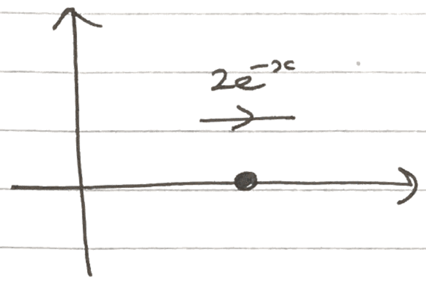
M3 June 2012 (MA)

Q1a)



$$v = 2e^{-x}$$

$$\frac{dv}{dx} = -2e^{-x}$$

$$a = v\frac{dv}{dx}$$

$$\therefore \quad a = (-2e^{-x})(v)$$

$$a = (2e^{-x})(-2e^{-x})$$

$$a = -4e^{-2x}$$

b) $$V = \frac{dx}{dt}$$

$$\therefore \quad \frac{dx}{dt} = 2e^{-x}$$

$$e^{x}\frac{dx}{dt} = 2$$

$$\int e^{x}\,dx = 2\int (1)\,dt$$

$\therefore e^x = 2t + c$

$\underline{t=0, x=0}$: $e^0 = 1 = 2(0) + c$

$\therefore c = 1$ //

so $e^x = 2t + 1$

$\Rightarrow x = \ln(2t+1)$

Q2a) $V_{max} = a\omega$. [speed is max at O]

$6 = a\omega$

$T = \frac{\pi}{2} \therefore \omega = \frac{2\pi}{\frac{\pi}{2}} = 4$ //

so $6 = 4a \quad \therefore a = \frac{6}{4} = \boxed{1.5m}$

= max distance from O.

b) $a_{max} = \omega^2 a = 1.5 \times (4)^2 = \boxed{24 ms^{-2}}$

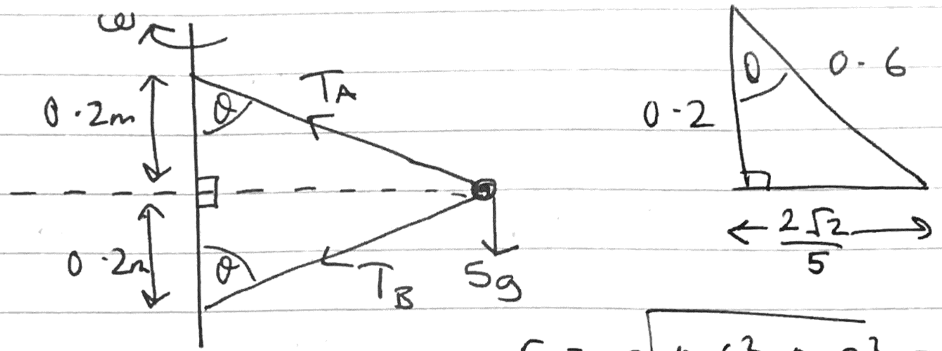
c) $x = 1.5\sin(4t)$

$1 = 1.5\sin(4t)$

$\frac{1}{1.5} = \sin(4t) \quad \therefore \sin^{-1}\frac{1}{1.5} = 4t = 0.7297..$

$\therefore t = \frac{0.7297..}{4} = \boxed{0.18}$

Q3i)



$r = \sqrt{0.6^2 - 0.2^2} = \frac{2\sqrt{2}}{5}$

$\therefore \cos\theta = \frac{1}{3}$

$\sin\theta = \frac{2\sqrt{2}}{3}$

$R(\updownarrow): \ T_A\cos\theta = 5g + T_B\cos\theta$

$\frac{1}{3}T_A = 5g + \frac{1}{3}T_B \quad \sim \ (1)$

$\overset{+}{\leftarrow}$ $\underline{N2L(Q)}: \ T_A\sin\theta + T_B\sin\theta = mr\omega^2$

