Edexcel Mechanics M4 – January 2005   **Answers**

1. (a) Impulse on *A* is in the direction of the line of centres.

   Impulse on $A = \Delta(mv) = m(-2\mathbf{i} + 5\mathbf{j}) - m(\mathbf{i} + 2\mathbf{j}) = m(-3\mathbf{i} + 3\mathbf{j})$.

   Therefore direction of line of centres is $(-\mathbf{i} + \mathbf{j})$. A unit vector in this direction is $\dfrac{(-\mathbf{i} + \mathbf{j})}{\sqrt{2}}$.

   (b) Let velocity of *b* after collision be $v_1\mathbf{i} + v_2\mathbf{j}$

   Momentum conserved: $m(\mathbf{i} + 2\mathbf{j}) + 5m(-\mathbf{i} + 3\mathbf{j}) = m(-2\mathbf{i} + 5\mathbf{j}) + 5m(v_1\mathbf{i} + v_2\mathbf{j})$

   Cancel *m* and equate coefficients: $\mathbf{i}$: $-4 = -2 + 5v_1$ $v_1 = -\tfrac{2}{5}$

   $\mathbf{j}$: $17 = 5 + 5v_2$ $v_2 = \tfrac{12}{5}$

   Velocity of *B* after collision $= -\tfrac{2}{5}\mathbf{i} + \tfrac{12}{5}\mathbf{j}$.

2. (a) Velocity of wind relative to man $= \mathbf{V}_{WM} = \mathbf{V}_W - \mathbf{V}_M$. $\therefore v(3\mathbf{i} - 4\mathbf{j}) = \mathbf{V}_W - u\mathbf{j}$

   Similarly $w\mathbf{i} = \mathbf{V}_W - \tfrac{1}{5}u(-3\mathbf{i} + 4\mathbf{j})$.

   Equate the two expressions for $\mathbf{V}_M$ that these produce: $v(3\mathbf{i} - 4\mathbf{j}) + u\mathbf{j} = w\mathbf{i} + \tfrac{1}{5}u(-3\mathbf{i} + 4\mathbf{j})$

   Equate coefficients: $\mathbf{i}$: $-3v = w - \tfrac{3}{5}u$

   $\mathbf{j}$: $-4v + u = \tfrac{4}{5}u$ $\therefore v = \tfrac{1}{20}u$

   (b) $\mathbf{V}_W = \tfrac{1}{20}u(3\mathbf{i} - 4\mathbf{j}) + u\mathbf{j} = \tfrac{1}{20}u(3\mathbf{i} + 16\mathbf{j})$

3. Treat *B* when $t = 0$ as the origin.

   $\mathbf{r}_A = 12t\mathbf{i} + 4\mathbf{j}$. $\quad \mathbf{r}_A = 16t\left(\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}\right)$

   $\mathbf{BA} = \mathbf{r}_A - \mathbf{r}_B = \mathbf{i}(12t - 8t\sqrt{3}) + \mathbf{j}(4 - 8t)$

   Length of $\mathbf{AB} = \sqrt{\left((12t - 8t\sqrt{3})^2 + (4 - 8t)^2\right)} = \sqrt{\left((144 - 192\sqrt{3} + 192)t^2 + 16 - 64t + 64t^2\right)}$

   Minimum when derivative of terms inside square root $= 0$:

   $2t(144 - 192\sqrt{3} + 192) - 64 + 128t = 0$, $t \approx 0.47$. (Minimum because this is a +ve quadratic.)

   Substitute back into length of $\mathbf{AB}$: $|\mathbf{AB}| \approx 0.90$ km.

4. Apply $F = ma$: $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{RU}{v} - R$

Separate the variables: $\displaystyle\int_{\frac{u}{4}}^{\frac{u}{2}} \dfrac{mv\mathrm{d}v}{R(U-v)} = \int_{0}^{T} \mathrm{d}t$.

$$\dfrac{m}{R}\int_{\frac{u}{4}}^{\frac{u}{2}}\left(-1+\dfrac{U}{(U-v)}\right)\mathrm{d}v = \left[t\right]_{0}^{T} = T$$

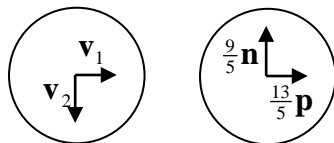$$\dfrac{m}{R}\left[-v - U\ln|U-v|\right]_{\frac{u}{4}}^{\frac{u}{2}} = T$$

$$T = \dfrac{m}{R}\left(\left(-\tfrac{1}{2}U - U\ln\left(\tfrac{1}{2}U\right)\right) - \left(-\tfrac{1}{4}U - U\ln\left(\tfrac{3}{4}U\right)\right)\right)$$

$$T = \dfrac{mU}{R}\left(-\tfrac{1}{4} + \ln\left(\tfrac{3}{2}\right)\right)$$

5. (Note: error in question: $\mathbf{p} = +\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}$.)

