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1.

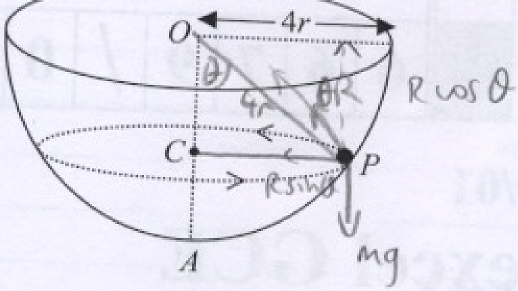

Figure 1

A hemispherical bowl of internal radius $4r$ is fixed with its circular rim horizontal. The centre of the circular rim is O and the point A on the surface of the bowl is vertically below O. A particle P moves in a horizontal circle, with centre C, on the smooth inner surface of the bowl. The particle moves with constant angular speed $\sqrt{\frac{3g}{8r}}$.

The point C lies on OA, as shown in Figure 1.

Find, in terms of r, the distance OC.

(9)

$(\updownarrow)\quad R\cos\theta = mg$

$(\rightarrow)\quad R\sin\theta = m \cdot CP \cdot \frac{3g}{8r}$

$\tan\theta = \frac{m\,CP\,3g}{8r \cdot mg}$

$\tan\theta = \frac{CP}{OC}$ so $\frac{CP}{OC} = \frac{CP \cdot 3}{8r}$

$\therefore$

$OC = \frac{8r}{3}$

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2. A particle P of mass m is fired vertically upwards from a point on the surface of the Earth and initially moves in a straight line directly away from the centre of the Earth. When P is at a distance x from the centre of the Earth, the gravitational force exerted by the Earth on P is directed towards the centre of the Earth and has magnitude $\frac{k}{x^2}$, where k is a constant.

At the surface of the Earth the acceleration due to gravity is g. The Earth is modelled as a fixed sphere of radius R.

(a) Show that $k = mgR^2$.

(2)

When P is at a height $\frac{R}{4}$ above the surface of the Earth, the speed of P is $\sqrt{\frac{gR}{2}}$.

Given that air resistance can be ignored,

(b) find, in terms of R, the greatest distance from the centre of the Earth reached by P.

(7)

a) At Surface of earth $F = mg$

Also, $x = R$ So $F = \frac{k}{R^2}$

$\therefore \frac{k}{R^2} = mg \Rightarrow k = mgR^2$

b) $F = ma = -\frac{k}{x^2}$

$\therefore \cancel{m}a = -\frac{\cancel{m}gR^2}{x^2}$

$v\frac{dv}{dx} = \frac{-gR^2}{x^2}$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + c$ (integrating w.r.t. x)

To find c, use $x = \frac{5R}{4}$, $v^2 = \frac{gR}{2}$

$\therefore \frac{gR}{4} = \frac{4gR}{5} + c \Rightarrow c = -\frac{11gR}{20}$

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Question 2 continued

So $\frac{1}{2}v^2 = \frac{gR^2}{x} - \frac{11gR}{20}$

Greatest distance $\Rightarrow v = 0$

$\therefore \frac{gR^2}{x} = \frac{11gR}{20}$ $\therefore 20\cancel{g}R^{\cancel{2}} = 11\cancel{g}\cancel{R}.x$

$\therefore x = \frac{20R}{11}$ is max distance fm centre of earth.

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3.

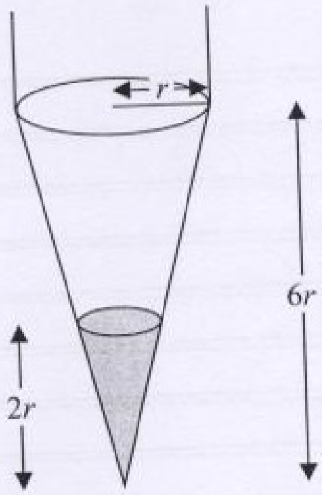

Figure 2

Figure 2 shows a container in the shape of a uniform right circular conical shell of height $6r$. The radius of the open circular face is r. The container is suspended by two vertical strings attached to two points at opposite ends of a diameter of the open circular face. It hangs with the open circular face uppermost and axis vertical. Molten wax is poured into the container. The wax solidifies and adheres to the container, forming a uniform solid right circular cone. The depth of the wax in the container is $2r$. The container together with the wax forms a solid S.

The mass of the container when empty is m and the mass of the wax in the container is $3m$.

(a) Find the distance of the centre of mass of the solid S from the vertex of the container. **(4)**

One of the strings is now removed and the solid S hangs freely in equilibrium suspended by the remaining vertical string.

