Chapter review 5

1 a $V = \int \pi y^2 \mathrm{d}x = \pi \int_0^4 4x \,\mathrm{d}x$

$= \pi \left[2x^2 \right]_0^4$

$= 32\pi \text{ cm}^3$

b $M\bar{x} = \rho \int x\pi y^2 \,\mathrm{d}x = \rho\pi \int_0^4 4x^2 \mathrm{d}x$

$= \rho\pi \left[\frac{4}{3}x^3 \right]_0^4$

$= \frac{256}{3}\rho\pi$

$\therefore 32\pi\rho\bar{x} = \frac{256}{3}\pi\rho$

$\therefore \bar{x} = \frac{8}{3} \text{cm}$

2 a $V = \int \pi y^2 \mathrm{d}x = \pi \int_1^2 \frac{1}{x^2} \mathrm{d}x$

$= \pi \left[\frac{-1}{x} \right]_1^2$

$= \pi \left[\frac{-1}{2} + 1 \right]$

$\text{Volume} = \frac{\pi}{2} \text{m}^3$

b $M\bar{x} = \rho \int x\pi y^2 \mathrm{d}x = \rho\pi \int_1^2 x \times \frac{1}{x^2} \mathrm{d}x$

$= \rho\pi \int_1^2 \frac{1}{x} \mathrm{d}x$

$= \rho\pi \left[\ln x \right]_1^2$

$= \rho\pi \ln 2$

$\therefore \frac{\pi}{2}\rho\bar{x} = \rho\pi \ln 2$

$\therefore \bar{x} = 2\ln 2$

So the distance of the centre of mass from the plane face $x = 1$ is $2\ln 2 - 1 = 0.386 \text{ m}$ (3 s.f.) i.e. 39 cm to the nearest cm.

3 Let the density of the solids be ρ. Let O be the centre of the circular base of the solid.

Shape	Mass	Ratio of masses	Distance of centre of mass from O
Cylinder	$\pi \times 40^2 \times 40\rho$	1	20 cm
Hemisphere	$\frac{2}{3}\pi\rho \times 40^3$	$\frac{2}{3}$	$\left(40 + \frac{3}{8} \times 40\right)$ cm
Solid	$\pi\rho \times 40^3\left(1 + \frac{2}{3}\right)$	$\frac{5}{3}$	$\bar{x}$

$$\circlearrowleft M(O)\ \frac{5}{3}\bar{x} = 1 \times 20 + \frac{2}{3} \times \left(40 + \frac{3}{8} \times 40\right)$$

$$= 20 + \frac{110}{3}$$

$$\therefore \bar{x} = \frac{170}{5}$$

$$= 34$$

$\therefore$ The centre of mass of the solid is at a height of 34 cm above the ground.

4 Let the mass per unit volume be ρ.

Shape	Mass	Mass ratios	Distance of centre of mass from plane face
Cylinder	$\pi\rho r^2 \times r$	1	$\frac{r}{2}$
Cone	$\frac{1}{3}\pi\rho r^2 \times 2r$	$\frac{2}{3}$	$r + \frac{2r}{4}$
Model	$\pi\rho r^2 \times 1\frac{2}{3}r$	$\frac{5}{3}$	$\bar{x}$

Note that the cylindrical base of this rocket has height r.

$$\circlearrowleft \text{M (plane face)}: 1\frac{2}{3}\bar{x} = 1 \times \frac{r}{2} + \frac{2}{3} \times \left(r + \frac{2r}{4}\right)$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{r}{2} + \frac{2r}{3} + \frac{r}{3}$$

$$\text{i.e. } \frac{5}{3}\bar{x} = \frac{3r}{2}$$

$$\therefore \bar{x} = \frac{9r}{10}$$

$\therefore$ The centre of mass is at a distance $\frac{9r}{10}$ from the plane face.

5 $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\frac{d}{dx}(x\sin 2x) = 2x\cos 2x + \sin 2x$$

$$x\sin 2x = 2\int x\cos 2x\,dx + \int \sin 2x\,dx$$

$$= 2\int x\cos 2x\,dx - \frac{1}{2}\cos 2x$$

$$2\int x\cos 2x\,dx = \frac{1}{2}\cos 2x + x\sin 2x$$

$$\int x\cos 2x\,dx = \frac{1}{4}\cos 2x + \frac{1}{2}x\sin 2x$$

$$\bar{x} = \frac{\int y^2 x\,dx}{\int y^2 dx} = \frac{\int_0^{\frac{\pi}{6}} x\sin^2 x\,dx}{\int_0^{\frac{\pi}{6}} \sin^2 x\,dx}$$

$$\int_0^{\frac{\pi}{6}} y^2 x\,dx = \int_0^{\frac{\pi}{6}} x\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} x(1 - \cos 2x)\,dx$$

