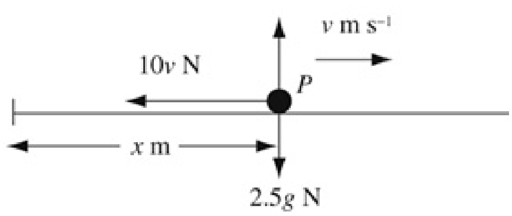
Solutionbank M4
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise A, Question 1

Question:

A particle P of mass 2.5 kg moves in a straight horizontal line. When the speed of P is v m s^{-1}, the resultant force acting on P is a resistance of magnitude $10v$ N. Find the time P takes to slow down from 24 m s^{-1} to 6 m s^{-1}.

Solution:



R(→) $\mathbf{F} = m\mathbf{a}$

$$-10v = 2.5\frac{dv}{dt}$$

Separating the variables

$$\int 4\,dt = -\int \frac{1}{v}\,dv$$

$$4t = A - \ln v$$

When $t = 0,\ v = 24$

$0 = A - \ln 24 \Rightarrow A = \ln 24$

Hence

$4t = \ln 24 - \ln v$

$$t = \frac{1}{4}\ln\left(\frac{24}{v}\right)$$

When $v = 6$

$$t = \frac{1}{4}\ln 4\ (\approx 0.347)$$

P takes $\frac{1}{4}\ln 4$ s(= 0.347 s, 3 d.p.) to slow from 24 m s^{-1} to 6 m s^{-1}.

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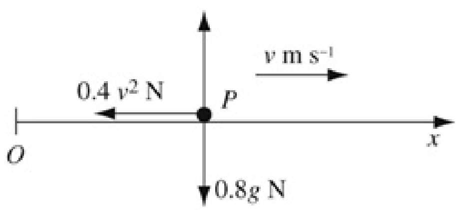
Resisted motion of a particle movig in a straight line
Exercise A, Question 2

Question:

A particle P of mass 0.8 kg is moving along the axis Ox in the direction of x-increasing. When the speed of P is $v \text{ m s}^{-1}$, the resultant force acting on P is a resistance of magnitude $0.4v^2$ N. Initially P is at O and is moving with speed 12 m s^{-1}. Find the distance P moves before its speed is halved.

Solution:



R($\rightarrow$) $\qquad \mathbf{F} = m\mathbf{a}$

$$-0.4v^2 = 0.8v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = -2\int \frac{1}{v}\,\mathrm{d}v$$

$$x = A - 2\ln v$$

At $x = 0$, $v = 12$

$0 = A - 2\ln 12 \Rightarrow A = 2\ln 12$

Hence

$$x = 2\ln 12 - 2\ln v = 2\ln\left(\frac{12}{v}\right)$$

When $v = 6$

$x = 2\ln 2$

The distance P moves before its speed is halved is $2\ln 2$ m $= 1.39$ m (3 s.f.).

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Resisted motion of a particle movig in a straight line
Exercise A, Question 3

Question:

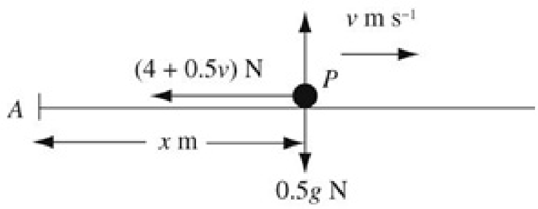
A particle P of mass 0.5 kg moves in a straight horizontal line against a resistance of magnitude $(4+0.5v)$N, where v m s^{-1} is the speed of P at time t seconds. When $t=0$, P is at a point A moving with speed 12 m s^{-1}. The particle P comes to rest at the point B. Find

a the time P takes to move from A to B,

b the distance AB.

Solution:

a



$R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5\frac{dv}{dt}$$

Separating the variables

$$\int 1 \, dt = -\int \frac{1}{8+v} \, dv$$

$$t = A - \ln(8+v)$$

When $t = 0$, $v = 12$

$0 = A - \ln 20 \Rightarrow A = \ln 20$

Hence

$$t = \ln 20 - \ln(8+v) = \ln\left(\frac{20}{8+v}\right)$$

When $v = 0$

$$t = \ln\left(\frac{20}{8}\right) = \ln 2.5$$

The time taken for P to move from A to B is $\ln 2.5 \text{ s} = 0.916 \text{ s}$ (3 d.p.).

b $R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$-(4+0.5v) = 0.5v\frac{dv}{dx}$$

Separating the variables

$$\int 1 \, dx = -\int \frac{v}{8+v} \, dv$$

$$\frac{v}{8+v} = \frac{8+v-8}{8+v} = 1 - \frac{8}{8+v}$$

Hence

$$\int 1 \, dx = -\int \left(1 - \frac{8}{8+v}\right) dv$$

$$x = A - v + 8\ln(8+v)$$

At $x = 0, v = 12$

$0 = A - 12 + 8\ln 20 \Rightarrow A = 12 - 8\ln 20$

Hence $\quad x = 12 - v - (8\ln 20 - 8\ln(8+v))$

$$= 12 - v - 8\ln\left(\frac{20}{8+v}\right)$$

When $v = 0$

$x = 12 - 8\ln 2.5$

$AB = (12 - 8\ln 2.5)\text{m} = 4.67 \text{ m}$ (3 s.f.)

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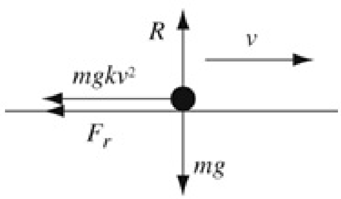
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise A, Question 4

Question:

A particle of mass m is projected along a rough horizontal plane with velocity u m s^{-1}. The coefficient of friction between the particle and the plane is μ. When the particle is moving with speed v m s^{-1}, it is also subject to an air resistance of magnitude $kmgv^2$, where k is a constant. Find the distance the particle moves before coming to rest.

Solution:



R(↑) $R = mg$

As friction is limiting

$$F_r = \mu R = \mu mg$$

R(→) $\mathbf{F} = m\mathbf{a}$

$$-F_r - kmgv^2 = ma$$

$$-\mu \cancel{m} g - k \cancel{m} gv^2 = \cancel{m} v \frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int g \, \mathrm{d}x = -\int \frac{v}{\mu + kv^2} \, \mathrm{d}v$$

$$gx = A - \frac{1}{2k} \ln(\mu + kv^2)$$

At $x = 0, v = u$

$$0 = A - \frac{1}{2k} \ln(\mu + ku^2) \Rightarrow A = \frac{1}{2k} \ln(\mu + ku^2)$$

Hence

$$x = \frac{1}{2kg} (\ln(\mu + ku^2) - \ln(\mu + kv^2)) = \frac{1}{2kg} \ln\left(\frac{\mu + ku^2}{\mu + kv^2}\right)$$

When $v = 0$

$$x = \frac{1}{2kg} \ln\left(\frac{\mu + ku^2}{\mu}\right)$$

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Resisted motion of a particle movig in a straight line
Exercise A, Question 5

Question:

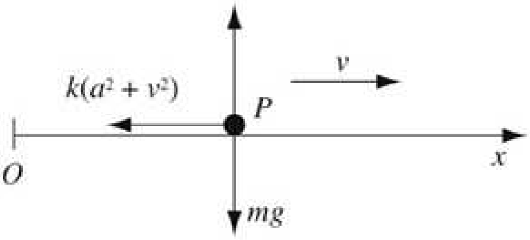
A particle P of mass m is moving along the axis Ox in the direction of x-increasing. At time t seconds, the velocity of P is v. The only force acting on P is a resistance of magnitude $k(a^2+v^2)$. At time $t=0$, P is at O and its speed is U. At time $t=T, v=\frac{1}{2}U$

a Show that $T=\frac{m}{ak}\left[\arctan\left(\frac{U}{a}\right)-\arctan\left(\frac{U}{2a}\right)\right]$.

b Find the distance travelled by P as its speed is reduced from U to $\frac{1}{2}U$.