(b) Find the size of the angle between the axis of the container and the downward vertical. **(3)**

a)

	Mass	C.O.M
Shell	m	$4r$
Wax	$3m$	$3/2 r$
Combined	$4m$	$\bar{x}$

$$4r\cancel{m} + \frac{9}{2}r\cancel{m} = 4\cancel{m}\bar{x}$$

$$\frac{17r}{2} = 4\bar{x} \therefore \bar{x} = \frac{17r}{8}$$

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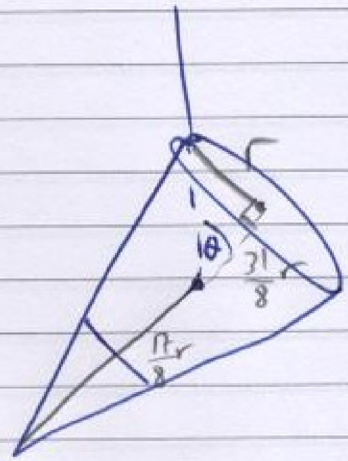
Question 3 continued

b)



$$\tan\theta = \frac{r}{\frac{31}{8}r} = \frac{8}{31}$$

$$\theta = 14.5^\circ \text{ (3sf)}$$

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4.

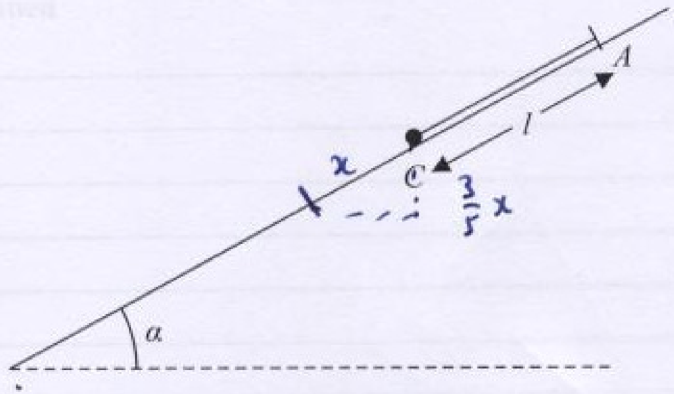

Figure 3

One end of a light elastic string, of natural length l and modulus of elasticity $3mg$, is fixed to a point A on a fixed plane inclined at an angle α to the horizontal, where $\sin \alpha = \frac{3}{5}$

A small ball of mass $2m$ is attached to the free end of the string. The ball is held at a point C on the plane, where C is below A and $AC = l$ as shown in Figure 3. The string is parallel to a line of greatest slope of the plane. The ball is released from rest. In an initial model the plane is assumed to be smooth.

(a) Find the distance that the ball moves before first coming to instantaneous rest.

(5)

In a refined model the plane is assumed to be rough. The coefficient of friction between the ball and the plane is μ. The ball first comes to instantaneous rest after moving a distance $\frac{2}{5}l$.

(b) Find the value of μ.

(6)

a) EPE : $\frac{3mgx^2}{2l}$ GPE : $2m.g.\frac{3}{5}x$

$\therefore \frac{\cancel{3}\cancel{mg}x^{\cancel{2}}}{2l} = 2\cancel{mg}.\frac{\cancel{3}}{5}\cancel{x}$

$\frac{x}{2l} = \frac{2}{5} \quad \therefore x = \frac{4l}{5}$

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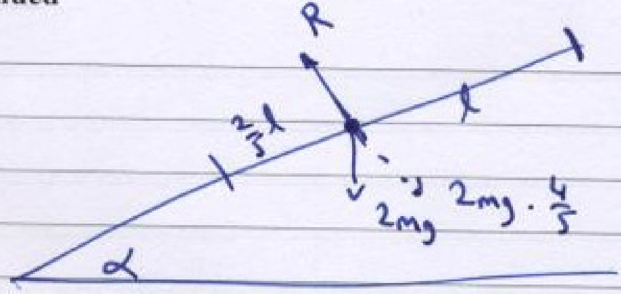
Question 4 continued

b)



$\cos\alpha = \frac{4}{5}$

WD by friction $= \frac{2}{5}l \cdot \mu R = \frac{2}{5} l \cdot \mu \cdot 2mg \frac{4}{5}$

$= \frac{16}{25} mgl\mu$

GPE : $2m \cdot g \cdot \frac{6}{25}l = \frac{12}{25}mgl$

EPE : $\frac{3mg}{2l} \cdot \frac{4l^2}{25} = \frac{6mgl}{25}$

Change in energy = WD

$\therefore \frac{16}{\cancel{25}} mgl\mu = \frac{12mgl}{\cancel{25}} - \frac{6mgl}{\cancel{25}}$

$16\mu = 6 \quad \therefore \mu = \frac{3}{8}$

"Leave blank"

5.