$$\int_0^{\frac{\pi}{6}} x\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} x - \frac{1}{2}\int_0^{\frac{\pi}{6}} x\cos 2x\,dx$$

$$= \frac{1}{4}\left[x^2\right]_0^{\frac{\pi}{6}} - \frac{1}{2}\left[\frac{1}{4}\cos 2x + \frac{1}{2}x\sin 2x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[x^2\right]_0^{\frac{\pi}{6}} - \frac{1}{8}\left[\cos 2x\right]_0^{\frac{\pi}{6}} - \frac{1}{4}\left[x\sin 2x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\times\frac{\pi^2}{36} - \frac{1}{8}\left[\frac{1}{2} - 1\right] - \frac{1}{4}\left[\frac{\pi}{6}\times\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi^2}{144} + \frac{1}{16} - \frac{\pi\sqrt{3}}{48}$$

$$\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 2x)\,dx$$

$$= \frac{1}{2}\left[x\right]_0^{\frac{\pi}{6}} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{12} - \frac{1}{4}\times\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

$$\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 dx}$$

$$= \frac{\frac{\pi^2}{144} + \frac{1}{16} - \frac{\pi\sqrt{3}}{48}}{\frac{\pi}{12} - \frac{\sqrt{3}}{8}}$$

$$= 0.390 \text{ (3 s.f.)}$$

6 $$\frac{d}{dx}(x^n \ln x) = \frac{x^n}{x} + nx^{n-1}$$

$$= x^{n-1} + nx^{n-1} \ln x$$

$$x^n \ln x = \int x^{n-1} dx + n\int x^{n-1} \ln x \, dx$$

$$= \frac{x^n}{n} + n\int x^{n-1} \ln x \, dx$$

$$n\int x^{n-1} \ln x \, dx = x^n \ln x - \frac{x^n}{n}$$

$$\int x^{n-1} \ln x \, dx = \frac{x^n \ln x}{n} - \frac{x^n}{n^2}$$

For $n = 1$:

$$\int \ln x \, dx = x \ln x - x = x(\ln x - 1)$$

For $n = 2$:

$$\int \ln x \, dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 dx}$$

$$= \frac{\int_1^2 x \ln x \, dx}{\int_1^2 \ln x \, dx}$$

$$= \frac{\frac{2^2 \ln 2}{2} - \frac{2^2}{4} - \left(\frac{1 \times \ln 1}{2} - \frac{1}{4}\right)}{2(\ln 2 - 1) - 1(\ln(1) - 1)}$$

$$= \frac{2\ln 2 - 1 + \frac{1}{4}}{2\ln 2 - 2 + 1}$$

$$= \frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1}$$

So the coordinate is $\left(\frac{2\ln 2 - \frac{3}{4}}{2\ln 2 - 1}, 0\right)$ or (1.647, 0) (3 s.f.)

Challenge

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin 2x)=\sin 2x+2x\cos 2x$$

$$x\sin 2x=\int\sin 2x\,\mathrm{d}x+2\int x\cos 2x\,\mathrm{d}x$$

$$=-\frac{1}{2}\cos 2x+2\int x\cos 2x\,\mathrm{d}x$$

$$2\int x\cos 2x\,\mathrm{d}x=x\sin 2x+\frac{1}{2}\cos 2x$$

$$\int x\cos 2x\,\mathrm{d}x=\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x$$

$$\bar{x}=\frac{\int y^2x\,\mathrm{d}x}{\int y^2\mathrm{d}x}$$

$$=\frac{\int_0^{\frac{\pi}{2}}x\cos^2 x\,\mathrm{d}x}{\int_0^{\frac{\pi}{2}}\cos^2 x\,\mathrm{d}x}$$

$$\int_0^{\frac{\pi}{2}}x\cos^2 x\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}x(\cos 2x+1)\,\mathrm{d}x$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,\mathrm{d}x+\frac{1}{2}\int_0^{\frac{\pi}{2}}x\cos 2x\,\mathrm{d}x$$

$$=\frac{1}{4}\left[x^2\right]_0^{\frac{\pi}{2}}+\frac{1}{4}\left[x\sin 2x\right]_0^{\frac{\pi}{2}}+\frac{1}{8}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\times\frac{\pi^2}{4}+\frac{1}{4}[0-0]+\frac{1}{8}[-1-1]$$

$$=\frac{\pi^2}{16}-\frac{1}{4}$$

$$\int_0^{\frac{\pi}{2}}\cos^2 x\,\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\cos 2x+1)\,\mathrm{d}x$$

$$=\frac{1}{4}\left[\sin 2x\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\left[x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}[0]+\frac{\pi}{4}$$

$$=\frac{\pi}{4}$$

$$\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 dx}$$

$$= \frac{\frac{\pi^2}{16} - \frac{1}{4}}{\frac{\pi}{4}}$$

$$= \frac{\pi^2 - 4}{4\pi}$$