$\frac{2\sqrt{2}}{3}(T_A + T_B) = 5\left(\frac{2\sqrt{2}}{5}\right)(10)^2$

$\underline{\div 2\sqrt{2}}: \quad \frac{1}{3}(T_A + T_B) = 100$ //

$\Rightarrow \quad \frac{1}{3}T_A + \frac{1}{3}T_B = 100$

from (1): $\frac{1}{3}T_A = 5g + \frac{1}{3}T_B$

so $\quad 5g + \frac{1}{3}T_B + \frac{1}{3}T_B = 100$

$\frac{2}{3}T_B = 100 - 5g$

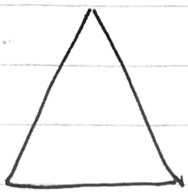
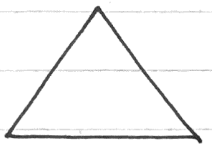
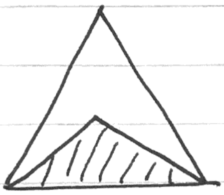
$T_B = \frac{100 - 5g}{\frac{2}{3}} = \boxed{76.5\text{N}}$

ii) and $\frac{1}{3}T_A + \frac{1}{3}T_B = 100$

$\therefore \frac{1}{3}T_A = 100 - \frac{1}{3}(76.5)$

$\Rightarrow T_A = 300 - 76.5 = \boxed{223.5N}$

Q4a)

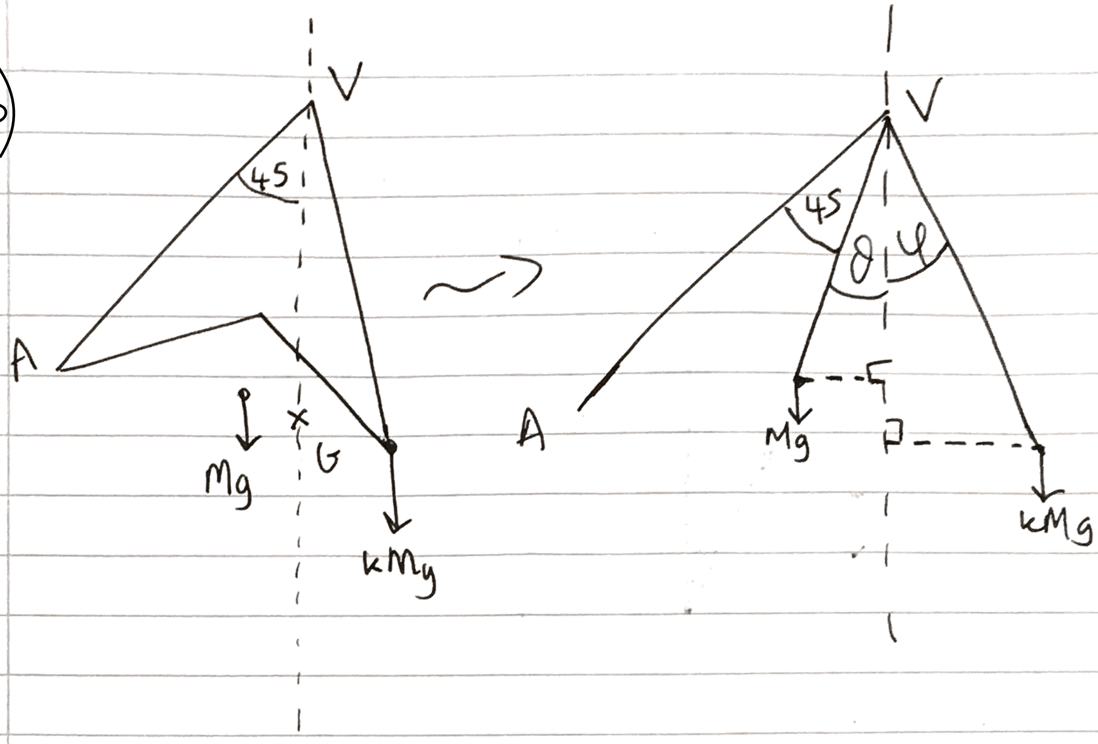
Shape	Mass (vol.)	Distance of c.o.m from V
	$\frac{1}{3}\pi(a)^2(2a) = \boxed{\frac{2\pi a^3}{3}}$	$\frac{3}{4} \times 2a = \boxed{\frac{3a}{2}}$
	$\frac{1}{3}\pi(a)^2(a) = \boxed{\frac{\pi a^3}{3}}$	$\frac{3}{4} \times a + a = \boxed{\frac{7a}{4}}$
	$\boxed{\frac{\pi a^3}{3}}$	$\boxed{\bar{y}}$