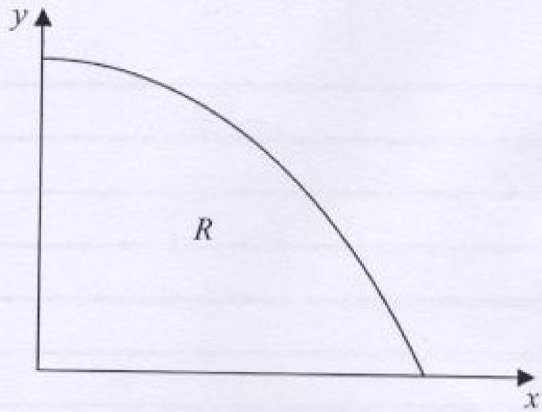


Figure 4

Figure 4 shows the region R bounded by part of the curve with equation $y = \cos x$, the x-axis and the y-axis. A uniform solid S is formed by rotating R through 2π radians about the x-axis.

(a) Show that the volume of S is $\frac{\pi^2}{4}$

(4)

(b) Find, using algebraic integration, the x coordinate of the centre of mass of S.

(7)

a) $V = \pi \int_0^{\pi/2} \cos^2 x \, dx = \pi \int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$

$= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin 2x \right]_0^{\pi/2}$

$= \pi \left[\left(\frac{\pi}{4} + 0\right) - (0 + 0) \right]$

$= \frac{\pi^2}{4}$, as req^d.

b) $\bar{x} = \dfrac{\pi \int_0^{\pi/2} x \cos^2 x \, dx}{V}$

By parts $\quad \int x \cos^2 x \, dx \qquad u = x \quad \frac{dv}{dx} = \cos^2 x$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}x + \frac{1}{4}\sin 2x$

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Question 5 continued

$$\therefore \int x\cos^2 x \, dx = \frac{1}{2}x^2 + \frac{1}{4}x\sin 2x - \int \frac{1}{2}x + \frac{1}{4}\sin 2x \, dx$$

$$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c.$$

$$\therefore \bar{x} = \frac{\pi\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2}}{\pi^2/4}$$

$$= \frac{4}{\pi}\left[\left(\frac{1}{4}\cdot\frac{\pi^2}{4} - \frac{1}{8}\right) - \left(0 + 0 + \frac{1}{8}\right)\right]$$

$$= \frac{4}{\pi}\left(\frac{\pi^2}{16} - \frac{4}{16}\right)$$

$$= \frac{4(\pi^2 - 4)}{16\pi} = \frac{\pi^2 - 4}{4\pi} = 0.467 \text{ (3sf)}$$

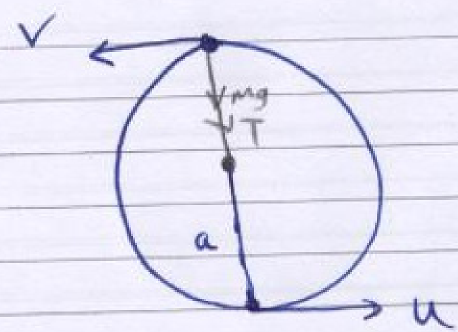
6. A particle P is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point. The particle is hanging freely at rest, with the string vertical, when it is projected horizontally with speed U. The particle moves in a complete vertical circle.

(a) Show that $U \geqslant \sqrt{5ag}$ **(8)**

As P moves in the circle the least tension in the string is T and the greatest tension is kT. Given that $U = 3\sqrt{ag}$

(b) find the value of k. **(5)**

a)



zero level

At top $T \geq 0$

	zero level	top
Energy: E_k	$\frac{1}{2}mu^2$	$\frac{1}{2}mv^2$
E_{GP}	0	$2mga$

$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + 2mga$

$u^2 = v^2 + 4ga$ ①

Resolving Radially at Top: $T + mg = \frac{mv^2}{a}$

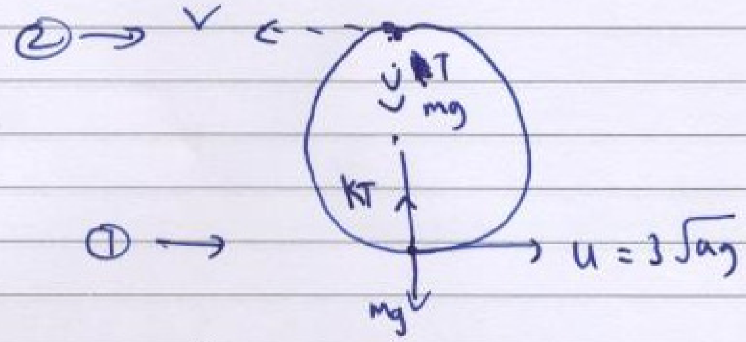
$\therefore \frac{mv^2}{a} - mg = T \geq 0$

