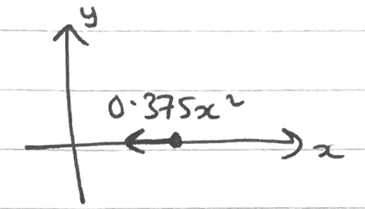
M3 June 2011 (MA)

Q1a)



$F = -0.375x^2 = ma$

$-0.375x^2 = 0.5\, v\frac{dv}{dx}$

$-\frac{3}{4}x^2 = v\frac{dv}{dx}$

$\int v\, dv = \frac{-3}{4}\int x^2\, dx$

$\frac{v^2}{2} = \frac{-3x^3}{12} + c$

$\underline{x = 8,\ v = -2}$: $\frac{4}{2} = -\frac{1}{4}(8)^3 + c$

$c = 128 + 2 = 130$ //

$\therefore \frac{v^2}{2} = 130 - \frac{x^3}{4}$

$\underline{\times 2}$: $v^2 = 260 - \frac{1}{2}x^3$

b) $\underline{v=5}$: $25 = 260 - \frac{1}{2}x^3$

$$\frac{x^3}{2} = 235$$

$$x^3 = 470$$

$$x = \sqrt[3]{470} \simeq \boxed{7.77\,m}$$

Q2) $V = \pi\int_0^3 (y^2)\,dx = \pi\int_0^3 (9-x^2)^2\,dx$

$$= \pi\int_0^3 \left[81 - 18x^2 + x^4\right] dx$$

$$= \pi\left[81x - 6x^3 + \frac{x^5}{5}\right]_0^3 = \pi\left[\frac{648}{5}\right]$$

$$M\bar{x} = \rho\pi\int_0^3 y^2 x\,dx = \rho\pi\int_0^3 \left[81x - 18x^3 + x^5\right] dx$$

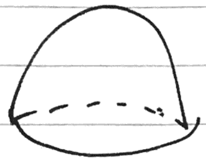
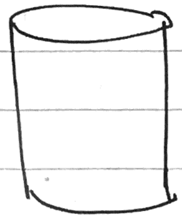
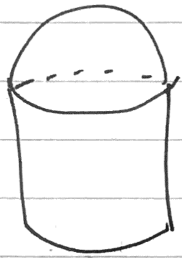
$$= \rho\pi\left[\frac{81x^2}{2} - \frac{18x^4}{4} + \frac{x^6}{6}\right]_0^3 = \rho\pi\left[\frac{729}{2} - \frac{729}{2} + \frac{243}{2}\right]$$

$M\bar{x} = \frac{243\rho\pi}{2}$ and $M = \frac{648\rho\pi}{5}$

$$\bar{x} = \frac{M\bar{x}}{M} = \frac{\frac{243\cancel{\rho\pi}}{2}}{\frac{648\cancel{\rho\pi}}{5}} = \boxed{\frac{15}{16}}$$

Q3a) density = $\frac{\text{mass}}{\text{volume}}$ $\therefore$ mass = density × volume

$m = \rho V$

Shape	Mass	$\updownarrow$ + Displacement of c.o.m from O
	$2\rho\left(\frac{2}{3}\pi(3l)^3\right) = \boxed{36\rho\pi l^3}$	$-\frac{3}{8}(3l) = \boxed{\frac{-9l}{8}}$
	$= \rho\pi(3l)^2(5l) = \boxed{45\rho\pi l^3}$	$\boxed{\frac{5l}{2}}$
	$\boxed{81\rho\pi l^3}$	$\boxed{\bar{y}}$

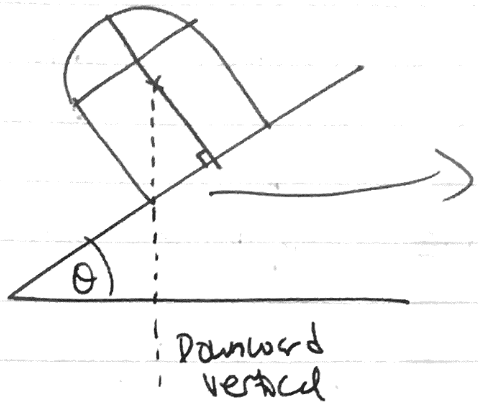
<u>taking moments about a diameter through O.</u>