(a) $\tfrac{9}{5}\mathbf{n} + \tfrac{13}{5}\mathbf{p} = \tfrac{9}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) + \tfrac{13}{5}\left(\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}\right) = \tfrac{25}{25}\mathbf{i} + \tfrac{75}{25}\mathbf{j} = \mathbf{i} + 3\mathbf{j}$

(b) Before   After

No impulse parallel to the wall so velocity parallel to wall unchanged: $\mathbf{v}_1 = \tfrac{13}{5}\mathbf{p}$

Newton's law of Restitution perpendicular to the wall: $e\mathbf{v}_2 = -\tfrac{9}{5}\mathbf{n}$

Put in values: $\tfrac{9}{16}\mathbf{v}_2 = -\tfrac{9}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right)$, $\mathbf{v}_2 = -\tfrac{16}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) = \tfrac{39}{25}\mathbf{i} - \tfrac{64}{25}\mathbf{j}$

$\mathbf{v}_1 + \mathbf{v}_2 = \tfrac{13}{5}\left(\tfrac{4}{5}\mathbf{i} + \tfrac{3}{5}\mathbf{j}\right) - \tfrac{16}{5}\left(-\tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{j}\right) = 4\mathbf{i} - \mathbf{i}$

(c) Change in KE $= \tfrac{1}{2}\times\tfrac{1}{2}\times\left(4^2 + 1^2\right) - \tfrac{1}{2}\times\tfrac{1}{2}\times\left(3^2 + 1^2\right) = 1.75$ J

6.   (a)   Take O as zero p.e.

Mechanical potential energy $(mgh) = -mga\cos 2\theta$

Elastic potential energy $\left(\dfrac{\lambda x^2}{2l}\right) = \frac{1}{2} \times \dfrac{4mg}{\frac{5}{4}a}\left(2a\cos 2\theta - \frac{5}{4}a\right)^2$

Total p.e. $= -mga\left(2\cos^2\theta - 1\right) + \dfrac{8mg}{5a}\left(\dfrac{8a\cos\theta - 5a}{4}\right)^2$

$= -2mga\cos^2\theta + mga + \dfrac{mga}{10}\left(8\cos\theta - 5\right)^2$

$= \dfrac{mga}{10}\left(8\cos\theta - 5\right)^2 - 2mga\cos^2\theta + c$ (change of constant with referral

of p.e. to any other zero position.)

(b)   Equilibrium when p.e. is max/min so $\dfrac{dE}{d\theta} = 0$

$\dfrac{dE}{d\theta} = \dfrac{mga}{10} \times 16 \times (8\cos\theta - 5)(-\sin\theta) + 4mga\cos\theta\sin\theta = 0$

$mga\sin\theta\left(-\frac{64}{5}\cos\theta + 8 + 4\cos\theta\right) = 0$

$mga\sin\theta\left(8 - \frac{44}{5}\cos\theta\right) = 0$

$\sin\theta = 0 ,\;\; \theta = 0$

or
$\cos\theta = \frac{10}{11} ,\;\; \theta = 24.6°$
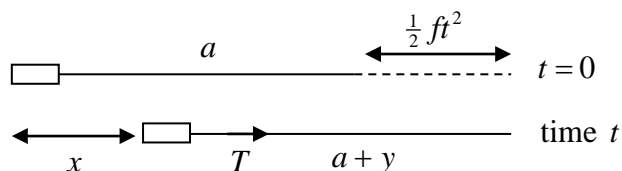
(c) $\dfrac{d^2E}{d\theta^2} = mga\sin\theta\left(\frac{44}{5}\sin\theta\right) + mga\cos\theta\left(8 - \frac{44}{5}\cos\theta\right)$

When $\theta = 0 , \dfrac{d^2E}{d\theta^2} = mga\left(8 - \frac{44}{5}\right) = -\frac{4}{5}mga$ which is < 0 so max E so unstable.

When $\theta = 24.6° , \dfrac{d^2E}{d\theta^2} = mga\left(\frac{44}{5}\left(1 - \left(\frac{10}{11}\right)^2\right) + \frac{10}{11}\left(8 - \frac{44}{5} \times \frac{10}{11}\right)\right) = \frac{84}{55}mga$

which is > 0 so min E so stable.

7. (a)

At time t:
- the particle has moved x,
- the string is length $(a + y)$,
- the end of the string has moved $\frac{1}{2} ft^2$. $\therefore a + \frac{1}{2} ft^2 = x + a + y$, $\therefore x + y = \frac{1}{2} ft^2$.

(b) $F = ma$ to particle: $T = m\ddot{x}$

$$\frac{man^2}{a} y = m\ddot{x}$$

$$n^2\left(\tfrac{1}{2} ft^2 - x\right) = \ddot{x}$$

$$\ddot{x} + n^2 x = \tfrac{1}{2} n^2 ft^2$$

(c) $x = 0, t = 0 \therefore 0 = A - \dfrac{f}{n^2}$, $A = \dfrac{f}{n^2}$

Differentiating the solution: $\dot{x} = -nA\sin nt + nB\cos nt + ft$ (or get $\ddot{x}$ from original d.e.)
$\dot{x} = 0, t = 0 \therefore 0 = nB$, $B = 0$

(d) Differentiating again: $\ddot{x} = -n^2 A\cos nt - n^2 B\sin nt + f = -f\cos nt + f$

$T = m\ddot{x} = mf(1 - \cos nt)$.

This takes max value of $2mf$ (when $\cos nt = -1$).