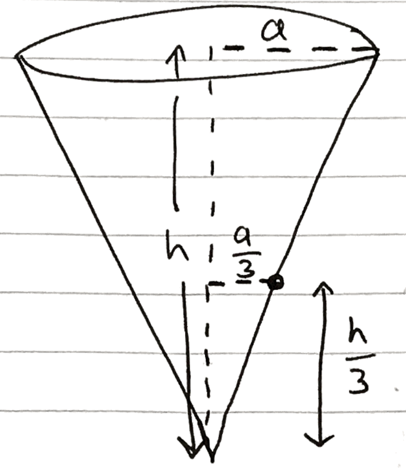
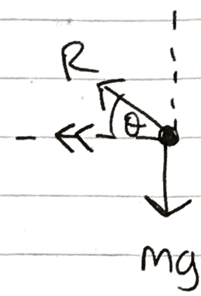
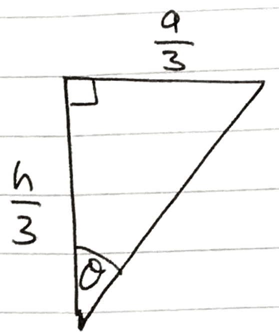
M3 June 2013(R) (MA)

Q1)







$$\tan\theta = \frac{a}{h}$$

$$\therefore \sin\theta = \frac{a}{\sqrt{a^2+h^2}}$$

$$\& \quad \cos\theta = \frac{h}{\sqrt{a^2+h^2}}$$

$R(\updownarrow)$: $R\sin\theta = mg$

$$\frac{Ra}{\sqrt{a^2+h^2}} = mg$$

$$\therefore R = \frac{mg\sqrt{a^2+h^2}}{a}$$

$\overset{+}{\leftarrow}$

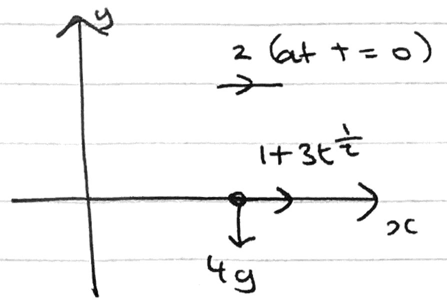
N2L (particle): $R\cos\theta = \frac{mv^2}{r}$

$r = \frac{a}{3}$: $\left(\frac{mg\sqrt{a^2+h^2}}{a}\right)\left(\frac{h}{\sqrt{a^2+h^2}}\right) = \frac{mv^2}{\frac{a}{3}}$

$$\Rightarrow \frac{gh}{a} = \frac{3v^2}{a}$$

$$\Rightarrow 3v^2 = gh \quad \therefore v^2 = \frac{gh}{3} \rightarrow \boxed{v = \sqrt{\frac{gh}{3}}}$$

Q2)



$$F = 1 + 3t^{\frac{1}{2}} = ma$$

$$1 + 3t^{\frac{1}{2}} = 4\frac{dv}{dt}$$

$$\int 4\, dv = \int 1 + 3t^{\frac{1}{2}}\, dt$$

$$4v = t + 3t^{\frac{3}{2}}\left(\frac{2}{3}\right) + c$$

$$4v = t + 2t^{\frac{3}{2}} + c$$

$\underline{t=0,\ v=2}$: $\quad 8 = c$

$$\therefore\ 4v = t + 2t^{3/2} + 8$$

$$v = \frac{1}{4}\left(t + 2t^{3/2} + 8\right)$$

Work Done $= \Delta KE$ from $t=0$ to $t=4$.

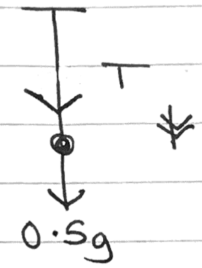
$\underline{\text{at } t=0}$: $\quad KE = 2v^2 = 2(4) = 8\text{ J}$.