Solution:

a



R$(\rightarrow)$ $\quad$ $\mathbf{F}=m\mathbf{a}$

$$-k(a^2+v^2)=m\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt=-\frac{m}{k}\int\frac{1}{a^2+v^2}\,dv$$

$$t=A-\frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t=0, v=U$

$$0=A-\frac{m}{ak}\arctan\left(\frac{U}{a}\right)\Rightarrow A=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)$$

Hence

$$t=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)-\frac{m}{ak}\arctan\left(\frac{v}{a}\right)$$

When $t=T, v=\frac{1}{2}U$

$$T=\frac{m}{ak}\arctan\left(\frac{U}{a}\right)-\frac{m}{ak}\arctan\left(\frac{\frac{1}{2}U}{a}\right)$$

$$T=\frac{m}{ak}\left[\arctan\left(\frac{U}{a}\right)-\arctan\left(\frac{U}{2a}\right)\right],\quad \text{as required}$$

b $\quad$ R$(\rightarrow)$ $\quad$ $\mathbf{F}=m\mathbf{a}$

$$-k(a^2+v^2)=mv\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx=-\frac{m}{k}\int\frac{v}{a^2+v^2}\,dv$$

$$x=A-\frac{m}{2k}\ln(a^2+v^2)$$

When $x=0, v=U$

$$0=A-\frac{m}{2k}\ln(a^2+U^2)\Rightarrow A=\frac{m}{2k}\ln(a^2+U^2)$$

Hence

$$x=\frac{m}{2k}\ln(a^2+U^2)-\frac{m}{2k}\ln(a^2+v^2)=\frac{m}{2k}\ln\left(\frac{a^2+U^2}{a^2+v^2}\right)$$

When $v=\frac{1}{2}U$

$$x=\frac{m}{2k}\ln\left(\frac{a^2+U^2}{a^2+\frac{1}{4}U^2}\right)=\frac{m}{2k}\ln\left(\frac{4a^2+4U^2}{4a^2+U^2}\right)$$

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise A, Question 6

Question:

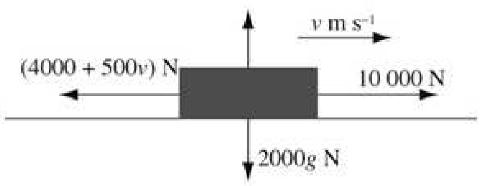
A lorry of mass 2000 kg travels along a straight horizontal road. The engine of the lorry produces a constant driving force of magnitude 10 000 N. At time t seconds, the speed of the lorry is v m s^{-1}. As the lorry moves, the total resistance to the motion of the lorry is of magnitude $(4000+500v)$ N. The lorry starts from rest. Find

a v in terms of t,

b the terminal speed of the lorry.

Solution:

a



$\text{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$

$10\,000-(4000+500v) = 2000a$

$6000-500v = 2000\dfrac{\mathrm{d}v}{\mathrm{d}t}$

Dividing throughout by 500

$12-v = 4\dfrac{\mathrm{d}v}{\mathrm{d}t}$

Separating the variables

$\int 1\,\mathrm{d}t = 4\int \dfrac{1}{12-v}\,\mathrm{d}v$

$t = A-4\ln(12-v)$

$\ln(12-v) = B-\dfrac{t}{4}$, where $B=\dfrac{1}{4}A$

$12-v = \mathrm{e}^{B-\frac{t}{4}} = \mathrm{e}^{B}\mathrm{e}^{-\frac{t}{4}} = C\mathrm{e}^{-\frac{t}{4}}$, where $C=\mathrm{e}^{B}$

Hence

$v=12-C\mathrm{e}^{-\frac{t}{4}}$

When $t=0, v=0$

$0=12-C \Rightarrow C=12$

Hence

$v=12\left(1-\mathrm{e}^{-\frac{t}{4}}\right)$

b As $t\rightarrow\infty, \mathrm{e}^{-\frac{t}{4}}\rightarrow 0$ and $v\rightarrow 12$

The terminal speed of the lorry is 12 m s^{-1}.

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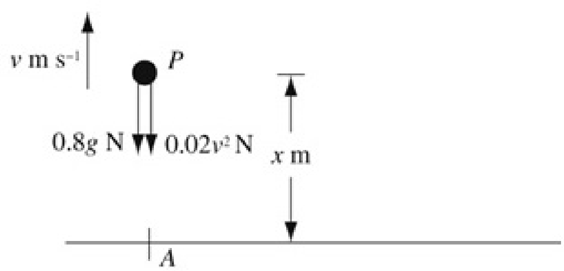
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 1

Question:

A particle P of mass 0.8 kg is projected vertically upwards with velocity 30 m s^{-1} from a point A on horizontal ground. Air resistance is modelled as a force of magnitude $0.02v^2\text{N}$, where $v \text{ m s}^{-1}$ is the velocity of P.
Find the greatest height above A attained by P.

Solution:



$\text{R}(\uparrow) \quad \mathbf{F} = m\mathbf{a}$

$-0.8g - 0.02v^2 = 0.8a$

$-7.84 - 0.02v^2 = 0.8v\dfrac{\text{d}v}{\text{d}x}$

Separating the variables

$\int 1 \, \text{d}x = -0.8\int \dfrac{v}{7.84 + 0.02v^2}\,\text{d}v$

$x = A - \dfrac{0.8}{0.04}\ln(7.84 + 0.02v^2)$

At $x = 0,\ v = 30$

$0 = A - 20\ln(7.84 + 18) \Rightarrow A = 20\ln 25.84$

Hence

$x = 20\ln 25.84 - 20\ln(7.84 + 0.02v^2)$

$= 20\ln\left(\dfrac{25.84}{7.84 + 0.02v^2}\right)$

At the greatest height, $v = 0$

$x = 20\ln\left(\dfrac{25.84}{7.84}\right) \approx 23.9$

The greatest height above A attained by P is 23.9 m (3 s.f).

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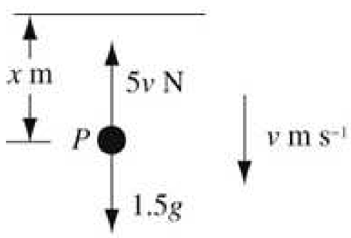
Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 2

Question:

A particle P of mass 1.5 kg is released from rest at time $t = 0$ and falls vertically through a liquid. The liquid resists the motion of P with a force of magnitude $5v$ N where v m s^{-1} is the speed of P at time t seconds.
Find the value of t when the speed of P is 2 m s^{-1}.

