Exercise 3B

1 $F = \dfrac{k}{d^2}$ where $d =$ distance from centre

distance $(x - R)$ above surface

$\Rightarrow$ distance x from centre

$\therefore F = \dfrac{k}{x^2}$

On surface $F = mg, x = R$

The magnitude of the gravitational force on a particle on the surface of the earth is the magnitude of the weight of the particle.

$\therefore mg = \dfrac{k}{R^2}$

$k = mgR^2$

$\therefore$ Magnitude of the gravitational force is $\dfrac{mgR^2}{x^2}$.

2 For a particle of mass m, distance x from the centre of the earth:

$F = ma$

$\dfrac{k}{x^2} = mA$

Use the inverse square law.

On the surface of the earth, $x = R, A = g$

$\therefore mg = \dfrac{k}{R^2}$

$k = mgR^2$

$\therefore mA = \dfrac{mgR^2}{x^2}$

$A = \dfrac{gR^2}{x^2}$

3 $F = ma$

$$\frac{mgR^2}{x^2} = -m\ddot{x}$$

S is moving away from the earth, so the acceleration is in the direction of decreasing *x*.

where x is the distance of S from the centre of the Earth.

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -g\frac{R^2}{x^2}$$

Use $\ddot{x} = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ as the acceleration is a function of x.

$$\int v\,\mathrm{d}v = -gR^2\int\frac{1}{x^2}\,\mathrm{d}x$$

$$\frac{1}{2}v^2 = g\frac{R^2}{x} + C$$

$$x = 2R:\quad v = \sqrt{gR}$$

$$\frac{1}{2}gR = \frac{gR^2}{2R} + C$$

$$C = 0$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x}$$

$$x = R: \frac{1}{2}v^2 = \frac{gr^2}{R}$$

$$v^2 = 2gR$$

$$v = \sqrt{2gR}$$

S was fired with speed $\sqrt{2gR}$.

4 $F = ma$

$$\frac{mgR^2}{x^2} = -m\ddot{x}$$

The acceleration is in the direction of decreasing x.

where x is the distance of the rocket from the centre of the Earth.

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v\,dv = -gR^2\int\frac{1}{x^2}dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$$x = R, \quad v = U$$

On the Earth's surface.

$$\frac{1}{2}U^2 = g\frac{R^2}{R} + C$$

$$C = \frac{1}{2}U^2 - gR$$

$$x = (X + R)$$

After travelling a distance X, the rocket is a distance $(X + R)$ from the centre of the Earth.

$$\frac{1}{2}v^2 = \frac{gR^2}{(X+R)} + \frac{1}{2}U^2 - gR$$

$$v^2 = \frac{2gR^2 + U^2(X+R) - 2gR(X+R)}{(X+R)}$$

$$v = \sqrt{\left[\frac{U^2X + U^2R - 2gRX}{(X+R)}\right]}$$

When it has travelled X meters, the speed of the rocket is $\sqrt{\left[\frac{U^2X + U^2R - 2gRX}{(X+R)}\right]}$

5 $\ddot{x} = -\frac{gR^2}{x^2}$

The acceleration is in the direction of decreasing x.

$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$\int v\,dv = -gR^2\int \frac{1}{x^2}\,dx$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$

$x = R \quad v^2 = 3gR$

$\therefore \frac{1}{2}\times 3gR = \frac{gR^2}{x} + C$

$C = \frac{1}{2}gR$

$\therefore v^2 = \frac{2gR^2}{x} + gR$

When $x = 5R$

At a height $4R$ above the Earth's surface, $x = 5R$.

$v^2 = \frac{2gR^2}{5R} + gR$

$v^2 = \frac{7gR}{5}$

$\therefore$ The speed at a height $4R$ above the Earth's surface is $\sqrt{\frac{7gR}{5}}$.

6 $$m\ddot{x} = \frac{-mg R^2}{x^2}$$

The acceleration is in the direction of decreasing x.

$$v\frac{dv}{dx} = \frac{-gR^2}{x^2}$$

Use $\ddot{x} = v\frac{dv}{dx}$ as the acceleration is a function of x.

$$\int v\,dv = -g R^2 \int \frac{1}{x^2}dx$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + C$$

$x = 3R, \quad v = 0$

$$C = \frac{-gR^2}{3R} = \frac{-gR}{3}$$

$$\therefore \frac{1}{2}v^2 = \frac{gR^2}{x} = \frac{gR}{3}$$

$$x = R \quad \frac{1}{2}v^2 = \frac{gR^2}{R} - \frac{gR}{3}$$

$x = R$ on the surface of the Earth.

$$\frac{1}{2}v^2 = \frac{2}{3}gR$$

$$v^2 = \frac{4}{3}gR$$

$$v = 2\sqrt{\frac{gR}{3}}$$

The particle hits the surface of the Earth's with speed $2\sqrt{\frac{gR}{3}}$.

7 a $F \propto \frac{1}{x^2}$

$$F = \frac{k}{x^2}$$

When $x = R, \; F = mg$

So $mg = \frac{k}{R^2}$

$$k = mgR^2$$

$$F = \frac{mgR^2}{x^2}$$

7 b Applying '$F = ma$'

$$mv\frac{dv}{dx} = -\frac{mgR^2}{x^2}$$

$$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$

Separating the variables and integrating:

$$\int_{\sqrt{2gR}}^{V} v\,dv = -\int_{4R}^{R} \frac{gR^2}{x^2}dx$$

$$\left[\frac{v^2}{2}\right]_{\sqrt{2gR}}^{V} = \left[\frac{gR^2}{x}\right]_{4R}^{R}$$

$$\frac{V^2}{2} - gR = \frac{gR^2}{R} - \frac{gR^2}{4R}$$

$$\frac{V^2}{2} - gR = gR - \frac{gR}{4}$$

$$\frac{V^2}{2} - gR = \frac{3gR}{4}$$

$$\frac{V^2}{2} = \frac{7gR}{4}$$

$$V^2 = \frac{7gR}{2}$$

$$V = \sqrt{\frac{7gR}{2}}$$

Challenge

a Consider a mass m resting on the earth's surface.
Suppose the earth has mass M_E.
Then by Newton's law of gravitation:

$$mg = \frac{GmM_E}{r^2}$$

Rearranging gives:

$$M_E = \frac{gr^2}{G}$$

$$M_E = \frac{9.81 \times \left(6.3781 \times 10^6\right)^2}{6.67 \times 10^{-11}}$$

$$M_E = 5.98 \times 10^{24} \text{ kg}$$

b $$\text{density} = \frac{\text{mass}}{\text{volume}}$$

$$= \frac{M_E}{\frac{4}{3}\pi r^3}$$

$$= \frac{5.983 \times 10^{24}}{\frac{4}{3}\pi \times \left(6.3781 \times 10^6\right)^3}$$

$$= 5500 \text{ kg m}^{-3} \text{ (3 s.f.)}$$