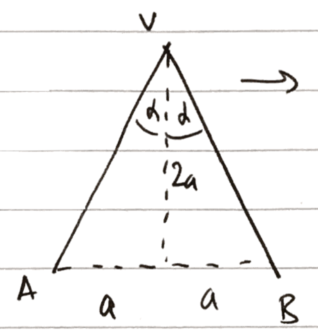
taking moments about V...

$$\frac{2}{3}\cancel{\pi a^3}\left(\frac{3a}{2}\right) - \frac{1}{3}\cancel{\pi a^3}\left(\frac{7a}{4}\right) = \frac{1}{3}\cancel{\pi a^3}(\bar{y})$$

$$\frac{a + \left(\frac{-7a}{12}\right)}{\frac{1}{3}} = \bar{y} = \frac{5a}{4}$$

b)





$\rightarrow \tan\alpha = \frac{1}{2} \therefore \alpha = \tan^{-1}\frac{1}{2} = 26.6°$

$\therefore \angle AVB = 53.1° \ (= 26.6 \times 2)$

$\therefore \theta = 45° - 26.6° = 18.44°$

and $\varphi = 53.1° - 45° = 8.13°$

so perpendicular distances from each mass to downward vertical will be...

For Mg : distance $= \frac{5a}{4}\sin(18.44°)$

For kMg : distance $= VB\sin(8.13°) = a\sqrt{5}\sin(8.13°)$

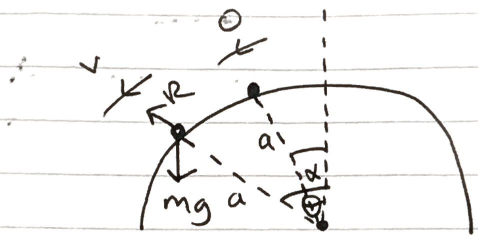
$\left(VB = \sqrt{4a^2 + a^2} = a\sqrt{5}\right)$

$\therefore$ moments about vertical through V

$$Mg\left(\frac{5a}{4}\right)\sin(18.44°) = kMg(a\sqrt{5})\sin(8.13°)$$

$$\therefore \mu = \frac{\frac{5}{4}\sin(18.44^\circ)}{\sqrt{5}\sin(8.13^\circ)} = \boxed{1.25}$$

Q5a)



<u>Initially</u> : $KE = 0$

$GPE = mga\cos\alpha$

<u>Finally</u> : $KE = \frac{1}{2}mv^2$

$GPE = mga\cos\theta$

<u>C.O.E</u> : $\cancel{m}ga\cos\alpha = \frac{\cancel{m}v^2}{2} + \cancel{m}ga\cos\theta$

$$v^2 = 2ag\cos\alpha - ag(2\cos\theta)$$

$$v^2 = \frac{6ag}{5} - 2ag\cos\theta$$

$$v^2 = \frac{2ga}{5}(3 - 5\cos\theta)$$

b) N2L $\searrow +$ at angle θ with upward vertical.

$$mg\cos\theta - R = \frac{mv^2}{a}$$

[when P loses contact] $R=0$:

$$\cancel{m}g\cos\theta = \frac{\cancel{m}v^2}{a}$$

$$ag\cos\theta = v^2$$

$$ag\cos\theta = \frac{2ga}{5}(3-5\cos\theta)$$

$$\frac{6ag}{5} - 2ag\cos\theta - ag\cos\theta = 0$$

$$3\cancel{a}g\cos\theta = \frac{6\cancel{a}g}{5}$$

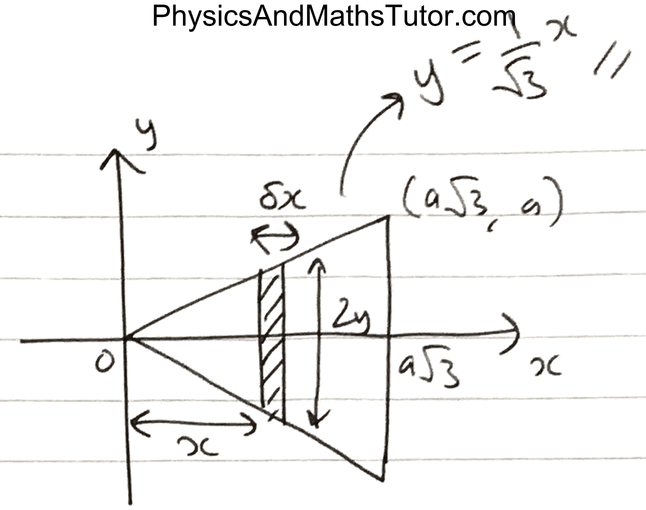
$$\therefore \cos\theta = \frac{2}{5} \quad //$$