Solution:



R(↓) $\mathbf{F} = m\mathbf{a}$

$1.5g - 5v = 1.5a$

$14.7 - 5v = 1.5\frac{dv}{dt}$

Separating the variables

$\int 1\,dt = 1.5\int \frac{1}{14.7 - 5v}\,dv$

$t = A - \frac{1.5}{5}\ln(14.7 - 5v)$

When $t = 0$, $v = 0$

$0 = A - 0.3\ln 14.7 \Rightarrow A = 0.3\ln 14.7$

Hence

$t = 0.3\ln 14.7 - 0.3\ln(14.7 - 5v)$

$= 0.3\ln\left(\frac{14.7}{14.7 - 5v}\right)$

When $v = 2$

$t = 0.3\ln\left(\frac{14.7}{14.7 - 10}\right) = 0.342$ s (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 3

Question:

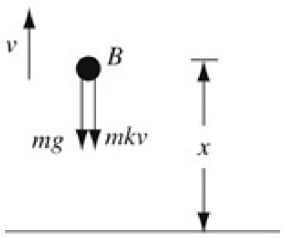
A small ball B of mass m is projected upwards from horizontal ground with speed u. Air resistance is modelled as a force of magnitude mkv, where v m s^{-1} is the velocity of P at time t seconds.

a Show that the greatest height above the ground reached by B is $\frac{u}{k}-\frac{g}{k^2}\ln\left(1+\frac{ku}{g}\right)$.

b Find the time taken to reach this height.

Solution:

a



$R(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$-mg - mkv = ma$

$-g - kv = v\dfrac{dv}{dx}$

Separating the variables

$$\int 1\,dx = -\int \frac{v}{g+kv}\,dv$$

$$= -\frac{1}{k}\int \frac{g+kv-g}{g+kv}\,dv$$

$$= -\frac{1}{k}\int\left(1-\frac{g}{g+kv}\right)dv$$

$$x = A - \frac{1}{k}\left[v - \frac{g}{k}\ln(g+kv)\right] = A - \frac{v}{k} + \frac{g}{k^2}\ln(g+kv)$$

At $x = 0, v = u$

$$0 = A - \frac{u}{k} + \frac{g}{k^2}\ln(g+ku) \Rightarrow A = \frac{u}{k} - \frac{g}{k^2}\ln(g+ku)$$

Hence

$$x = \frac{u}{k} - \frac{v}{k} - \left[\frac{g}{k^2}\ln(g+ku) - \frac{g}{k^2}\ln(g+kv)\right]$$

$$= \frac{1}{k}(u-v) - \frac{g}{k^2}\ln\left(\frac{g+ku}{g+kv}\right)$$

At the greatest height, $v = 0$

$$x = \frac{u}{k} - \frac{g}{k^2}\ln\left(\frac{g+ku}{g}\right) = \frac{u}{k} - \frac{g}{k^2}\ln\left(1+\frac{ku}{g}\right)$$, as required

b $-g - kv = \frac{dv}{dt}$

Separating the variables

$$\int 1\,dt = -\int \frac{1}{g+kv}\,dt$$

$$t = B - \frac{1}{k}\ln(g+kv)$$

When $t = 0, v = u$

$$0 = B - \frac{1}{k}\ln(g+ku) \Rightarrow B = \frac{1}{k}\ln(g+ku)$$

Hence

$$t = \frac{1}{k}\ln(g+ku) - \frac{1}{k}\ln(g+kv) = \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right)$$

At the greatest height, $v = 0$

$$t = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right) = \frac{1}{k}\ln\left(1+\frac{ku}{g}\right)$$

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 4

Question:

A parachutist of mass 60 kg falls vertically from rest from a fixed balloon. For the first 3 s of her motion, her fall is resisted by air resistance of magnitude $20v$N where $v\text{ m s}^{-1}$ is her velocity.

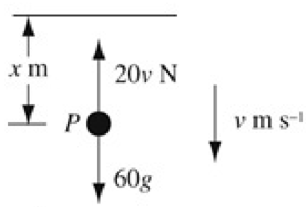
a Find the velocity of the parachutist after 3 s.

After 3 s, her parachute opens and her further motion is resisted by a force of magnitude $(20v+60v^2)$N.

b Find the terminal speed of the parachutist.

Solution:

a



R(↓) $\mathbf{F}=m\mathbf{a}$

$$60g-20v=60a$$

$$588-20v=60\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt=60\int\frac{1}{588-20v}\,dv$$

$$t=A-\frac{60}{20}\ln(588-20v)$$

When $t=0, v=0$

$0=A-3\ln 588 \Rightarrow A=3\ln 588$

Hence

$$t=3\ln 588-3\ln(588-20v)=3\ln\left(\frac{588}{588-20v}\right)$$

When $t=3$

$$3=3\ln\left(\frac{588}{588-20v}\right)\Rightarrow \ln\left(\frac{588}{588-20v}\right)=1$$

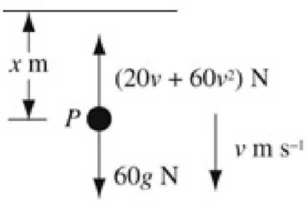
$$\frac{588}{588-20v}=e$$

$$588-20v=588e^{-1}$$

$$v=\frac{588}{20}(1-e^{-1})\approx 18.6$$

The velocity of the parachutist after 3 s is 18.6 m s^{-1} (3 s.f.)

b



$R(\downarrow)\quad \mathbf{F} = m\mathbf{a}$

$60g - (20v + 60v^2) = 60a$

At the terminal speed $a = 0$

$588 - 20v - 60v^2 = 0$

$60v^2 + 20v - 588 = 0$

Only the positive root need be considered

$v = \dfrac{-20 + \sqrt{(20^2 + 4 \times 60 \times 588)}}{120} \approx 2.97$

The terminal speed of the parachutist is 2.97 m s^{-1} (3 s.f.)

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise B, Question 5

Question:

A particle P of mass m is projected vertically upwards with speed u from a point A on horizontal ground. The particle P is subject to air resistance of magnitude $mgkv$, where v is the speed of P and k is a positive constant.

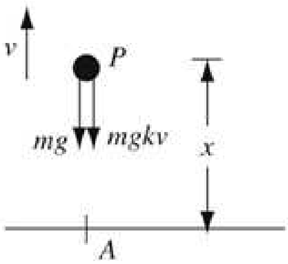
a Find the greatest height above A reached by P.

Assuming P has not reached the ground,

b find an expression for the speed of the particle t seconds after it has reached its greatest height.

Solution:

a



$R(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$-mg - mgkv = ma$

$-g - gkv = v\frac{\mathrm{d}v}{\mathrm{d}x}$

Separating the variables

$$-\int g\,\mathrm{d}x = \int \frac{v}{1+kv}\,\mathrm{d}v$$

$$= \frac{1}{k}\int \frac{1+kv-1}{1+kv}\,\mathrm{d}v$$

$$= \frac{1}{k}\int \left(1 - \frac{1}{1+kv}\right)\mathrm{d}v$$

$$-gx = \frac{1}{k}\left[v - \frac{1}{k}\ln(1+kv)\right] + A$$

At $x = 0, v = u$

$$0 = \frac{1}{k}\left[u - \frac{1}{k}\ln(1+ku)\right] + A \Rightarrow A = -\frac{u}{k} + \frac{1}{k^2}\ln(1+ku)$$

Hence

$$-gx = \frac{v}{k} - \frac{1}{k^2}\ln(1+kv) - \frac{u}{k} + \frac{1}{k^2}\ln(1+ku)$$

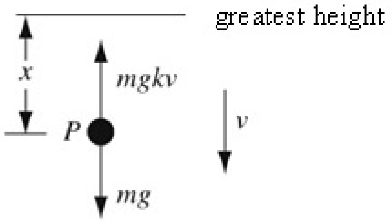
$$x = \frac{1}{gk}(u-v) - \frac{1}{gk^2}\ln\left(\frac{1+ku}{1+kv}\right)$$

At the greatest height $v = 0$

$$x = \frac{u}{gk} - \frac{1}{gk^2}\ln(1+ku) = \frac{1}{gk^2}(ku - \ln(1+ku))$$

The greatest height above A reached by P is $\frac{1}{gk^2}(ku - \ln(1+ku))$.

b



$\mathrm{R}(\downarrow) \qquad \mathbf{F} = m\mathbf{a}$

$mg - mgkv = ma$

$g(1-kv) = \frac{dv}{dt}$

Separating the variables

$\int g \, dt = \int \frac{1}{1-kv} \, dv$

$gt = A - \frac{1}{k}\ln(1-kv)$

When $t = 0, v = 0$

$0 = A - \ln 1 \Rightarrow A = 0$

Hence

$gt = -\frac{1}{k}\ln(1-kv)$

$-kgt = \ln(1-kv)$

$1 - kv = e^{-kgt}$

$v = \frac{1}{k}(1 - e^{-kgt})$

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Resisted motion of a particle movig in a straight line
Exercise B, Question 6

Question:

A particle of mass m is projected vertically upwards from a point A on horizontal ground with speed u. The particle reaches its greatest height above the ground at the point B.

a Ignoring air resistance, find the distance AB.

