 0, v = 0$

$$0 = -\ln c + A \Rightarrow A = \ln c$$

$$kt = \ln c - \ln(c - v) = \ln\left(\frac{c}{c-v}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{c}{c-v}\right)$$

When $v = \dfrac{1}{2}c$

$$t = \frac{1}{k}\ln\left(\frac{c}{c-\frac{1}{2}c}\right) = \frac{1}{k}\ln\left(\frac{c}{\frac{1}{2}c}\right)$$

$= \dfrac{1}{k}\ln 2$, as required

c Writing $a = v\frac{dv}{dx}$ equation ② in part **b** becomes

$kc - kv = v\frac{dv}{dx}$

Separating the variables

$\int k\,dx = \int \frac{v}{c-v}\,dv$

Let $\frac{v}{c-v} = A + \frac{B}{c-v}$

$\frac{v}{c-v}$ is an improper fraction and, before integrating, you must express it as a constant + a proper fraction. If preferred, you could use long division rather than the method shown here.

Multiply throughout by $c - v$

$v = A(c-v) + B$

Equating coefficients of v

$1 = -A \Rightarrow A = -1$

Let $v \to c$

$c = B$

Hence

$\int k\,dx = \int \left(-1 + \frac{c}{c-v}\right) dv$

$kx = -v - c\ln(c-v) + D$

At $x = 0, v = 0$

$0 = -c\ln c + D \Rightarrow D = c\ln c$

Hence

$kx = c\ln c - c\ln(c-v) - v$

$= c(\ln c - \ln(c-v)) - v = c\ln\left(\frac{c}{c-v}\right) - v$

Using $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$.

$x = \frac{c}{k}\ln\left(\frac{c}{c-v}\right) - \frac{v}{k}$

When $v = \frac{1}{2}c$

$x = \frac{c}{k}\ln\left(\frac{c}{c-\frac{1}{2}c}\right) - \frac{\frac{1}{2}c}{k}$

$= \frac{c}{k}\left(\ln 2 - \frac{1}{2}\right)$

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Review Exercise 1
Exercise A, Question 55

Question:

A particle of mass m is falling vertically under gravity in a resisting medium. The particle is released from rest. The speed v of the particle at a distance x from rest is given by $v^2 = 2kg\left[1 - e^{-\frac{x}{k}}\right]$, where k is a positive constant.

a Show that the magnitude of the resistance is $\frac{mv^2}{2k}$.

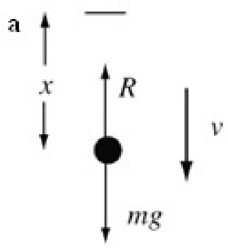
The particle is projected upwards in the same medium with speed $\sqrt{(2kg)}$.

b Show that the maximum height reached by the particle above the point of projection is $k\ln 2$.

c Find the time taken to reach the maximum height above the point of projection. **[E]**

Solution:

a



Let the resistance be R.

$\text{R}(\downarrow)\ \mathbf{F} = m\mathbf{a}$

$$mg - R = ma = mv\frac{dv}{dx}$$

$$R = mg - mv\frac{dv}{dx} \quad ①$$

$$v^2 = 2kg\left[1 - e^{-\frac{x}{k}}\right] \quad ②$$

Differentiate ② with respect to x

$$2v\frac{dv}{dx} = 2kg \times \left(\frac{1}{k}\right)e^{-\frac{x}{k}}$$

$$v\frac{dv}{dx} = ge^{-\frac{x}{k}} \quad ③$$

You differentiate v^2 implicitly with respect to x.
$\frac{d}{dx}(v^2) = \frac{d}{dv}(v^2) \times \frac{dv}{dx} = 2v\frac{dv}{dx}$.

From ②

$$1 - e^{-\frac{x}{k}} = \frac{v^2}{2kg} \Rightarrow e^{-\frac{x}{k}} = 1 - \frac{v^2}{2kg} \quad ④$$

To complete this part, you use the equation given in the question to obtain $e^{-\frac{x}{k}}$ in terms of v.