$\underline{\text{at } t=4}$: $\quad v = \frac{1}{4}\left(4 + 2(4)^{3/2} + 8\right) = 7\text{ ms}^{-1}$

$$\therefore\ KE = \frac{1}{2}(4)(7^2) = 98\text{ J}$$

$\Delta KE = (98 - 8)\,J = \boxed{90J} =$ Work Done by the force

Q3a)

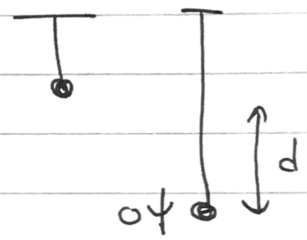


$\downarrow^{+}$ N2L (P) : $0.5g + T = 0.5a$

$T = \frac{\lambda x}{l} = \frac{20}{2}(1) = 10$

$\frac{0.5g + 10}{0.5} = a = \boxed{29.8\,ms^{-2}}$

b) Initially :
$KE = 0$
$GPE = 0.5gd$
$EPE = \frac{20}{4}(1)$



Finally :
$KE = 0$
$GPE = 0$
$EPE = \frac{20}{4}(d-1)^2$

C.O.E : $0.5gd + \frac{20}{4} = \frac{20}{4}(d^2 - 2d + 1)$

$0.5gd + 5 = 5(d^2 - 2d + 1)$

$5d^2 - 10d + \cancel{5} = \cancel{5} + 0.5gd$

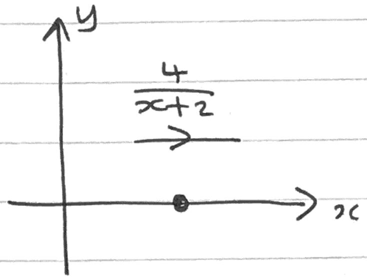
$5d^2 - 10d - 0.5gd = 0$

$d(5d - 10 - 4.9) = 0$

$d \neq 0, \quad 5d = 14.9$

$\therefore d = \frac{14.9}{5} = \boxed{2.98m}$

Q4a)



$V = \frac{4}{x+2} = \frac{dx}{dt}$

$(x+2)\frac{dx}{dt} = 4$

$\therefore \int (x+2)\,dx = \int (4)\,dt$

$\frac{x^2}{2} + 2x = 4t + c$

$t=0, x=0: \quad 0 = c$ //

$\therefore \frac{x^2}{2} + 2x = 4t$ //

$\underline{t=2}$: $\frac{x^2}{2} + 2x = 8$

$\frac{x^2}{2} + 2x - 8 = 0$

$\underline{\times 2}$: $x^2 + 4x - 16 = 0$

<u>By Quadratic formula,</u> $x = -2 \pm 2\sqrt{5}$

$x > 0$ as P is always moving <u>away</u> from O.

so $\boxed{x = -2 + 2\sqrt{5}}$ //

$= 2.47m.$

b) $v = 4(x+2)^{-1}$

$\frac{dv}{dx} = -4(x+2)^{-2}$

$a = v\frac{dv}{dx} = v(-4(x+2)^{-2}) = \frac{4}{x+2} \times \frac{-4}{(x+2)^2}$

$a = \frac{-16}{(x+2)^3}$ //

$\underline{\text{at } t=2}$, $x = -2+2\sqrt{5}$ from (a).

$\therefore a = \frac{-16}{(-2+2+2\sqrt{5})^3} = \underline{\underline{-0.18 ms^{-2}}}$

so $|a| = 0.18 \, ms^{-2}$

Direction is towards O. (as $a < 0$)

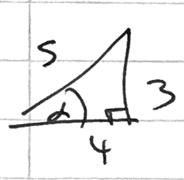
Q5a) At A : $KE = \frac{1}{2} M (\sqrt{gr})^2$

$GPE = 0$

At B : $KE = \frac{1}{2} M v^2$

$GPE = Mgr(1 - \cos\alpha)$

C.O.E : $\frac{Mgr}{2} = \frac{Mv^2}{2} + Mgr(1 - \cos\alpha)$



$\tan\alpha = \frac{3}{4} \quad \therefore \cos\alpha = \frac{4}{5}$

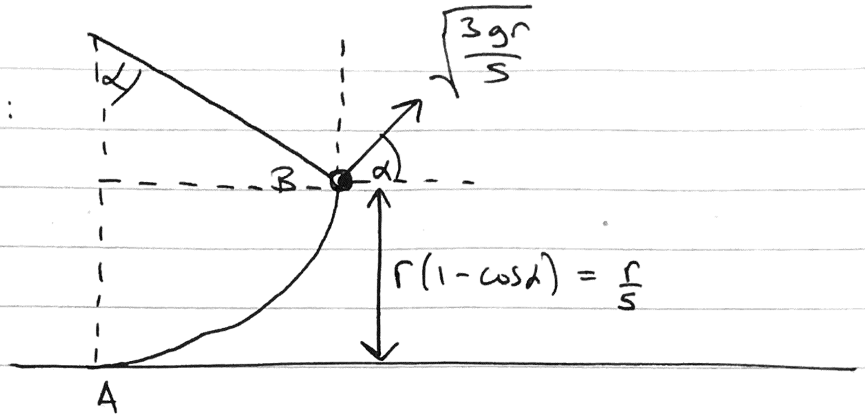
so $\frac{gr}{2} = \frac{v^2}{2} + gr\left(1 - \frac{4}{5}\right)$