Instead of ignoring air resistance, it is modelled as a resisting force of magnitude mkv^2, where $v \text{ m s}^{-1}$ is the velocity of the particle and k is a positive constant. Using this model find

b the distance AB,

c the work done by air resistance against the motion of the particle as it moves from A to B.

Solution:

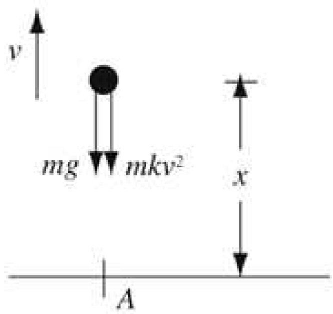
a $v^2 = u^2 + 2as$

At the greatest height, $v = 0$

$0 = u^2 - 2g \times AB$

$AB = \dfrac{u^2}{2g}$

b



$R(\uparrow)\quad \mathbf{F} = m\mathbf{a}$

$-mg - mkv^2 = ma$

$-g - kv^2 = v\dfrac{dv}{dx}$

$\displaystyle\int 1\,dx = -\int \frac{v}{g + kv^2}\,dv$

$x = A - \dfrac{1}{2k}\ln(g + kv^2)$

At $x = 0, v = u$

$0 = A - \dfrac{1}{2k}\ln(g + ku^2) \Rightarrow A = \dfrac{1}{2k}\ln(g + ku^2)$

Hence

$x = \dfrac{1}{2k}\ln(g + ku^2) - \dfrac{1}{2k}\ln(g + kv^2) = \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g + kv^2}\right)$

At the greatest height $v = 0$ and $x = AB$

$AB = \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g}\right) = \dfrac{1}{2k}\ln\left(1 + \dfrac{ku^2}{g}\right)$

c The work done by air resistance is the difference between the potential energies of the particle at the greatest heights in parts **a** and **b** and is given by

$mg \times \dfrac{u^2}{2g} - mg \times \dfrac{1}{2k}\ln\left(1 + \dfrac{ku^2}{g}\right)$

$= mg\left(\dfrac{u^2}{2g} - \dfrac{1}{2k}\ln\left(1 + \dfrac{ku^2}{g}\right)\right)$

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Resisted motion of a particle movig in a straight line
Exercise B, Question 7

Question:

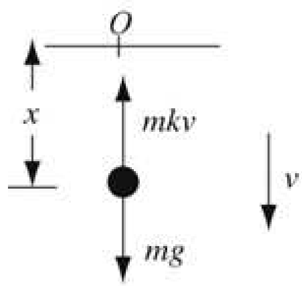
A particle P of mass m is projected vertically downwards from a fixed point O with speed $\frac{g}{2k}$, where k is a constant. At time t seconds after projection, the displacement of P from O is x and the velocity of P is v. The particle P is subject to a resistance of magnitude mkv.

a Show that $v = \frac{g}{2k}(2 - e^{-kt})$.

b Find x when $t = \frac{\ln 4}{k}$.

Solution:

a



R(↓) $\quad \mathbf{F} = m\mathbf{a}$

$$mg - mkv = ma$$

$$g - kv = \frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = \int \frac{1}{g - kv}\,dv$$

$$t = A - \frac{1}{k}\ln(g - kv)$$

$$kt = kA - \ln(g - kv)$$

$$\ln(g - kv) = kA - kt$$

$$g - kv = e^{kA - kt} = Be^{-kt}, \quad \text{where} \quad B = e^{kA}$$

$$kv = g - Be^{-kt}$$

When $t = 0, v = \frac{g}{2k}$

$$k \times \frac{g}{2k} = g - B \Rightarrow B = g - \frac{g}{2} = \frac{g}{2}$$

Hence

$$kv = g - \frac{g}{2}e^{-kt} = \frac{g}{2}(2 - e^{-kt})$$

$$v = \frac{g}{2k}(2 - e^{-kt}), \text{ as required}$$

b From part **a**

$$v = \frac{dx}{dt} = \frac{g}{2k}(2 - e^{-kt})$$

$$x = \int \frac{g}{2k}(2 - e^{-kt})\,dt$$

$$= \frac{g}{2k}\left(2t + \frac{1}{k}e^{-kt}\right) + B$$

When $t = 0, x = 0$

$$0 = \frac{g}{2k^2} + B \Rightarrow B = -\frac{g}{2k^2}$$

Hence

$$x = \frac{g}{k}t + \frac{g}{2k^2}(e^{-kt} - 1)$$

When $t = \frac{\ln 4}{k}$

$$x = \frac{g}{k^2}\ln 4 + \frac{g}{2k^2}(e^{-\ln 4} - 1) = \frac{2g}{k^2}\ln 2 + \frac{g}{2k^2}\left(\frac{1}{4} - 1\right)$$

$$= \frac{2g}{k^2}\ln 2 - \frac{3g}{8k^2}$$

$$= \frac{g}{8k^2}(16\ln 2 - 3)$$

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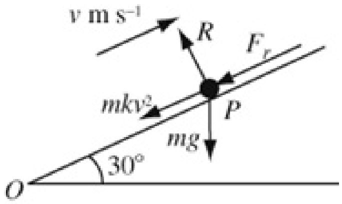
Resisted motion of a particle movig in a straight line
Exercise B, Question 8

Question:

A particle P of mass m is projected with speed U up a rough plane inclined at an angle $30°$ to the horizontal. The coefficient of friction between P and the plane is $\frac{\sqrt{3}}{4}$. The particle P is subject to an air resistance of magnitude mkv^2, where v is the speed of P and k is a positive constant.
Find the distance P moves before coming to rest.

Solution:



R(↖) $R = mg\cos 30°$

Friction is limiting

$$F_r = \mu R = \frac{\sqrt{3}}{4} mg\cos 30° = \frac{\sqrt{3}}{4} mg \times \frac{\sqrt{3}}{2} = \frac{3}{8} mg$$

R(↗) $\mathbf{F} = m\mathbf{a}$

$$-F_r - mg\sin 30° - mkv^2 = ma$$

$$-\frac{3}{8} mg - \frac{1}{2} mg - mkv^2 = mv\frac{dv}{dx}$$

Dividing throughout by m and multiplying throughout by 8

$$-7g - 8kv^2 = 8v\frac{dv}{dx}$$

Separating the variables

$$\int 1\, dx = -\int \frac{8v}{7g + 8kv^2}\, dv$$

$$x = A - \frac{1}{2k}\ln(7g + 8kv^2)$$

At $x = 0, v = U$

$$0 = A - \frac{1}{2k}\ln(7g + 8kU^2) \Rightarrow A = \frac{1}{2k}\ln(7g + 8kU^2)$$

Hence

$$x = \frac{1}{2k}\ln(7g + 8kU^2) - \frac{1}{2k}\ln(7g + 8kv^2)$$

$$= \frac{1}{2k}\ln\left(\frac{7g + 8kU^2}{7g + 8kv^2}\right)$$

When $v = 0$

$$x = \frac{1}{2k}\ln\left(\frac{7g + 8kU^2}{7g}\right) = \frac{1}{2k}\ln\left(1 + \frac{8kU^2}{7g}\right)$$

The distance P moves before coming to rest is $\frac{1}{2k}\ln\left(1 + \frac{8kU^2}{7g}\right)$.