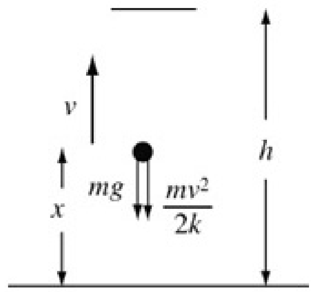
Substituting ④ into ③

$$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{2kg}\right) = g - \frac{v^2}{2k}$$

Substituting for $v\frac{dv}{dx}$ into ①

$$R = mg - mg + \frac{mv^2}{2k} = \frac{mv^2}{2k}\text{, as required}$$

b



R(↑) $\mathbf{F} = ma$

$$-mg - \frac{mv^2}{2k} = ma = mv\frac{dv}{dx}$$

Dividing throughout by m

$$-g - \frac{v^2}{2k} = v\frac{dv}{dx}$$

$$-\frac{2kg + v^2}{2k} = v\frac{dv}{dx}$$

Before separating the variables, you need to put the left hand side of the equation over the common denominator $2k$.

Separating the variables

$$\int 1\,dx = -\int \frac{2kv}{2kg + v^2}\,dv$$

$$x = -k\ln(2kg + v^2) + A$$

As $\frac{d}{dv}(2kg + v^2) = 2v$, $\int \frac{2v}{2kg + v^2}\,dv = \ln(2kg + v^2) + \text{a constant}$.

At $x = 0, v = \sqrt{(2kg)}$

$$0 = -k\ln(2kg + 2kg) + A \Rightarrow A = k\ln(4kg)$$

When $x = 0$, $v = \sqrt{2kg}$

Hence

$$x = k\ln(4kg) - k\ln(2kg + v^2) = k\ln\left(\frac{4kg}{2kg + v^2}\right)$$

Let the greatest height above the point of projection reached by the particle be h.

At $x = h, v = 0$

$$h = k\ln\left(\frac{4kg}{2kg}\right) = k\ln 2, \text{ as required}$$

At the greatest height, the velocity of the particle is 0.

c From part **b**

$$-mg - \frac{mv^2}{2k} = ma = m\frac{dv}{dt}$$

Divide by m and put the left hand side of the equation over a common denominator.

$$-\frac{2kg + v^2}{2k} = \frac{dv}{dt}$$

Separating the variables

$$\int 1 \, dt = -2k \int \frac{1}{2kg + v^2} \, dv$$

$\int \frac{1}{a^2 + v^2} \, dv = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + A$.
Here $a^2 = 2kg$, so $a = \sqrt{(2kg)}$.

$$t = -\frac{2k}{\sqrt{(2kg)}} \arctan\left(\frac{v}{\sqrt{(2kg)}}\right) + B$$

When $t = 0, v = \sqrt{(2kg)}$

$$0 = -\frac{2k}{\sqrt{(2kg)}} \arctan\left(\frac{\sqrt{(2kg)}}{\sqrt{(2kg)}}\right) + B$$

$$B = \frac{2k}{\sqrt{(2kg)}} \arctan 1 = \frac{2k}{\sqrt{(2kg)}} \frac{\pi}{4}$$

Hence

$$t = \frac{2k}{\sqrt{(2kg)}} \left[\frac{\pi}{4} - \arctan\left(\frac{v}{\sqrt{(2kg)}}\right)\right]$$

At the maximum height, $v = 0$

$$t = \frac{2k}{\sqrt{(2kg)}} \left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{2} \sqrt{\left(\frac{k}{2g}\right)}$$

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Review Exercise 1
Exercise A, Question 56

Question:

A ship of mass m is propelled in a straight line through the water by a propeller which develops a constant force of magnitude F. When the speed of the ship is v, the water causes a drag of magnitude kv, where k is a constant, to act on the ship. The ship starts from rest at time $t = 0$.

