M3 June 2008 (MA)

Q1(a) Initially: $KE = 0$

$$EPE = \frac{\lambda\left(\frac{L}{2}\right)^2}{2L}$$

As it passes through the end: $KE = \frac{1}{2}m(2gL)$

$EPE = 0$

$\Delta KE = \Delta EPE$:
$$\frac{\frac{\lambda L^{\cancel{2}}}{4}}{2\cancel{L}} = mgL$$

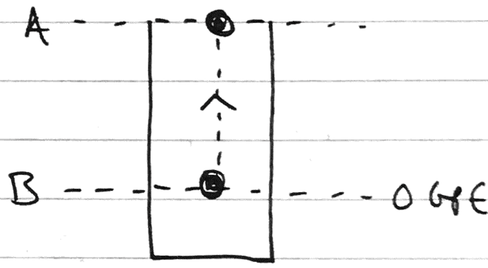
$$\Rightarrow \frac{\lambda \cancel{L}}{8} = mg\cancel{L}$$

$$\overset{\times 8}{\Rightarrow} \lambda = 8mg$$

b) At B: $KE = 0$

$GPE = 0$

$$EPE = \frac{\lambda\left(\frac{L}{2}\right)^2}{2L}$$



At A: $KE = \frac{1}{2}mu^2$

$GPE = mg\left(\frac{L}{2}\right)$

$EPE = 0$

C.O.E:
$$\frac{\lambda L}{8} = \frac{mu^2}{2} + \frac{mgL}{2}$$

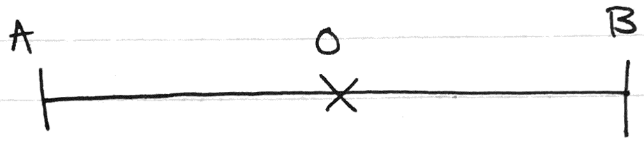
$$\cancel{m}gL = \frac{\cancel{m}u^2}{2} + \frac{\cancel{m}gL}{2}$$

$$\Rightarrow \quad \frac{gL}{2} = \frac{u^2}{2}$$

$$\Rightarrow \quad u^2 = \frac{2gL}{2} = gL$$

hence $\boxed{u = \sqrt{gL}}$

Q2a)



$T = 1.5 \times 2 = 3$ $\therefore$ $\omega = \frac{2\pi}{3}$ & $a = \frac{0.24}{2} = 0.12$

at O, P will have max speed.

$$V_{max} = a\omega = u$$

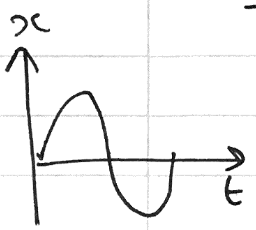
$$(0.12)\left(\frac{2\pi}{3}\right) = u = \boxed{\frac{2\pi}{25}} \ ms^{-1}$$

$$= (0.25 \ ms^{-1})$$

b) at $t=0$, P passes through O.

so $x = 0.12 \sin\left(\frac{2\pi t}{3}\right)$

$\underline{t=2}$: $x = 0.12 \sin\left(\frac{4\pi}{3}\right) = -0.104\ldots$



x is negative so P is between O and B.

$\therefore$ distance from B $= 0.12 - 0.104$

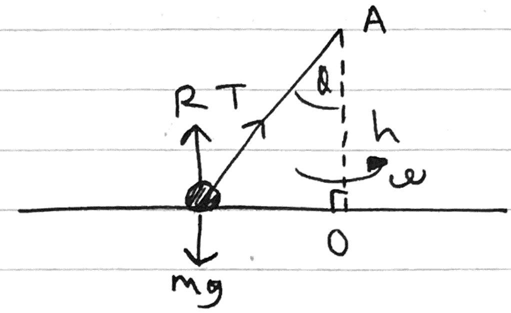
$= \boxed{0.016m}$

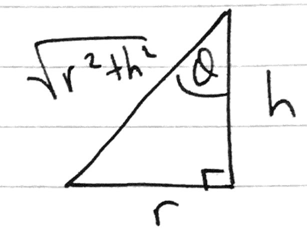
c) $v^2 = \omega^2(a^2 - x^2)$

$$v = \omega\sqrt{a^2 - x^2} = \frac{2\pi}{3}\sqrt{0.12^2 - (0.104..)^2} = \frac{\pi}{25}$$

$$= \boxed{0.13\,m/s}$$

Q3a)





$\tan\theta = \frac{r}{h}$.

