Solutionbank M2

Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 1

Question:

Whenever a numerical value of g is required, take $g = 9.8 \text{ m s}^{-2}$.

[*In this question* **i** *and* **j** *are perpendicular unit vectors in a horizontal plane.*]

A ball has a mass 0.2 kg. It is moving with velocity $(30\mathbf{i})$ m s^{-1} when it is struck by a bat. The bat exerts an impulse of $(-4\mathbf{i}+4\mathbf{j})$ Ns on the ball. Find

a the velocity of the ball immediately after the impact,

b the angle through which the ball is deflected as a result of the impact,

c the kinetic energy lost by the ball in the impact.

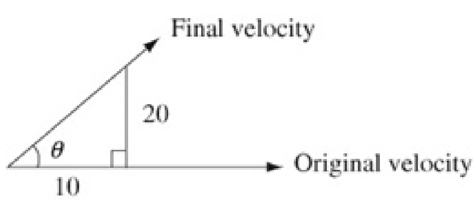
Solution:

a $-4\mathbf{i}+4\mathbf{j} = 0.2\mathbf{v} - 0.2\times 30\mathbf{i}$

$\mathbf{v} = 10\mathbf{i}+20\mathbf{j}$ m s^{-1}

Impulse = change of momentum

b



$\tan\theta = \frac{20}{10}$

$\theta = 63.4°$

c K. E. lost

$= \frac{1}{2}\times 0.2\times 30^2 - \frac{1}{2}\times 0.2(10^2+20^2)$

$= 40$ J

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2 Review Exercise
Exercise A, Question 2

Question:

A particle P of mass 0.75 kg is moving under the action of a single force $\mathbf{F}$ newtons. At time t seconds, the velocity $\mathbf{v}$ m s^{-1} of P is given by

$$\mathbf{v} = (t^2 + 2)\mathbf{i} - 6t\mathbf{j}.$$

a Find the magnitude of $\mathbf{F}$ when $t = 4$.

When $t = 5$, the particle P receives an impulse of magnitude $9\sqrt{2}$ Ns in the direction of the vector $\mathbf{i} - \mathbf{j}$.

b Find the velocity of P immediately after the impulse.

Solution:

a $\mathbf{v} = (t^2 + 2)\mathbf{i} - 6t\mathbf{j}$

$\mathbf{a} = 2t\mathbf{i} - 6\mathbf{j}$ — Differentiate to find $\mathbf{a}$.

$\mathbf{F} = 0.75(2t\mathbf{i} - 6\mathbf{j})$ — Use $\mathbf{F} = m\mathbf{a}$ to find $\mathbf{F}$.

$t = 4$ — Make $t = 4$.

$\mathbf{F} = 0.75(8\mathbf{i} - 6\mathbf{j})$

$= 6\mathbf{i} - 4.5\mathbf{j}$

$|\mathbf{F}| = \sqrt{(6^2 + 4.5^2)} = 7.5$ N — The magnitude of $\mathbf{F}$ is the modulus of the vector.

b

$\mathbf{I} = p\mathbf{i} - p\mathbf{j}$ — Impulse is parallel to $\mathbf{i} - \mathbf{j}$.

$|\mathbf{I}| = \sqrt{(p^2 + p^2)} = p\sqrt{2}$

But $|\mathbf{I}| = 9\sqrt{2} \therefore p = 9$

$9\mathbf{i} - 9\mathbf{j} = 0.75(\mathbf{v} - (27\mathbf{i} - 30\mathbf{j}))$ — Impulse = change of momentum

$36\mathbf{i} - 36\mathbf{j} = 3\mathbf{v} - 81\mathbf{i} + 90\mathbf{j}$ — For the initial velocity, $t = 5$.