and $v^2 = \frac{2ag}{5}(3-5\cos\theta)$

$$v^2 = \frac{2ag}{5}(3-2) = \frac{2ag}{5}$$

hence $v = \boxed{\sqrt{\frac{2ag}{5}}}$

Q6a)



Mass of lamina $= \rho \times \text{area} = \frac{1}{2} \times \rho \times 2a \times \sqrt{3}a$

$= \rho a^2\sqrt{3}$

Mass of one thin vertical strip $= \rho \times 2y\,\delta x$

Distance of c.o.m of one thin vertical strip from y-axis $= x$

from M2, $\underline{\sum m_i x_i = \bar{x} \sum m_i}$.

$$\sum_{x=0}^{a\sqrt{3}} 2\rho y x\,\delta x = \bar{x}\left(\rho a^2\sqrt{3}\right)$$

$$\lim_{\delta x \to 0} \sum_{0=x}^{a\sqrt{3}} 2\rho y x\,\delta x = 2\rho\int_0^{a\sqrt{3}} [xy]\,dx = 2\rho\int_0^{a\sqrt{3}} \left[\frac{x^2}{\sqrt{3}}\right] dx$$

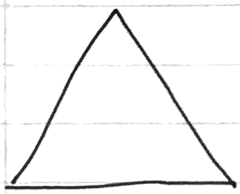
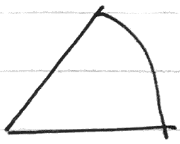
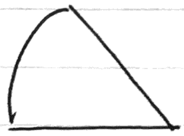
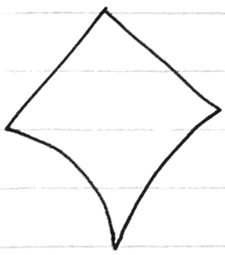
$$= 2\rho\left[\frac{x^3}{3\sqrt{3}}\right]_0^{a\sqrt{3}} = 2\rho\left[\frac{3\cancel{\sqrt{3}}a^3}{3\cancel{\sqrt{3}}}\right] = 2\rho a^3$$

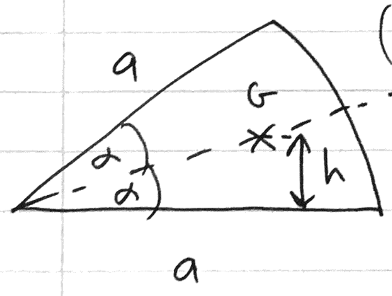
$$\therefore 2\rho a^3 = \bar{x}\left(\rho a^2\sqrt{3}\right)$$

$\underline{\div \rho a^2}$: $2a = \bar{x}(\sqrt{3}) \quad \therefore \bar{x} = \frac{2a}{\sqrt{3}}$

$$\bar{x} = \frac{2a}{\sqrt{3}} = \frac{2a\sqrt{3}}{3}$$

b)

Shape	Mass	Distance of c.o.m from PUT.
	$\boxed{a^2\sqrt{3}}$	$a\sqrt{3} - \frac{2\sqrt{3}}{3}a = \boxed{\frac{\sqrt{3}}{3}a}$
(−) 	$\frac{1}{2}r^2\theta = \boxed{\frac{\pi a^2}{6}}$	$d = \frac{2a\sin\alpha}{3\alpha} = \frac{2a}{\pi}$ $h = \frac{2a}{\pi} \times \frac{1}{2} = \boxed{\frac{a}{\pi}}$
(−) 	$\boxed{\frac{\pi a^2}{6}}$	$\boxed{\frac{a}{\pi}}$
	$\boxed{a^2\left(\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)}$	$\boxed{\bar{y}}$



$\left(\alpha = \frac{\pi}{6}\right)$ as its an equilateral triangle.