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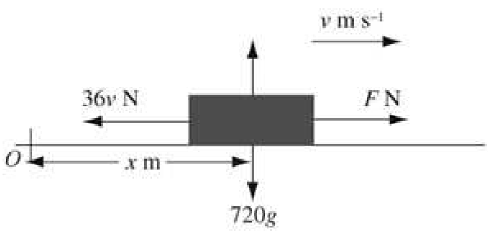
Resisted motion of a particle movig in a straight line
Exercise C, Question 1

Question:

A car of mass 720 kg is moving along a straight horizontal road with the engine of the car working at 30 kW. At time $t = 0$, the car passes a point A moving with speed 12 m s^{-1}. The total resistance to the motion of the car is $36v$ N, where $v\text{ m s}^{-1}$ is the speed of the car at time t seconds.
Find the time the car takes to double its speed.

Solution:



$30\text{ kW} = 30\,000\text{W}$

Let the tractive force generated by the engine be F N.

$$P = Fv$$

$$30\,000 = Fv \Rightarrow F = \frac{30\,000}{v}$$

$$\text{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$

$$F - 36v = 720a$$

$$\frac{30\,000}{v} - 36v = 720\frac{dv}{dt}$$

$$30\,000 - 36v^2 = 720v\frac{dv}{dt}$$

Separating the variables

$$\int 1\,dt = \int \frac{720v}{30\,000 - 36v^2}\,dv$$

$$t = A - 10\ln(30\,000 - 36v^2)$$

When $t = 0, v = 12$

$$0 = A - 10\ln(30\,000 - 36\times 12^2) \Rightarrow A = 10\ln 24\,816$$

Hence

$$t = 10\ln 24\,816 - 10\ln(30\,000 - 36v^2) = 10\ln\left(\frac{24\,816}{30\,000 - 36v^2}\right)$$

When $v = 24$

$$t = 10\ln\left(\frac{24\,816}{30\,000 - 36\times 24^2}\right) = 10\ln\left(\frac{24\,816}{9264}\right) \approx 9.85$$

The time the car takes to double its speed is 9.85s (3 s.f.)

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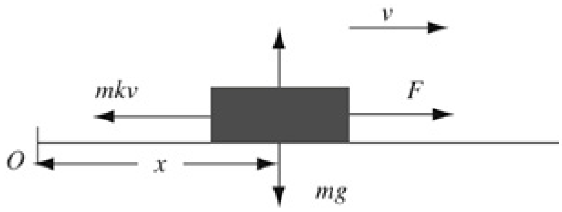
Resisted motion of a particle movig in a straight line
Exercise C, Question 2

Question:

A train of mass m is moving along a straight horizontal track with its engine working at a constant rate of $16mkU^2$, where k and U are constants. The resistance to the motion of the train has magnitude mkv, where v is the speed of the train.
Find the time the train takes to increase its speed from U to $3U$.

Solution:



Let the tractive force generated by the engine be F.

$$P = Fv$$
$$16mkU^2 = Fv$$
$$F = \frac{16mkU^2}{v}$$

R($\rightarrow$) $\quad \mathbf{F} = m\mathbf{a}$

$$\frac{16mkU^2}{v} - mkv = ma$$
$$\frac{16kU^2}{v} - kv = \frac{dv}{dt}$$
$$k(16U^2 - v^2) = v\frac{dv}{dt}$$

Separating the variables

$$\int k \, dt = \int \frac{v}{16U^2 - v^2} dv$$
$$kt = A - \frac{1}{2}\ln(16U^2 - v^2)$$

Let $t = 0$ when $v = U$

$$0 = A - \frac{1}{2}\ln(16U^2 - U^2) \Rightarrow A = \frac{1}{2}\ln(15U^2)$$

Hence

$$kt = \frac{1}{2}\ln(15U^2) - \frac{1}{2}\ln(16U^2 - v^2)$$
$$t = \frac{1}{2k}\ln\left(\frac{15U^2}{16U^2 - v^2}\right)$$

When $v = 3U$

$$t = \frac{1}{2k}\ln\left(\frac{15U^2}{16U^2 - 9U^2}\right) = \frac{1}{2k}\ln\left(\frac{15}{7}\right)$$

The time the train takes to increase its speed from U to $3U$ is $\frac{1}{2k}\ln\left(\frac{15}{7}\right)$.

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Resisted motion of a particle movig in a straight line
Exercise C, Question 3

Question:

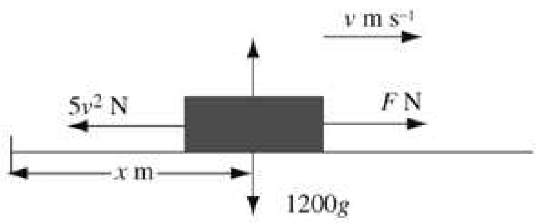
A van of mass 1200 kg is moving along a horizontal road with its engine working at a constant rate of 40 kW. The resistance to motion of the van is of magnitude of $5v^2$ N, where v m s^{-1} is the speed of the van. Find

a the terminal speed of the van,

b the distance the van travels while increasing its speed from 10 m s^{-1} to 15 m s^{-1}.

Solution:

a



40 kW = 40 000 W

Let the tractive force generated by the engine be F N.

$$P = Fv$$

$$40\,000 = Fv \Rightarrow F = \frac{40\,000}{v}$$

R(→) $\mathbf{F} = m\mathbf{a}$

$$\frac{40\,000}{v} - 5v^2 = 1200a \quad *$$

At the terminal speed $a = 0$

$$\frac{40\,000}{v} - 5v^2 = 0 \Rightarrow v^3 = 8000 \Rightarrow v = 20$$

The terminal speed of the van is 20 m s^{-1}.

b Equation * can be written

$$\frac{40\,000}{v} - 5v^2 = 1200v\frac{dv}{dx}$$

Dividing throughout by 5 and multiplying throughout by v

$$8000 - v^3 = 240v^2\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = 240\int \frac{v^2}{8000 - v^3}\,dv$$

$$x = A - \frac{240}{3}\ln(8000 - v^3)$$

Let $x = 0$ when $v = 10$

$$0 = A - 80\ln(8000 - 1000) \Rightarrow A = 80\ln 7000$$

Hence

$$x = 80\ln 7000 - 80\ln(8000 - v^3) = 80\ln\left(\frac{7000}{8000 - v^3}\right)$$

When $v = 15$

$$x = 80\ln\left(\frac{7000}{8000 - 15^3}\right) = 80\ln\left(\frac{7000}{4625}\right) \approx 33.2$$

The distance the van travels while increasing its speed from 10 m s^{-1} to 15 m s^{-1} is 33.2 m (3 s.f.)