$3\mathbf{v} = 117\mathbf{i} - 126\mathbf{j}$

$\mathbf{v} = 39\mathbf{i} - 42\mathbf{j}$ m s^{-1}

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2 Review Exercise
Exercise A, Question 3

Question:

A tennis ball of mass 0.2 kg is moving with velocity $(-10\mathbf{i})$ m s^{-1} when it is struck by a tennis racket. Immediately after being struck, the ball has velocity $(15\mathbf{i}+15\mathbf{j})$ m s^{-1}. Find

a the magnitude of the impulse exerted by the racket on the ball,

b the angle, to the nearest degree, between the vector **i** and the impulse exerted by the racket,

c the kinetic energy gained by the ball as a result of being struck.

Solution:

a

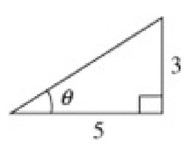
$$\mathbf{I} = 0.2((15\mathbf{i}+15\mathbf{j})-(-10\mathbf{i}))$$
$$= 5\mathbf{i}+3\mathbf{j}$$

Use impulse = change of momentum.

$$|\mathbf{I}| = \sqrt{(5^2+3^2)} = \sqrt{34}$$
$$= 5.83 \text{ Ns}$$

The magnitude of the impulse is the modulus of the vector.

b



$$\tan\theta = \frac{3}{5}$$
$$\theta = 31° \text{ (nearest degree)}$$

c K. E. gained

$$= \frac{1}{2}\times 0.2\times(15^2+15^2) - \frac{1}{2}\times 0.2\times 10^2$$
$$= 35 \text{ J}$$

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2 Review Exercise
Exercise A, Question 4

Question:

At time t seconds the acceleration, $\mathbf{a}$ m s^{-2}, of a particle P relative to a fixed origin O, is given by $\mathbf{a} = 2\mathbf{i} + 6t\mathbf{j}$. Initially the velocity of P is $(2\mathbf{i} - 4\mathbf{j})$ m s^{-1}.

a Find the velocity of P at time t seconds.

At time $t = 2$ seconds the particle P is given an impulse $(3\mathbf{i} - 1.5\mathbf{j})$ Ns.

Given that the particle P has mass 0.5 kg,

b find the speed of P immediately after the impulse has been applied.

Solution:

a

$\mathbf{v} = \int (2\mathbf{i} + 6t\mathbf{j})\, dt$

$= 2t\mathbf{i} + 3t^2\mathbf{j} + \mathbf{c}$ ← Don't forget the (vector) constant of integration.

$t = 0$ $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$

$\therefore \mathbf{c} = 2\mathbf{i} - 4\mathbf{j}$

$\mathbf{v} = (2t + 2)\mathbf{i} + (3t^2 - 4)\mathbf{j}$

b

$t = 2$ $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j}$ ← Find the velocity immediately before the impact.

$3\mathbf{i} - 1.5\mathbf{j} = 0.5(\mathbf{V} - (6\mathbf{i} + 8\mathbf{j}))$ ← Impulse = change of momentum.

$6\mathbf{i} - 3\mathbf{j} = \mathbf{V} - 6\mathbf{i} - 8\mathbf{j}$

$\mathbf{V} = 12\mathbf{i} + 5\mathbf{j}$

$|\mathbf{V}| = \sqrt{(12^2 + 5^2)} = 13$ ← Speed is the modulus of the velocity.

Speed = 13 m s^{-1}

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2 Review Exercise
Exercise A, Question 5

Question:

The unit vectors **i** and **j** lie in a vertical plane, **i** being horizontal and **j** vertical.

A ball of mass 0.1 kg is hit by a bat which gives it an impulse of $(3.5\mathbf{i}+3\mathbf{j})$ Ns. The velocity of the ball immediately after being hit is $(10\mathbf{i}+25\mathbf{j})$ m s^{-1}.

a Find the velocity of the ball immediately before it is hit.

In the subsequent motion the ball is modelled as a particle moving freely under gravity. When it is hit the ball is 1 m above horizontal ground.

b Find the greatest height of the ball above the ground in the subsequent motion.

The ball is caught when it is again 1 m above the ground.

c Find the distance from the point where the ball is hit to the point where it is caught.