$\underline{N2L(B)}\uparrow^{+}$: $R + T\cos\theta - mg = 0$

$$R = mg - T\cos\theta \rightarrow T\cos\theta = mg - R \sim 1$$

$\underline{N2L(B)}\xrightarrow{+}$: $T\sin\theta = mr\omega^2 \sim 2$

$$\frac{2}{1} : \frac{\cancel{T}\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg - R} = \tan\theta \mathbin{/\!/} = \frac{r}{h}$$

$$mr\omega^2 = \frac{r}{h}(mg) - \frac{r}{h}(R)$$

$$\frac{hm\cancel{r}\omega^2}{\cancel{r}} = mg - R$$

$$R = mg - hm\omega^2$$

notice that B must stay on the ground for this circular motion to continue.

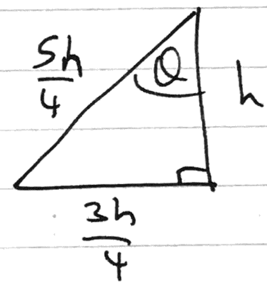
This means $R \geq 0$.

$R \geq 0$: $mg - hm\omega \geq 0$

$\Rightarrow \quad \omega h \leq g$

$\Rightarrow \quad \omega \leq \dfrac{g}{h}$

b) $\tan\theta = \frac{3}{4}$: $\frac{r}{h} = \frac{3}{4}$



so $\quad r = \frac{3}{4}h$

$$\sqrt{\left(\frac{3h}{4}\right)^2 + (h)^2} = \frac{5h}{4} = AB$$

$$T = \frac{\lambda x}{l} = \frac{2mg\left(\frac{5h}{4} - h\right)}{h} = \frac{mg}{2}$$

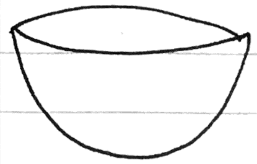
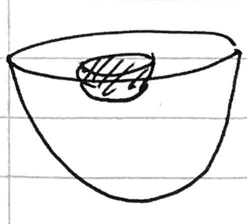
② : $T\sin\theta = mr\omega^2$

$$\frac{mg}{2}\left(\frac{3}{5}\right) = m\left(\frac{3h}{4}\right)\omega^2$$

$$\frac{3g}{10} = \frac{3h}{4}\omega^2 \qquad \therefore \quad \omega^2 = \frac{12g}{30h} = \frac{2g}{5h}$$

hence $\boxed{\omega = \sqrt{\frac{2g}{5h}}}$

Q4a)

	Shape	Mass (vol. ratio)	Distance of c.o.m from O
		$\frac{2}{3}\pi(6a)^3 = \boxed{144\pi a^3}$	$3\left(\frac{6a}{8}\right) = \boxed{\frac{9a}{4}}$
(−)		$\frac{2}{3}\pi(2a)^3 = \boxed{\frac{16}{3}\pi a^3}$	$3\left(\frac{2a}{8}\right) = \boxed{\frac{3a}{4}}$
		$= \left(144 - \frac{16}{3}\right)\pi a^3 = \boxed{\frac{416}{3}\pi a^3}$	$\boxed{\bar{y}}$

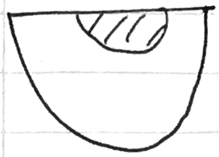
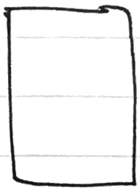
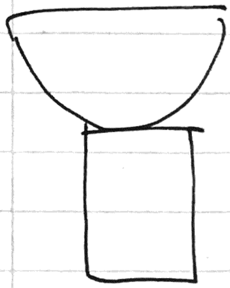
taking moments about O . . .

$$144\cancel{\pi a^3}\left(\frac{9a}{4}\right) - \frac{16}{3}\cancel{\pi a^3}\left(\frac{3a}{4}\right) = \frac{416}{3}\cancel{\pi a^3}(\bar{y})$$

$$324a - 4a = \frac{416}{3}\bar{y} = 320a$$

$$\therefore \bar{y} = \frac{320a \times 3}{416} = \frac{30a}{13}$$

b)