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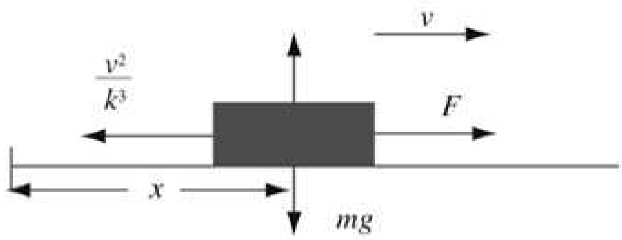
Resisted motion of a particle movig in a straight line
Exercise C, Question 4

Question:

A car of mass m is moving along a straight horizontal road with its engine working at a constant rate D^3. The resistance to the motion of the car is of magnitude $\frac{v^2}{k^3}$, where v is the speed of the car and k is a positive constant.

Find the distance travelled by the car as its speed increases from $\frac{kD}{4}$ to $\frac{kD}{2}$.

Solution:



Let the tractive force generated by the engine be F.

$$P = Fv$$

$$D^3 = Fv \Rightarrow F = \frac{D^3}{v}$$

$\mathrm{R}(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$\frac{D^3}{v} - \frac{v^2}{k^3} = mv\frac{dv}{dx}$$

Multiplying throughout by k^3v

$$k^3D^3 - v^3 = mk^3v^2\frac{dv}{dx}$$

Separating the variables

$$\int 1\, dx = mk^3\int \frac{v^2}{k^3D^3 - v^3}\, dv$$

$$x = A - \frac{mk^3}{3}\ln(k^3D^3 - v^3)$$

Let $x = 0$ when $v = \frac{kD}{4}$

$$0 = A - \frac{mk^3}{3}\ln\left(k^3D^3 - \frac{k^3D^3}{64}\right) \Rightarrow A = \frac{mk^3}{3}\ln\left(\frac{63k^3D^3}{64}\right)$$

Hence

$$x = \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln(k^3D^3 - v^3)\right)$$

When $v = \frac{kD}{2}$

$$x = \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln\left(k^3D^3 - \frac{k^3D^3}{8}\right)\right)$$

$$= \frac{mk^3}{3}\left(\ln\left(\frac{63k^3D^3}{64}\right) - \ln\left(\frac{7k^3D^3}{8}\right)\right)$$

$$= \frac{mk^3}{3}\ln\left(\frac{63k^3D^3}{64} \times \frac{8}{7k^3D^3}\right) = \frac{mk^3}{3}\ln\left(\frac{9}{8}\right)$$

The distance travelled by the car as its speed increases from $\frac{kD}{4}$ to $\frac{kD}{2}$ is

$$\frac{mk^3}{3}\ln\left(\frac{9}{8}\right)$$

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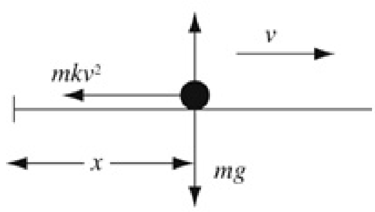
Resisted motion of a particle movig in a straight line
Exercise D, Question 1

Question:

A particle of mass m moves in a straight line on a smooth horizontal plane in a medium which exerts a resistance of magnitude mkv^2, where v is the speed of the particle and k is a positive constant. At time $t = 0$ the particle has speed U.
Find, in terms of k and U, the time at which the particle's speed is $\frac{3}{4}U$. **[E]**

Solution:



$$\text{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$
$$-mkv^2 = ma$$
$$-kv^2 = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Separating the variables

$$\int k \, \mathrm{d}t = -\int v^{-2} \, \mathrm{d}v$$
$$kt = -\frac{v^{-1}}{-1} + A = \frac{1}{v} + A$$

At $t = 0, v = U$

$$0 = \frac{1}{U} + A \Rightarrow A = -\frac{1}{U}$$

Hence

$$t = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{U}\right)$$

When $v = \frac{3}{4}U$

$$t = \frac{1}{k}\left(\frac{1}{\frac{3}{4}U} - \frac{1}{U}\right) = \frac{1}{k}\left(\frac{4}{3U} - \frac{1}{U}\right) = \frac{1}{3kU}$$

The time at which the particle's speed is $\frac{3}{4}U$ is $\frac{1}{3kU}$.

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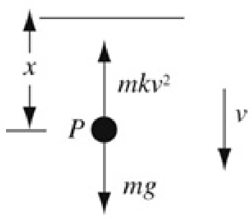
Resisted motion of a particle movig in a straight line
Exercise D, Question 2

Question:

A small pebble of mass m is placed in a viscous liquid and sinks vertically from rest through the liquid. When the speed of the particle is v the magnitude of the resistance due to the liquid is modelled as mkv^2, where k is a positive constant.
Find the speed of the pebble after it has fallen a distance D through the liquid. **[E]**

Solution:



R(↓) $\quad \mathbf{F} = m\mathbf{a}$

$$mg - mkv^2 = ma$$

$$g - kv^2 = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separating the variables

$$\int 1\,\mathrm{d}x = \int \frac{v}{g - kv^2}\,\mathrm{d}v$$

$$x = A - \frac{1}{2k}\ln(g - kv^2)$$

When $x = 0, v = 0$

$$0 = A - \frac{1}{2k}\ln g \Rightarrow A = \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g - kv^2) = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$$

$$\ln\left(\frac{g}{g - kv^2}\right) = 2kx$$

$$\frac{g}{g - kv^2} = \mathrm{e}^{2kx}$$

$$g - kv^2 = g\,\mathrm{e}^{-2kx}$$

$$v^2 = \frac{g}{k}(1 - \mathrm{e}^{-2kx})$$

When $x = D$

$$v^2 = \frac{g}{k}(1 - \mathrm{e}^{-2kD})$$

$$v = \left(\frac{g}{k}\right)^{\frac{1}{2}}(1 - \mathrm{e}^{-2kD})^{\frac{1}{2}}$$

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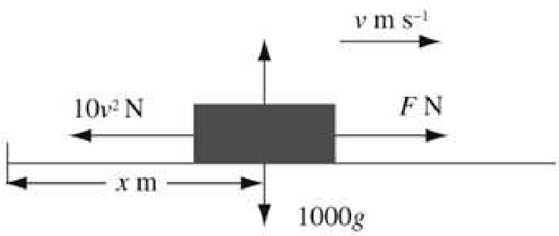
Resisted motion of a particle movig in a straight line
Exercise D, Question 3

Question:

A car of mass 1000 kg is driven by an engine which generates a constant power of 12 kW. The only resistance to the car's motion is air resistance of magnitude $10v^2$ N, where v m s^{-1} is the speed of the car.
Find the distance travelled as the car's speed increases from 5 m s^{-1} to 10 m s^{-1}. **[E]**

Solution:



$12\text{ kW} = 12\,000\text{ W}$

$P = Fv$

$F = \dfrac{12\,000}{v}$

$\text{R}(\rightarrow) \qquad \mathbf{F} = m\mathbf{a}$

$F - 10v^2 = 1000a$

$\dfrac{12\,000}{v} - 10v^2 = 1000v\dfrac{dv}{dx}$

Dividing throughout by 10 and multiplying throughout by v

$1200 - v^3 = 100v^2\dfrac{dv}{dx}$

Separating the variables

$\int 1\,dx = 100\int \dfrac{v^2}{1200 - v^3}\,dv$

$x = A - \dfrac{100}{3}\ln(1200 - v^3)$

Let $x = 0$ when $v = 5$

$0 = A - \dfrac{100}{3}\ln(1200 - 125) \Rightarrow A = \dfrac{100}{3}\ln 1075$

Hence

$x = \dfrac{100}{3}\ln 1075 - \dfrac{100}{3}\ln(1200 - v^3) = \dfrac{100}{3}\ln\left(\dfrac{1075}{1200 - v^3}\right)$

When $v = 10$

$x = \dfrac{100}{3}\ln\left(\dfrac{1075}{1200 - 10^3}\right) = \dfrac{100}{3}\ln\left(\dfrac{1075}{200}\right) \approx 56.1$

The distance travelled as the car's speed increases from 5 m s^{-1} to 10 m s^{-1} is 56.1 m (3 s.f.).