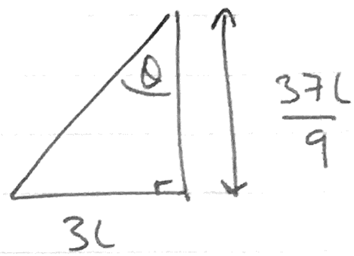
$$36\left(-\frac{9l}{8}\right) + 45\left(\frac{5l}{2}\right) = 81\,\bar{y}$$

$$\therefore \bar{y} = \frac{\frac{225}{2}l - \frac{81l}{2}}{81} = \boxed{\frac{8l}{9}}$$

$$5l - \frac{8l}{9} = \frac{37l}{9}$$

b)





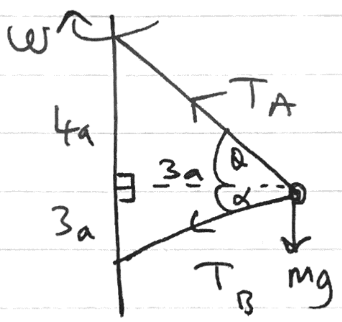
$$\tan\theta = \frac{3}{\frac{37}{9}} = \frac{27}{37}$$

On point of toppling ∴ downward vertical passes through lowest point of contact with plane

$$\therefore \theta = \tan^{-1}\left(\frac{27}{37}\right)$$

$$= \boxed{36 \cdot 1^\circ}$$

Q4a)



$\sin\theta = \frac{4}{5}$ $\quad \sin\alpha = \frac{\sqrt{2}}{2}$

$\cos\theta = \frac{3}{5}$

$\cos\alpha = \frac{\sqrt{2}}{2}$

$$R(\updownarrow): \; T_A \sin\theta = mg + T_B \sin\alpha$$

$$\therefore \; \frac{4}{5}T_A = mg + T_B\frac{\sqrt{2}}{2} \quad \text{— (1)}$$

$\overset{+}{\leftarrow}$

<u>N2L (P)</u>: $T_A\cos\theta + T_B\cos\alpha = m(3a)\omega^2$

$$\frac{3}{5}T_A + \frac{\sqrt{2}}{2}T_B = 3ma\omega^2$$

from (1), $\quad T_B\frac{\sqrt{2}}{2} = \frac{4}{5}T_A - mg$

$$\therefore \quad \frac{3}{5}T_A + \frac{4}{5}T_A - mg = 3ma\omega^2$$

$$\frac{7}{5}T_A = m(g + 3a\omega^2)$$

$$\therefore \quad T_A = \frac{5m}{7}(3a\omega^2 + g)$$

b) $$T_B\frac{\sqrt{2}}{2} = \frac{4}{5}\left(\frac{5m}{7}\right)(3a\omega^2 + g) - mg$$

$$T_B\frac{\sqrt{2}}{2} = \frac{4m}{7}(3a\omega^2) + \frac{4mg}{7} - mg$$

$$T_B\frac{\sqrt{2}}{2} = \frac{12ma\omega^2}{7} - \frac{3mg}{7} = \frac{3m}{7}(4a\omega^2 - g)$$

$$\therefore \quad T_B = \frac{6m}{7\sqrt{2}}(4a\omega^2 - g)$$

$$= \frac{3m\sqrt{2}}{7}(4a\omega^2 - g)$$

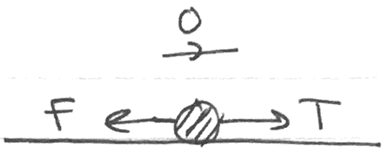
c) $T_B \geq 0$ for P to remain in circular motion.

$$\therefore \quad (4a\omega^2 - g)\frac{3m\sqrt{2}}{7} \geq 0$$

$$4a\omega^2 - g \geq 0$$

$$\omega^2 \geq \frac{g}{4a} \qquad \therefore \quad \omega \geq \frac{1}{2}\sqrt{\frac{g}{a}}$$