Solution:

a

$3.5\mathbf{i}+3\mathbf{j} = 0.1[(10\mathbf{i}+25\mathbf{j})-(u\mathbf{i}+v\mathbf{j})]$

$u\mathbf{i}+v\mathbf{j} = (-25\mathbf{i}-5\mathbf{j})\ \text{m s}^{-1}$

Impulse = change of momentum .

b Vertical motion ↑ +ve :

$u = 25\ \text{m s}^{-1}$

$v = 0\ \text{m s}^{-1}$

$a = -9.8\ \text{m s}^{-2}$

$v^2 = u^2 + 2as$

$0 = 625 - 2\times 9.8s$

$s = \frac{625}{19.6}$

The **j** component is the vertical velocity.

Height above ground

$= \frac{625}{19.6} + 1 = 32.9\ \text{m}$

c Vertical motion ↑ +ve :

Use the vertical motion to find the time taken.

$s = 0$

$a = -9.8\ \text{m s}^{-2}$

$u = 25\ \text{m s}^{-1}$

$s = ut + \frac{1}{2}at^2$

$0 = 25t - \frac{1}{2}\times 9.8t^2$

$0 = t(25 - 4.9t)$

$t = \frac{25}{4.9}\ (\text{or } t = 0)$

At start.

Horizontal motion:

This will give you the distance.

$s = 10\times\frac{25}{4.9}$

$= 51\ \text{m}$

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2 Review Exercise
Exercise A, Question 6

Question:

Two particles, A and B, of mass m and $3m$ respectively, lie at rest on a smooth horizontal table. The coefficient of restitution between the particles is 0.25.

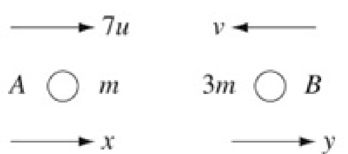
The particles A and B are given speeds of $7u$ and u respectively towards each other so that they collide directly. Find

a the speeds of A and B after the collision,

b the loss in kinetic energy due to the collision.

Solution:

a



$e = 0.25$ ← Draw a diagram!

$$7mu - 3mu = mx + 3my$$ ← Conservation of momentum.

$$4u = x + 3y \quad (1)$$

$$0.25 \times (7u + u) = y - x$$ ← Newton's Law of Restitution.

$$2u = y - x \quad (2)$$

$$(1)+(2) \quad 6u = 4y$$

$$y = \frac{3u}{2}$$

Solve (1) and (2) simultaneously.

$$\text{In (2)} \quad 2u = \frac{3u}{2} - x$$

$$x = -\frac{u}{2}$$ ← The minus sign shows the arrow in the diagram is pointing in the wrong direction.

A has speed $\frac{u}{2}$.

B has speed $\frac{3u}{2}$. ← Speed is always positive.

b K.E. lost

$$= \frac{1}{2} \times m \times (7u)^2 + \frac{1}{2} \times 3m \times u^2$$

$$- \left(\frac{1}{2} m \times \left(\frac{u}{2}\right)^2 + \frac{1}{2} \times 3m \times \left(\frac{3u}{2}\right)^2 \right)$$

$$= \frac{1}{2} m \times 49u^2 + \frac{3}{2} mu^2 - \left(\frac{mu^2}{8} + \frac{27mu^2}{8} \right)$$

$$= \frac{45}{2} mu^2$$

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2 Review Exercise

Exercise A, Question 7

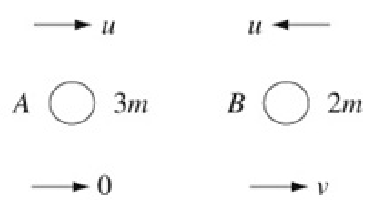
Question:

Two uniform smooth spheres A and B are of equal size and have masses $3m$ and $2m$ respectively. They are both moving in the same straight line with speed u, but in opposite directions, when they are in direct collision with each other. Given that A is brought to rest by the collision, find

a the coefficient of restitution between the spheres,

b the kinetic energy lost in the impact.