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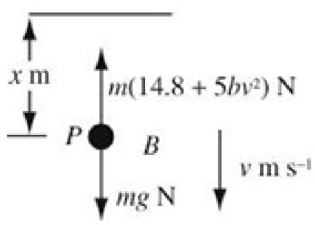
Resisted motion of a particle movig in a straight line
Exercise D, Question 4

Question:

A bullet B, of mass m kg, is fired vertically downwards into a block of wood W which is fixed in the ground. The bullet enters W with speed U m s^{-1} and W offers a resistance of magnitude $m(14.8+5bv^2)$ N, where v m s^{-1} is the speed of B and b is a positive constant. The path of B in W remains vertical until B comes to rest after travelling a distance d metres into W.
Find d in terms of b and U. **[E]**

Solution:



R(↓) $\quad \mathbf{F} = m\mathbf{a}$

$$mg - m(14.8+5bv^2) = ma$$

$$9.8 - 14.8 - 5bv^2 = v\frac{dv}{dx}$$

$$-5(1+bv^2) = v\frac{dv}{dx}$$

Separating the variables

$$\int 1\,dx = -\frac{1}{5}\int \frac{v}{1+bv^2}\,dv$$

$$x = A - \frac{1}{10b}\ln(1+bv^2)$$

At $x=0, v=U$

$$0 = A - \frac{1}{10b}\ln(1+bU^2) \Rightarrow A = \frac{1}{10b}\ln(1+bU^2)$$

$$x = \frac{1}{10b}\ln(1+bU^2) - \frac{1}{10b}\ln(1+bv^2) = \frac{1}{10b}\ln\left(\frac{1+bU^2}{1+bv^2}\right)$$

When $v=0, x=d$

$$d = \frac{1}{10b}\ln(1+bU^2)$$

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Resisted motion of a particle movig in a straight line
Exercise D, Question 5

Question:

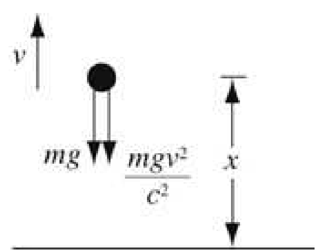
A particle of mass m is projected vertically upwards, with speed V, in a medium which exerts a resisting force of magnitude $\frac{mgv^2}{c^2}$, where v is the speed of the particle and c is a positive constant.

a Show that the greatest height attained above the point of projection is $\frac{c^2}{2g}\ln\left(1+\frac{V^2}{c^2}\right)$.

b Find an expression, in terms of V, c and g, for the time to reach this height. **[E]**

Solution:

a



R(↑) $\mathbf{F} = m\mathbf{a}$

$$-mg-\frac{mgv^2}{c^2}=ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right)=v\frac{dv}{dx}$$

$$\int g\,dx=-c^2\int\frac{v}{c^2+v^2}\,dv$$

$$gx=A-\frac{c^2}{2}\ln(c^2+v^2)$$

At $x=0, v=V$

$$0=A-\frac{c^2}{2}\ln(c^2+V^2)\Rightarrow A=\frac{c^2}{2}\ln(c^2+V^2)$$

Hence

$$gx=\frac{c^2}{2}\ln(c^2+V^2)-\frac{c^2}{2}\ln(c^2+v^2)=\frac{c^2}{2}\ln\left(\frac{c^2+V^2}{c^2+v^2}\right)$$

$$x=\frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{c^2+v^2}\right)$$

At the greatest height $v=0$

$$x=\frac{c^2}{2g}\ln\left(\frac{c^2+V^2}{c^2}\right)=\frac{c^2}{2g}\ln\left(1+\frac{V^2}{c^2}\right)\text{, as required.}$$

b $R(\uparrow)$ $\mathbf{F} = m\mathbf{a}$

$$-mg - \frac{mgv^2}{c^2} = ma$$

$$-g\left(\frac{c^2+v^2}{c^2}\right) = \frac{dv}{dt}$$

Separating the variables

$$\frac{g}{c^2}\int 1\,dt = -\int \frac{1}{c^2+v^2}\,dv$$

$$\frac{gt}{c^2} = A - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

When $t = 0, v = V$

$$0 = A - \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow A = \frac{1}{c}\arctan\left(\frac{V}{c}\right)$$

Hence

$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) - \frac{1}{c}\arctan\left(\frac{v}{c}\right)$$

At the greatest height $v = 0$

$$\frac{gt}{c^2} = \frac{1}{c}\arctan\left(\frac{V}{c}\right) \Rightarrow t = \frac{c}{g}\arctan\left(\frac{V}{c}\right)$$

The time taken to reach the greatest height is $\frac{c}{g}\arctan\left(\frac{V}{c}\right)$.

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Resisted motion of a particle movig in a straight line
Exercise D, Question 6

Question:

A particle is projected vertically upwards with speed U in a medium in which the resistance is proportional to the square of the speed. Given that U is also the speed for which the resistance offered by the medium is equal to the weight of the particle show that

a the time of ascent is $\frac{\pi U}{4g}$,

b the distance ascended is $\frac{U^2}{2g}\ln 2$. **[E]**

Solution:

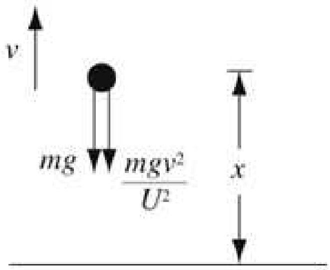
a Let the mass of the particle be m.

Let the resistance be kv^2, where k is a constant of proportionality.

If U is the speed for which the resistance is equal to the weight of the particle the

$$kU^2 = mg \Rightarrow k = \frac{mg}{U^2}$$

Hence the resistance is $\frac{mgv^2}{U^2}$.



R(↑) $\mathbf{F} = m\mathbf{a}$

$$-mg - \frac{mgv^2}{U^2} = ma$$

$$-\frac{g(U^2+v^2)}{U^2} = \frac{dv}{dt} \quad *$$

Separating the variables

$$\int g \, dt = -U^2 \int \frac{1}{U^2+v^2} \, dv$$

$$gt = A - U^2 \times \frac{1}{U} \arctan\left(\frac{v}{U}\right)$$

When $t = 0, v = U$

$$0 = A - U \arctan 1 \Rightarrow A = U \arctan 1 = \frac{\pi U}{4}$$

Hence

$$gt = \frac{\pi U}{4} - U \arctan\left(\frac{v}{U}\right)$$

$$t = \frac{\pi U}{4g} - \frac{U}{g} \arctan\left(\frac{v}{U}\right)$$

Let the time of ascent be T.