$gr = v^2 + \frac{2gr}{5}$

$v^2 = \frac{3gr}{5}$

$\therefore v = \sqrt{\frac{3gr}{5}}$ = speed at B.

b) At B :



$\vec{u} = \sqrt{\frac{3gr}{5}} \cos\alpha = \frac{4}{5}\sqrt{\frac{3gr}{5}} \xrightarrow{r=0.4} \vec{u} = \frac{4}{5}\sqrt{\frac{6g}{25}}$

$u\uparrow = \sqrt{\frac{3gr}{5}} \sin\alpha = \frac{3}{5}\sqrt{\frac{3gr}{5}} \xrightarrow{r=0.4} u\uparrow = \frac{3}{5}\sqrt{\frac{6g}{25}}$

suvat from B till ground :

$\uparrow +$
$s = -\frac{0.4}{5}$
$u = \frac{3}{5}\sqrt{\frac{6g}{25}}$
$v =$
$a = -g$
$t =$

$s = ut + \frac{1}{2}at^2$

$-\frac{2}{25} = \frac{3}{5}t\sqrt{\frac{6g}{25}} - \frac{g}{2}t^2$

$-0.08 = 0.9202t - 4.9t^2$

$4.9t^2 - 0.9202t - 0.08 = 0$

By Quadratic formula, $t = \underline{\underline{0.252..}}$, $-0.0647.$

reject $(t \geq 0)$

so M takes 0.252s to reach C.

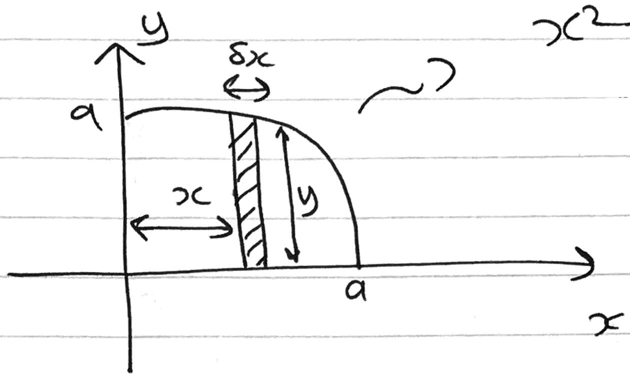
now consider horizontal motion of M,

$\vec{s} = ut$: $\vec{BC} = \frac{4}{5}\sqrt{\frac{6g}{25}} \times 0.252 \simeq 0.309\text{m}$

so $AC = r\sin\alpha + 0.309$

$$\simeq 0.4\left(\frac{3}{5}\right) + 0.309 \simeq \boxed{0.55m}$$

Q6a)



$x^2 + y^2 = a^2$

$(x \geq 0,\ y \geq 0)$

split up this lamina into thin vertical strips of thickness δx.

mass of one strip $= \delta m = \rho \times \text{area} = \rho y \delta x$.

distance of c.o.m of one strip from y-axis $= x$

$\therefore m_i x_i = \rho x y \delta x$

Total mass of lamina $= \rho \times \left(\frac{\pi a^2}{4}\right) = \frac{\rho\pi a^2}{4}$.

recall from M2 that $\underline{\bar{x}\sum m_i = \sum m_i x_i}$

$$\therefore \bar{x}\left(\frac{\rho\pi a^2}{4}\right) = \sum_{x=0}^{a} \rho x y \delta x$$

$$\lim_{\delta x \to 0} \sum_{x=0}^{a} \rho x y \delta x = \rho\int_0^a (xy)\,dx$$

$$p\int_0^a [xy]\,dx = p\int_0^a [x]\left[\sqrt{a^2-x^2}\right]dx$$

By Pattern.