$\frac{v^2}{a} \geq g \quad v^2 \geq ag$ ②

① & ② $\Rightarrow u^2 = v^2 + 4ga \geq 5ag \quad \therefore u^2 \geq 5ag$

$\therefore u \geq \sqrt{5ag}$

Question 6 continued



Using $u^2 = v^2 + 4ga \Rightarrow v^2 = 5ag$

① $kT - mg = \frac{m \cdot 9ag}{a} = 9mg \qquad \therefore kT = 10mg$

② $T + mg = \frac{m \cdot 5ag}{a} \qquad T = 4mg$

$\cancel{\therefore k = \frac{5}{2}}$

$\frac{kT}{T} = \frac{10mg}{4mg}$

$\therefore k = \frac{5}{2}$

Leave blank

7. A particle P of mass m is attached to one end of a light elastic spring of natural length l. The other end of the spring is attached to a fixed point A. The particle is hanging freely in equilibrium at the point B, where $AB = 1.5l$

(a) Show that the modulus of elasticity of the spring is $2mg$.

(3)

The particle is pulled vertically downwards from B to the point C, where $AC = 1.8l$, and released from rest.

(b) Show that P moves in simple harmonic motion with centre B.

(6)

(c) Find the greatest magnitude of the acceleration of P.

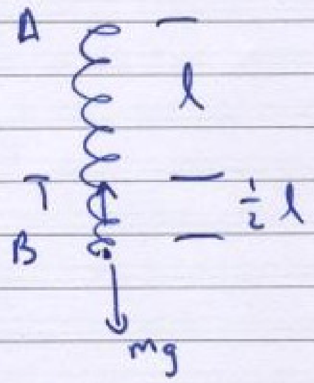
(2)

The midpoint of BC is D. The point E lies vertically below A and $AE = 1.2l$

(d) Find the time taken by P to move directly from D to E.

(4)

a)



$T = \dfrac{\lambda x}{l} = mg$ in equilib

$\dfrac{\lambda \cdot \frac{1}{2} l}{l} = mg \quad \therefore \quad \lambda = 2mg$ as req^{d}.

b) Pulled x from equilib

$\therefore \quad mg - T = m\ddot{x} \quad$ (Newton II)

$mg - \left(\underset{\uparrow}{mg} + \dfrac{\lambda x}{l} \right) = m\ddot{x}$

equilib tension

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Question 7 continued

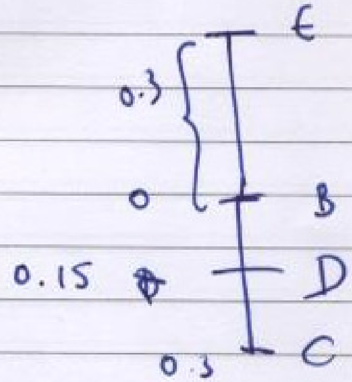
$$\therefore \quad \ddot{x} = -\frac{\lambda x}{ml} = -\frac{2g}{l}x$$

This shows particle moves in SHM

c) $|\ddot{x}| \Rightarrow x$ at end pts $x = 0.3l$

$$\therefore \left| -\frac{2g}{l} \times 0.3l \right| = 0.6g = 5.88\ \text{m/s}^2$$

d)



E is at end of $\frac{1}{2}$ period of motion.

Time from D to E is (Time from C to E) − (Time from C to D)

C to E is $\frac{1}{2}$ period $= \frac{\pi}{\omega}$

$\omega^2 = \frac{2g}{l} \quad \therefore \quad \omega = \sqrt{\frac{2g}{l}} \quad \therefore \frac{1}{2}$ Period $= \pi\sqrt{\frac{l}{2g}}$

C to D $\Rightarrow$ using $x = a\cos\omega t$

$$0.15 = 0.3\cos\omega t$$

$$\frac{1}{2} = \cos\omega t \qquad \omega t = \frac{\pi}{3}$$

$$t = \frac{\pi}{3\omega}$$

$\therefore$ Time from D to E $= \frac{3\pi}{3\omega} - \frac{\pi}{3\omega} = \frac{2}{3}\frac{\pi}{\omega} = \frac{2}{3}\pi\sqrt{\frac{l}{2g}}$