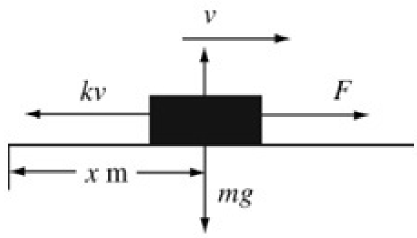
a Show that the ship reaches half of its theoretical maximum speed of $\frac{F}{k}$ when $t = \frac{m\ln 2}{k}$.

When the ship in moving with speed $\frac{F}{2k}$, an emergency occurs and the captain reverses the engines so that the propeller force, which remains of magnitude F, acts backwards.

b Show that the ship covers a further distance $\frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right]$ on its original course, which may be assumed to remain unchanged, before being brought to rest. **[E]**

Solution:

a



R(→) $\mathbf{F} = m\mathbf{a}$

$$F - kv = ma = m\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = \int \frac{m}{F - kv}\,dv$$

$$t = -\frac{m}{k}\ln(F - kv) + A$$

As $\int \frac{-k}{F-kv}\,dv = \ln(F - kv) + \text{a constant}$,
$\int \frac{1}{F-kv}\,dv = -\frac{1}{k}\ln(F - kv) + \text{a constant}$.

When $t = 0, v = 0$

$$0 = -\frac{m}{k}\ln F + A \Rightarrow A = \frac{m}{k}\ln F$$

Hence

$$t = \frac{m}{k}\ln F - \frac{m}{k}\ln(F - kv)$$

$$t = \frac{m}{k}\ln\left(\frac{F}{F - kv}\right)$$

When $v = \frac{F}{2k}$

The limiting (or terminal) velocity is, in this case, the theoretical maximum speed of the ship. It is given by substituting $a = 0$ into $F - kv = ma$, which gives $v = \frac{F}{k}$. In this question, you are not asked to prove this result but you should know how to prove it. Half of the theoretical maximum speed is $\frac{F}{2k}$.

$$t = \frac{m}{k}\ln\left(\frac{F}{F - \frac{1}{2}F}\right) = \frac{m}{k}\ln\left(\frac{F}{\frac{1}{2}F}\right)$$

$= \frac{m\ln 2}{k}$, as required

b $-F - kv = ma = mv\frac{dv}{dx}$

The force developed by the propeller is now reversed, so you change the sign of F in the equation of motion you found in part a.

Separating the variables

$$-\int 1\,dx = m\int \frac{v}{F + kv}\,dv$$

$$= \frac{m}{k}\int \frac{kv}{F + kv}\,dv = \frac{m}{k}\int \frac{F + kv - F}{F + kv}\,dv$$

$$-x = \frac{m}{k}\int\left(1 - \frac{F}{F + kv}\right)dv$$

$\frac{v}{F+kv}$ is an improper fraction and must be transformed into an expression involving a proper fraction before integration. You may use any appropriate method to do this.

$$= \frac{m}{k}\left(v - \frac{F}{k}\ln(F + kv)\right) + B$$

When $x = 0, v = \frac{F}{2k}$

$$0 = \frac{m}{k}\left(\frac{F}{2k} - \frac{F}{k}\ln\left(F + \frac{F}{2}\right)\right) + B$$

$$B = -\frac{mF}{k^2}\left(\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right)$$

The algebra here is complicated and it is worth taking the factor $\frac{mF}{k^2}$ outside the bracket as both this and $\ln\left(\frac{3}{2}\right)$ appear in the printed answer. Always keep in mind what you are aiming for and try to work towards it.

Hence

$$x = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right] - \frac{m}{k}\left[v - \frac{F}{k}\ln(F + kv)\right]$$

When $v = 0$

$$x = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2}\right)\right] + \frac{mF}{k^2}\ln F$$

$$= \frac{mF}{k^2}\left[\frac{1}{2} - \left(\ln\left(\frac{3F}{2}\right) - \ln F\right)\right] = \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3F}{2F}\right)\right]$$

$$= \frac{mF}{k^2}\left[\frac{1}{2} - \ln\left(\frac{3}{2}\right)\right],$$ as required