$$= p\left[-\frac{1}{2}\left(\frac{(a^2-x^2)^{3/2}}{\frac{3}{2}}\right)\right]_0^a$$

$$= \left[-\frac{1}{2}p\left[\frac{2}{3}\right]\left[a^2-x^2\right]^{3/2}\right]_0^a$$

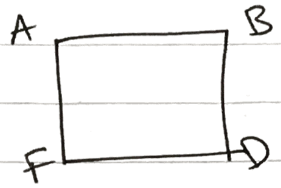
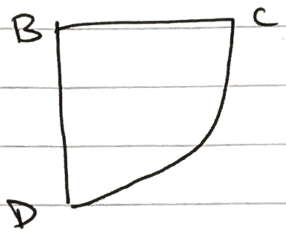
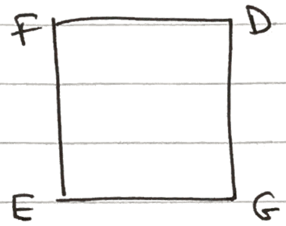
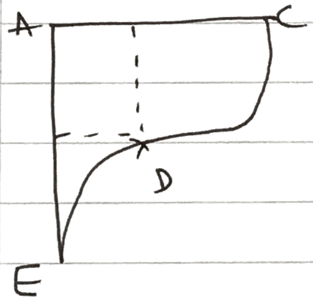
$$= \left[-\frac{p}{3}(a^2-x^2)^{3/2}\right]_0^a = [0] + \frac{p}{3}\left[a^2\right]^{3/2}$$

$$= \frac{p}{3}a^3$$

$$\therefore \bar{x}\left(\frac{p\pi a^2}{4}\right) = \frac{pa^3}{3}$$

$$\Rightarrow \bar{x} = \frac{4a^3\cancel{p}}{3\pi a^2\cancel{p}} = \frac{4a}{3\pi} = \text{distance from each edge.}$$

b)

	Shape	Mass	Distance of c.o.m from AE
$+$		$\boxed{a^2}$	$\boxed{\frac{a}{2}}$
$+$		$\boxed{\frac{\pi a^2}{4}}$	$\boxed{a + \frac{4a}{3\pi}}$
$+$		$\boxed{a^2}$	$\boxed{\frac{a}{2}}$
$-$		$\boxed{\frac{\pi a^2}{4}}$	$\boxed{a - \frac{4a}{3\pi}}$
$=$		$\boxed{2a^2}$	$\boxed{\bar{x}}$

taking moments about side AE...

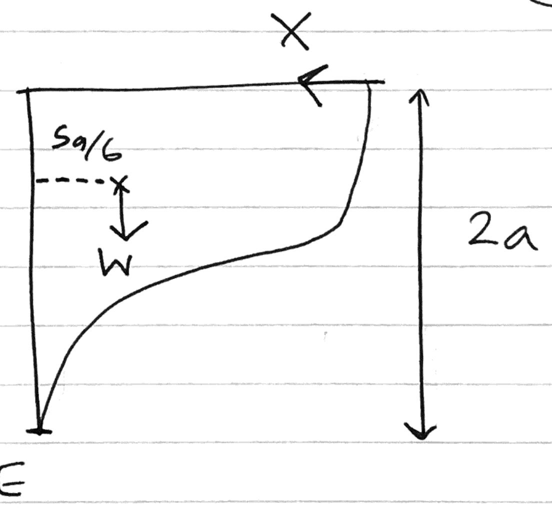
$$a^2\left(\frac{a}{2}\right) + \frac{\pi a^2}{4}\left(a + \frac{4a}{3\pi}\right) + a^2\left(\frac{a}{2}\right) - \frac{\pi a^2}{4}\left(a - \frac{4a}{3\pi}\right) = 2a^2\bar{x}$$

$$\Rightarrow a + \frac{\pi}{4}\left(a + \frac{4a}{3\pi} - a + \frac{4a}{3\pi}\right) = 2\bar{x}$$

$$\Rightarrow a + \frac{\cancel{\pi}}{4}\left(\frac{8a}{3\cancel{\pi}}\right) = 2\bar{x}$$

$$\Rightarrow \frac{a + \frac{2a}{3}}{2} = \bar{x} = \boxed{\frac{5a}{6}}$$

c)