Q5a)



$$F = T$$

$$T = \frac{\lambda x}{l} = \frac{3mg\left(\frac{l}{6}\right)}{l} = \frac{mg}{2}$$

$$\therefore \quad F = \frac{mg}{2}$$

but $F \leq \mu R$

$$\therefore \quad \frac{mg}{2} \leq \mu R$$

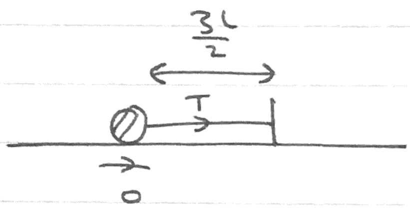
$R = mg$ (resolve perpendicular to table)

$$\therefore \quad \frac{mg}{2} \leq \mu(mg)$$

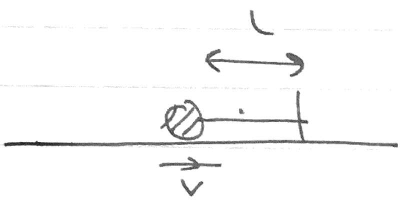
$\div \underline{mg} \implies \quad \mu \geq \frac{1}{2}$

b) <u>Initially</u> : $KE = 0$

$$EPE = \frac{3mg\left(\frac{l}{2}\right)^2}{2l}$$



<u>When string becomes slack</u> :



$$KE = \frac{mv^2}{2}$$

$$EPE = \frac{3mg(0)^2}{2l} = 0 \quad //$$

$+$ W.D by friction $= \mu R s = \frac{1}{2}mg\left(\frac{l}{2}\right) = \frac{mgl}{4} \quad //$

$$\therefore \underbrace{\frac{3mgl}{8}}_{\text{total energy at start}} = \underbrace{\frac{mv^2}{2}}_{\text{KE at end}} + \underbrace{\frac{mgl}{4}}_{\text{work done by friction}}$$

$$\Rightarrow \left(\frac{3}{8} - \frac{1}{4}\right) gl = \frac{v^2}{2} = \frac{gl}{8}$$

$$\therefore v^2 = \frac{gl}{4} \qquad \therefore v = \frac{\sqrt{gl}}{2}$$

c) string will be slack so no more EPE. only energy that exists now is KE.

s

$\therefore$ Total KE = Work Done by Friction (from slack point to rest)

$$\Rightarrow \frac{m\left(\frac{gl}{4}\right)}{2} = \frac{1}{2} mg d$$

$$\Rightarrow \div mg : \frac{l}{8} = \frac{d}{2}$$

$$\therefore d = \frac{l}{4}$$

so total distance moved $= \frac{l}{4} + \frac{l}{2} = \boxed{\frac{3l}{4}}$

Q6a) At A: $KE = \frac{9mag}{2}$

$GPE = 0$

At B: $KE = \frac{mv^2}{2}$

$GPE = mga$

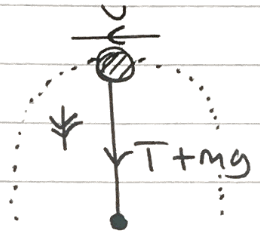
$\Delta KE = \Delta GPE$

$$\frac{9a\cancel{m}g}{2} = \frac{\cancel{m}v^2}{2} + a\cancel{m}g$$

$$\frac{v^2}{2} = \frac{9ag}{2} - ag$$

$$v^2 = 7ag \text{ at } B.$$

N2L (P at B) $\downarrow +$: $T + mg = \frac{mv^2}{a}$



$$T = \frac{m(7ag)}{a} - mg$$

$$T = 6mg > 0 \quad \text{at } B$$

hence P will pass through B

[(T > 0 at top) must be true.]

b) $\Delta GPE = \Delta KE$: $\frac{\cancel{m}v^2}{2} - \frac{\cancel{m}u^2}{2} = \cancel{m}ga$

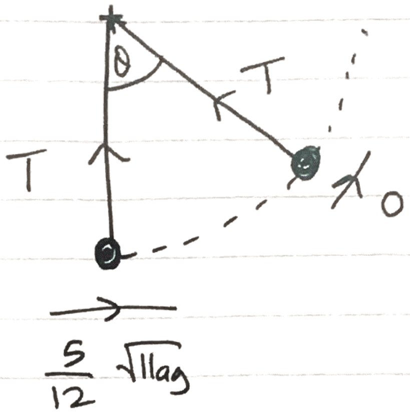
(from A to C)

$$\frac{v^2}{2} = ag + \frac{u^2}{2} \leftarrow \text{speed at } A.$$

$$v^2 = 2ag + (9ag) = 11ag$$

$$\therefore v = \sqrt{11ag}$$

c)



let angle with downward vertical be θ when P comes to rest.