$h =$ distance required $= \frac{2a}{\pi}\sin\alpha$

$= \frac{a}{\pi}$ //

$$d = \frac{2a\sin\alpha}{3\alpha} = \frac{2(a)(\sin\frac{\pi}{6})}{\frac{3\pi}{6}} = \frac{2a}{\pi}$$ //

taking moments about O

$$\cancel{a^2}\sqrt{3}\left(\frac{\sqrt{3}a}{3}\right) - 2\times\frac{\cancel{\pi}a^2}{6}\left(\frac{a}{\cancel{\pi}}\right) = \cancel{a^2}\left(\sqrt{3}-\frac{\pi}{3}\right)\bar{y}$$

$$a - \frac{a}{3} = \left(\sqrt{3} - \frac{\pi}{3}\right)\bar{y}$$

$$\bar{y} = \frac{\frac{2a}{3}}{\sqrt{3}-\frac{\pi}{3}} = \frac{2a}{3\sqrt{3}-\pi} \quad \begin{matrix}(\times 3)\\(\times 3)\end{matrix}$$

Q7a) $R(\updownarrow)$: $T = 0.5g$

$$\frac{\lambda x}{l} = 0.5g$$

$$\frac{24.5e}{0.75} = 0.5g$$

$$e = \frac{0.5g \times 0.75}{24.5} = \frac{3}{20}\text{ m}$$

$$\therefore AE = 0.75 + \frac{3}{20} = \boxed{0.9\text{m}}$$

b) <u>At A</u> : $KE = 0$

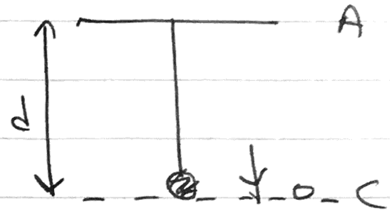
$GPE = 0.5gd$

$EPE = \frac{24.5}{1.5}(0) = 0$

<u>At C</u> : $KE = 0$

$GPE = 0$

$EPE = \frac{24.5}{1.5}(d-0.75)^2$



<u>Conservation of energy</u> : $0.5gd = \frac{24.5}{1.5}(d^2 - 1.5d + 0.75^2)$

$\Rightarrow 0.5gd = \frac{49}{3}\left(d^2 - 1.5d + \frac{9}{16}\right)$

$\Rightarrow 0.5gd = \frac{49}{3}d^2 - \frac{49}{2}d + \frac{147}{16}$

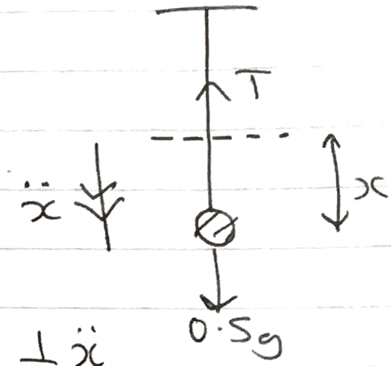
$\Rightarrow \frac{49}{3}d^2 - \frac{147}{5}d + \frac{147}{16} = 0$

By Quadratic formula, $d = 1.3975...$, $d = 0.403...$

$d > 0.75$ as string must be taut when B comes to rest.

So $\boxed{AC = 1.40\,m}$

c) $\underline{\text{N2L(B)}} \downarrow_+ : \ 0.5g - T = 0.5\ddot{x}$



$$\Rightarrow 0.5g - \frac{24.5}{0.75}(x + 0.15) = \frac{\ddot{x}}{2}$$

$$\Rightarrow 0.5g - \frac{24.5}{0.75}(x) - \frac{24.5}{0.75}(0.15) = \frac{1}{2}\ddot{x}$$

$$\Rightarrow 4.9 - \frac{98}{3}x - 4.9 = 0.5\ddot{x}$$

$$\Rightarrow -\frac{98}{3}x = 0.5\ddot{x}$$

$$\Rightarrow -\frac{196}{3}x = \ddot{x}$$

hence P moves with S.H.M centre B. while string is taut.

d) $V_{max} = a\omega \longrightarrow \omega = \sqrt{\frac{196}{3}}$.

$$a = (AC - AE) \simeq 1.4 - 0.9 \simeq 0.5$$

$$\therefore V_{max} \simeq 0.5\sqrt{\frac{196}{3}} = \boxed{4.0 \text{ ms}^{-1}}$$