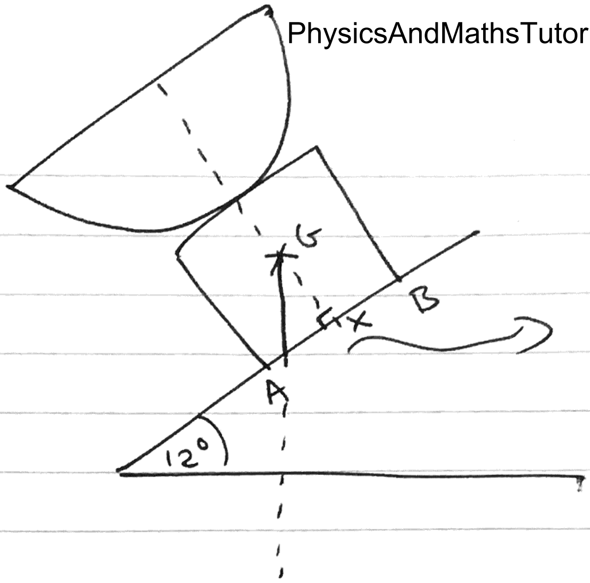
Shape	Mass (vol.) ratio	Distance of c.o.m from O
	$\boxed{\frac{416}{3}\pi a^3}$	$\boxed{\frac{30a}{13}}$
(+) 	$\pi(2a)^2(6a) = \boxed{24\pi a^3}$	$6a + 3a = \boxed{9a}$
	$\boxed{\frac{488}{3}\pi a^3}$	$\boxed{\bar{y}}$

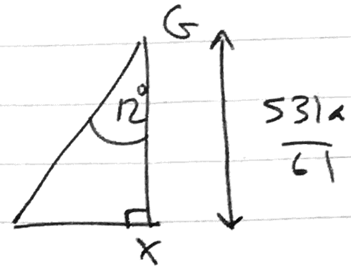
Moments about O . . .

$$\frac{416}{3}\left(\frac{30a}{13}\right) + 24(9a) = \frac{488}{3}(\bar{y})$$

$$\therefore \bar{y} = \frac{\left[\frac{416}{3}\left(\frac{30}{13}\right) + 24(9)\right]a}{\frac{488}{3}} = \boxed{\frac{201}{61}a}$$

c)





$$GX = 12a - \frac{201a}{61} = \frac{531a}{61}$$

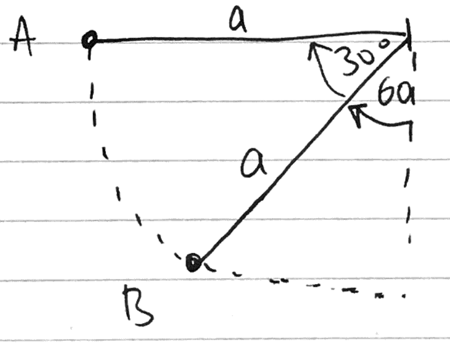
$$\tan 12 = \frac{\text{opposite}}{\frac{531a}{61}}$$

$$\therefore \text{opposite} = \frac{531a}{61} \times \tan 12 = 1.85a$$

$1.85a < 2a \therefore$ S won't topple.

[length AX = 2a. if S is on the point of toppling then the length of the 'opposite' side will be 2a.]

Q5a)



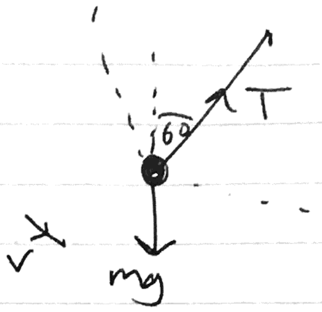
Energy from A to B:

$$\underbrace{\frac{mv^2}{2} - \frac{mu^2}{2}}_{\Delta KE} = \underbrace{mas\text{in}30}_{\Delta GPE}$$

$$(u=0) \Rightarrow \frac{v^2}{2} = a\sin 30$$

$$\therefore v^2 = 2a\sin 30 = a$$

At B :



N2L ↗+ : $T - mg\cos 60 = \dfrac{mv^2}{a}$

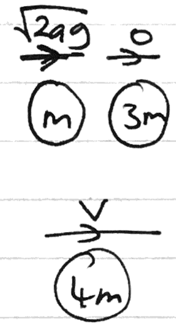
$$T = \frac{mg}{2} + \frac{m}{a}(a) = \boxed{\frac{3mg}{2}}$$

b) finding speed of P at lowest point via energy :

$\Rightarrow \quad mga = \dfrac{mv^2}{2} \qquad (\Delta GPE = \Delta KE).$

$\Rightarrow \quad v^2 = 2ag \qquad \therefore v = \sqrt{2ag}$.

Before :



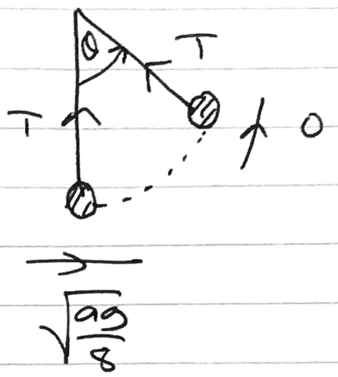
After :

considering the collision.