Energy at C : $KE = \frac{1}{2}m\left(\frac{25}{144}\right)(11ag) = \frac{275agm}{288}$

$GPE = 0$

Energy at rest : $KE = 0$

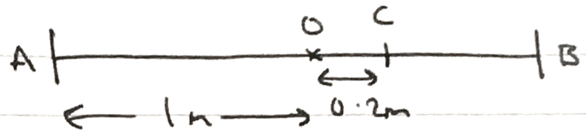
$GPE = mga(1-\cos\theta)$

C.O.E : $\frac{275\cancel{a}\cancel{g}\cancel{m}}{288} = \cancel{a}\cancel{g}\cancel{m}(1-\cos\theta)$

$\therefore \cos\theta = 1 - \frac{275}{288} = \frac{13}{288}$

$\therefore \theta = \cos^{-1}\frac{13}{288} = \boxed{87.4°}$

Q7a)



$$T_B = \frac{\lambda x}{l} = \frac{2\,(2-1-x-0.7)}{0.7} = \frac{2(0.3-x)}{0.7}$$

$$= \frac{2\,(3-10x)}{7}$$

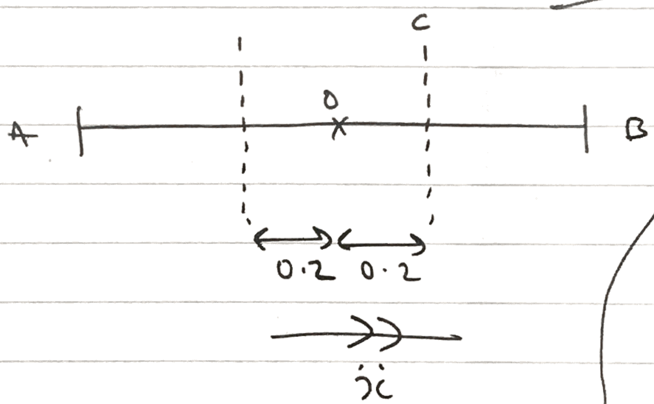
×10 to top and bottom.

[$l = 0.7$m as it is half of the whole string]

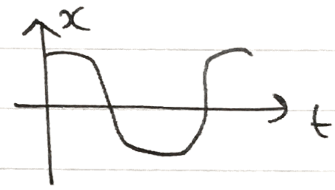
b) $T_A = \frac{2}{0.7}(1+x-0.7) = \frac{2}{0.7}(0.3+x)$

$$= \frac{2\,(3+10x)}{7}$$

c)



P starts at an endpoint (C), so $x = a\cos\omega t$ applies



$\therefore$ x is positive in the direction OC. So $\ddot{x}$ is positive in this direction.

$\overrightarrow{\text{N2L(P)}}^{+}$: $T_B - T_A = \frac{1}{2}\ddot{x}$

$$\frac{2}{7}(3-10x) - \frac{2}{7}(3+10x) = \frac{1}{2}\ddot{x}$$

$$\frac{6}{7} - \frac{6}{7} - \frac{40x}{7} = \frac{1}{2}\ddot{x}$$

So $\ddot{x} = -\frac{80}{7}x$ hence P moves with S.H.M.

$$\ddot{x} = \frac{-80x}{7} \longrightarrow \omega^2 = \frac{80}{7}$$

$$\therefore \omega = \sqrt{\frac{80}{7}}$$

$$T = 2\pi \times \frac{1}{\omega} = 2\pi\sqrt{\frac{7}{80}}$$

d) $V_{max} = a\omega = 0.2\sqrt{\frac{80}{7}} = \boxed{0.68}\ ms^{-1}$

e) $x = 0.2\cos\left(t\sqrt{\frac{80}{7}}\right)$

at D, $x = -0.1$

$$\frac{-0.1}{0.2} = \cos\left(t\sqrt{\frac{80}{7}}\right) = -\frac{1}{2}$$

$$\cos^{-1}\left(-\frac{1}{2}\right) = t\sqrt{\frac{80}{7}} = \frac{2\pi}{3}$$

$$\therefore t = \frac{2\pi}{3}\sqrt{\frac{7}{80}}$$