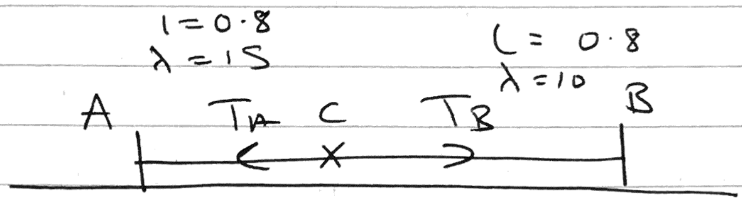
X must act to the left to counteract the turning effect of the weight W since the entire lamina is in equilibrium.

$$\underline{M(E)}: \; X(2a) = W\left(\frac{5a}{6}\right)$$

$$X = \frac{\frac{5}{6}W}{2} \rightarrow X = \frac{5W}{12}$$

[We take moments about E as this is where the axis passes through - we can then ignore the forces exerted by the axis on the lamina allowing us to deal only with W and X.]

(Q7a)



$$T_A = T_B$$

$$\frac{15}{0.8}(AC - 0.8) = \frac{10}{0.8}(4 - AC - 0.8)$$

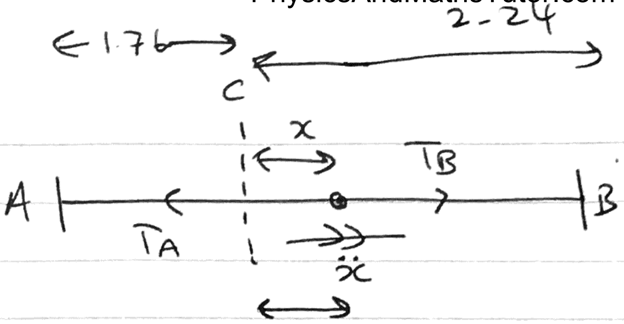
$$\frac{75}{4}(AC - 0.8) = \frac{25}{2}(3.2 - AC)$$

$$\frac{75}{4}AC - 15 = 40 - \frac{25}{2}AC$$

$$\left(\frac{75}{4} + \frac{25}{2}\right)AC = 55$$

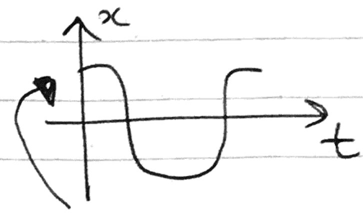
$$\therefore AC = \frac{55}{\frac{125}{4}} = \boxed{1.76\text{m}}$$

b)



$2.16-1.76 = 0.4m$

P starts at an endpoint so $x = a\cos\omega t$ applies



N2L(P) : $T_B - T_A = 0.2\ddot{x}$ (→ +)

$\Rightarrow \frac{10}{0.8}(2.24 - x - 0.8) = T_B$

$\Rightarrow \frac{15}{0.8}(1.76 + x - 0.8) = T_A$

hence x is max at $t=0$. so x is positive in the direction CB. so $\ddot{x}$ is also positive in the direction CB.

$\Rightarrow 12.5(1.44 - x) - 18.75(0.96 + x) = 0.2\ddot{x}$

$\Rightarrow 18 - 12.5x - 18 - 18.75x = 0.2\ddot{x}$

$\Rightarrow -\frac{125}{4}x = 0.2\ddot{x}$

$\Rightarrow -\frac{625}{4}x = \ddot{x}$

hence P moves with S.H.M

c) at C, speed is maximum.

$$V_{max} = a\omega = 0.4 \times \sqrt{\frac{625}{4}} = \boxed{5\,m/s}$$

d) $x = 0.4\cos\left(t\sqrt{\frac{625}{4}}\right)$

$$\dot{x} = -0.4\sqrt{\frac{625}{4}}\sin\left(t\sqrt{\frac{625}{4}}\right) = v$$

the first instance when the speed will be 2m/s will be when P is moving in the direction BC so v will be $\underline{\underline{-2}}$ ms^{-1}.

$$-2 = -0.4\sqrt{\frac{625}{4}}\sin\left(t\sqrt{\frac{625}{4}}\right)$$

$$\frac{2}{0.4\sqrt{\frac{625}{4}}} = \sin\left(t\sqrt{\frac{625}{4}}\right) = \frac{2}{5}$$

$$t\sqrt{\frac{625}{4}} = \sin^{-1}\left(\frac{2}{5}\right)$$

$$\therefore t = \frac{\sin^{-1}\left(\frac{2}{5}\right)}{\sqrt{\frac{625}{4}}} = \boxed{0.033s}$$