$\overrightarrow{C.L.M}$: $m(\sqrt{2ag}) + 0 = 4mV$

$$V = \frac{\sqrt{2ag}}{4} = \sqrt{\frac{ag}{8}}$$

ci)



<u>Using energy</u> : initially : $KE = \frac{1}{2}(4m)\left(\frac{ag}{8}\right)$

$GPE = 0$

finally : $KE = 0$

$GPE = 4mg(a - a\cos\theta)$

Total energy remains constant...

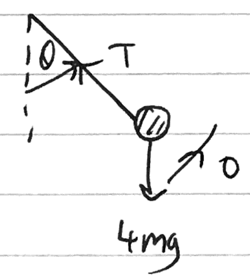
$$\therefore \quad 2m\left(\frac{ag}{8}\right) = 4mga(1-\cos\theta)$$

$$\frac{\cancel{a}m\cancel{g}}{4} = 4\cancel{a}m\cancel{g}(1-\cos\theta)$$

$$\frac{1}{16} = 1-\cos\theta$$

$$\therefore \cos\theta = \frac{15}{16} \rightarrow \theta = \cos^{-1}\frac{15}{16} = \boxed{20^\circ}$$

ii)



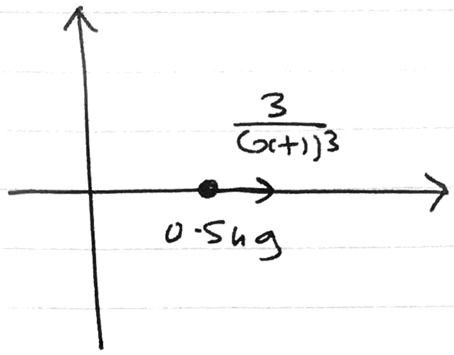
<u>N2L (particles)</u> $\nwarrow +$: $T - 4mg\cos\theta = 0$

$\left(\frac{mv^2}{r} = 0\right)$

$$\therefore T = 4mg \times \frac{15}{16}$$

$$T = \boxed{\frac{15mg}{4}}$$

Q6a)



$$F = 0.5a = \frac{3}{(x+1)^3}$$

$$0.5v\frac{dv}{dx} = \frac{3}{(x+1)^3}$$

$$\int(v)\,dv = 6\int(x+1)^{-3}\,dx$$

$$\frac{v^2}{2} = 6\left[-\frac{(x+1)^{-2}}{2}\right] + c$$

$$\frac{v^2}{2} = -3(x+1)^{-2} + c$$

$x=0,\ v=0$: $0 = -3(1) + c$

$$c = 3$$

$$\therefore \quad \frac{v^2}{2} = \frac{-3}{(x+1)^2} + 3$$

$\times 2$: $v^2 = 6 - \frac{6}{(x+1)^2} = 6\left(1 - \frac{1}{(x+1)^2}\right)$

b) as $x \to \infty$, $v^2 \to 6$ (as $\frac{1}{(x+1)^2} \to 0$)

hence $v \to \sqrt{6}$

so v can never reach $\sqrt{6}$ ms^{-1}

or you could say: $\underbrace{\forall x}$ $v^2 < 6$ hence $v < \sqrt{6}$.

this means 'for all x'.

c) $v = \sqrt{6} \times \sqrt{1 - \frac{1}{(x+1)^2}} = \frac{dx}{dt}$

$$\sqrt{6}\int (1)\, dt = \int \frac{1}{\sqrt{1-\left(\frac{1}{x+1}\right)^2}}\, dx = \int \frac{1}{\sqrt{\frac{x^2+2x+1-1}{(x+1)^2}}}\, dx$$

$$\int \frac{x+1}{\sqrt{x^2+2x}}\, dx = (\sqrt{6})t + c$$

$$\Rightarrow \frac{1}{2}\int (2x+2)(x^2+2x)^{-\frac{1}{2}}\, dx = \sqrt{6}\, t + c.$$

By Pattern.

$$\Rightarrow \frac{1}{2}\left[\frac{(x^2+2x)^{\frac{1}{2}}}{\frac{1}{2}}\right] = t\sqrt{6} + c$$

$$\Rightarrow \sqrt{x^2+2x} = t\sqrt{6} + c$$

$\underline{t=0, x=0}$: $c = 0$ //

hence $\Rightarrow$ $t\sqrt{6} = \sqrt{x^2+2x}$.

$\underline{t=2}$: $2\sqrt{6} = \sqrt{x^2+2x}$

$\Rightarrow$ $(2\sqrt{6})^2 = x^2+2x = 24$

$\Rightarrow$ $x^2 + 2x - 24 = 0$

$\Rightarrow$ $(x+6)(x-4) = 0$

$\underline{x=4}$ $(x \geqslant 0$ for all $t)$.