When $t = T, v = 0$

$$T = \frac{\pi U}{4g} - \frac{U}{g} \arctan 0$$

$$= \frac{\pi U}{4g}, \text{ as required}$$

b Equation * in part **a** can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{dv}{dx}$$

Separating the variables

Equation * in part **a** can be written as

$$-\frac{g(U^2+v^2)}{U^2}=v\frac{dv}{dx}$$

Separating the variables

$$\int g\ dx=-U^2\int\frac{v}{U^2+v^2}\ dv$$

$$gx=B-\frac{U^2}{2}\ln(U^2+v^2)$$

When $x=0, v=U$

$$0=B-\frac{U^2}{2}\ln(2U^2)\Rightarrow B=\frac{U^2}{2}\ln(2U^2)$$

Hence

$$gx=\frac{U^2}{2}\ln(2U^2)-\frac{U^2}{2}\ln(U^2+v^2)$$

$$x=\frac{U^2}{2g}\ln\left(\frac{2U^2}{U^2+v^2}\right)$$

Let the total distance ascended be H.

When $h=H, v=0$

$$H=\frac{U^2}{2g}\ln\left(\frac{2U^2}{U^2}\right)=\frac{U^2}{2g}\ln 2\text{, as required}$$

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Edexcel AS and A Level Modular Mathematics

Resisted motion of a particle movig in a straight line
Exercise D, Question 7

Question:

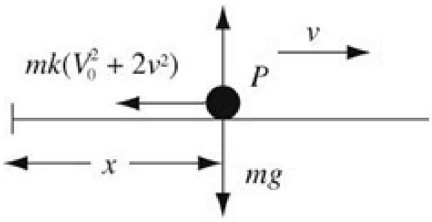
At time t, a particle P, of mass m, moving in a straight line has speed v. The only force acting is a resistance of magnitude $mk(V_0^2 + 2v^2)$, where k is a positive constant and V_0 is the speed of P when $t = 0$.

a Show that, as v reduces from V_0 to $\frac{1}{2}V_0$, P travels a distance $\frac{\ln 2}{4k}$.

b Express the time P takes to cover this distance in the form $\frac{\lambda}{kV_0}$, giving the value of λ to two decimal places. **[E]**

Solution:

a



$R(\rightarrow)$ $\qquad \mathbf{F} = m\mathbf{a}$

$$-mk(V_0^2 + 2v^2) = ma$$

$$-k(V_0^2 + 2v^2) = v\frac{dv}{dx}$$

Separating the variables

$$\int k \, dx = -\int \frac{v}{V_0^2 + 2v^2} dv$$

$$kx = A - \frac{1}{4}\ln(V_0^2 + 2v^2)$$

At $x = 0, v = V_0$

$$0 = A - \frac{1}{4}\ln(V_0^2 + 2V_0^2) \Rightarrow A = \frac{1}{4}\ln(3V_0^2)$$

Hence

$$kx = \frac{1}{4}\ln(3V_0^2) - \frac{1}{4}\ln(V_0^2 + 2v^2)$$

$$x = \frac{1}{4k}\ln\left(\frac{3V_0^2}{V_0^2 + 2v^2}\right)$$

When $v = \frac{1}{2}V_0$

$$x = \frac{1}{4k}\ln\left(\frac{3V_0^2}{V_0^2 + \frac{1}{2}V_0^2}\right) = \frac{1}{4k}\ln\left(\frac{3V_0^2}{\frac{3}{2}V_0^2}\right)$$

$$= \frac{\ln 2}{4k}, \text{ as required}$$

b Equation * can be written as

$$-k(V_0^2 + 2v^2) = \frac{dv}{dt}$$

Separating the variables

$$\int k \, dt = -\int \frac{1}{V_0^2 + 2v^2} \, dv = -\frac{1}{2}\int \frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)^2 + v^2} \, dv$$

$$kt = B - \frac{1}{2} \times \frac{1}{\left(\frac{V_0}{\sqrt{2}}\right)} \arctan \frac{v}{\left(\frac{V_0}{\sqrt{2}}\right)}$$

When $t = 0, v = V_0$

$$0 = B - \frac{\sqrt{2}}{2V_0} \arctan\left(\frac{\sqrt{2}V_0}{V_0}\right) \Rightarrow B = \frac{\sqrt{2}}{2V_0} \arctan \sqrt{2}$$

Hence

$$t = \frac{\sqrt{2}}{2kV_0}\left(\arctan \sqrt{2} - \arctan\left(\frac{\sqrt{2}v}{V_0}\right)\right)$$

$$v = \frac{1}{2}V_0$$

$$t = \frac{\sqrt{2}}{2kV_0}\left(\arctan \sqrt{2} - \arctan\left(\frac{\sqrt{2} \times \frac{1}{2}V_0}{V_0}\right)\right)$$

$$= \frac{1}{kV_0}\left[\frac{\sqrt{2}}{2}\left(\arctan \sqrt{2} - \arctan\left(\frac{\sqrt{2}}{2}\right)\right)\right]$$

This has the form $\frac{\lambda}{kV_0}$, as required, where

$$\lambda = \frac{\sqrt{2}}{2}\left(\arctan \sqrt{2} - \arctan\left(\frac{\sqrt{2}}{2}\right)\right) \approx 0.24 \text{ (2 d.p.)}$$

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Resisted motion of a particle movig in a straight line
Exercise D, Question 8

Question:

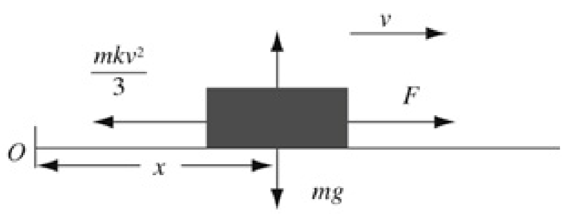
A car of mass m is moving along a straight horizontal road. When displacement of the car from a fixed point O is x, its speed is v. The resistance to the motion of the car has magnitude $\frac{mkv^2}{3}$, where k is a positive constant. The engine of the car is working at a constant rate P.

a Show that $3mv^2 \frac{dv}{dx} = 3P - mkv^3$.

When $t = 0$, the speed of the car is half of its limiting speed.

b Find x in terms of m, k, P and v.

Solution:



a $P = Fv \Rightarrow F = \frac{P}{v}$

$R(\rightarrow)$ $\quad \mathbf{F} = m\mathbf{a}$

$$F - \frac{mkv^2}{3} = ma$$

$$\frac{P}{v} - \frac{mkv^2}{3} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

Multiplying throughout by $3v$

$$3P - mkv^3 = 3mv^2\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$3mv^2\frac{\mathrm{d}v}{\mathrm{d}x} = 3P - mkv^3\text{, as required}$$

b The limiting speed is given by $a = v\frac{\mathrm{d}v}{\mathrm{d}x} = 0$

$$0 = 3P - mkv^3 \Rightarrow v^3 = \frac{3P}{mk} \Rightarrow v = \left(\frac{3P}{mk}\right)^{\frac{1}{3}}$$

Separating the variables in the answer to part **a**

$$\int 1\,\mathrm{d}x = \int \frac{3mv^2}{3P - mkv^3}\,\mathrm{d}v$$

$$x = A - \frac{1}{k}\ln(3P - mkv^3)$$

When $x = 0, v = \frac{1}{2}\left(\frac{3P}{mk}\right)^{\frac{1}{3}} \Rightarrow v^3 = \frac{3P}{8mk}$

$$0 = A - \frac{1}{k}\ln\left(3P - \frac{3P}{8}\right) \Rightarrow A = \frac{1}{k}\ln\left(\frac{21P}{8}\right)$$

Hence

$$x = \frac{1}{k}\ln\left(\frac{21P}{8}\right) - \frac{1}{k}\ln(3P - mkv^3)$$

$$= \frac{1}{k}\ln\left(\frac{21P}{8(3P - mkv^3)}\right)$$