Solution:



a

$$3mu - 2mu = 2mv$$

$$mu = 2mv$$

Conservation of momentum.

$$e(u+u) = v$$

$$2ue = v$$

Newton's Law of Restitution.

Eliminating v:

$$mu = 2m(2ue)$$

$$e = \frac{1}{4}$$

b K.E. lost

$$= \frac{1}{2}\times 3mu^2 + \frac{1}{2}\times 2mu^2 - \left(0 + \frac{1}{2}\times 2m\left(\frac{1}{2}u\right)^2\right)$$

$$v = 2ue = 2u\times\frac{1}{4} = \frac{1}{2}u$$

$$= \frac{5}{2}mu^2 - \frac{1}{4}mu^2$$

$$= \frac{9}{4}mu^2$$

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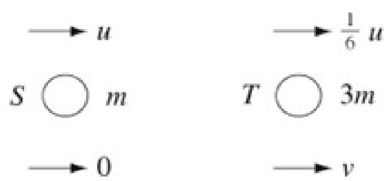
2 Review Exercise
Exercise A, Question 8

Question:

A smooth sphere S of mass m is moving on a smooth horizontal plane with speed u. It collides directly with another smooth sphere T, of mass $3m$, whose radius is the same as S. The sphere T is moving in the same direction as S with speed $\frac{1}{6}u$. The sphere S is brought to rest by the impact.

Find the coefficient of restitution between the spheres.

Solution:



$$mu + 3m \times \frac{1}{6}u = 3mv$$ ← Conservation of momentum.

$$\frac{3}{2}u = 3v$$

$$u = 2v$$

$$e(u - \frac{1}{6}u) = v$$ ← Newton's Law of Restitution.

$$\therefore \frac{5}{6}eu = \frac{1}{2}u$$

$$e = \frac{6}{5} \times \frac{1}{2}$$

$$e = \frac{3}{5}$$

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2 Review Exercise
Exercise A, Question 9

Question:

A smooth sphere S of mass m is moving with speed u on a smooth horizontal plane. The sphere S collides with another smooth sphere T, of equal radius to S but of mass km, moving in the same straight line and in the same direction with speed λu, $0 < \lambda < \frac{1}{2}$. The coefficient of restitution between S and T is e.

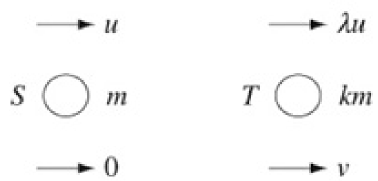
Given that S is brought to rest by the impact,

a show that $e = \dfrac{1+k\lambda}{k(1-\lambda)}$.

b Deduce that $k > 1$.

Solution:

a



$$mu + km\lambda u = kmv$$ ← Conservation of momentum.

$$u(1+k\lambda) = kv \quad (1)$$

$$v = e(u - \lambda u) = eu(1-\lambda) \quad (2)$$ ← Newton's Law of Restitution.

Eliminate v from (1) and (2)

$$u(1+k\lambda) = keu(1-\lambda)$$

$$e = \frac{1+k\lambda}{k(1-\lambda)}$$

b

$$e \le 1$$

$$\Rightarrow \quad 1+k\lambda \le k - k\lambda$$ ← Any coefficient of restitution satisfies $0 \le e \le 1$.

$$\frac{1}{1-2\lambda} \le k$$

but $0 < \lambda < \frac{1}{2} \Rightarrow 0 < 1 - 2\lambda < 1$ and $k > 1$.

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2 Review Exercise
Exercise A, Question 10

Question:

A particle A, of mass $2m$, is moving with speed u on a horizontal table when it collides directly with a particle B, of mass $3m$, which is at rest. The coefficient of restitution between the particles is e.

a Find, in terms of e and u, the velocities of A and B immediately after the collision.

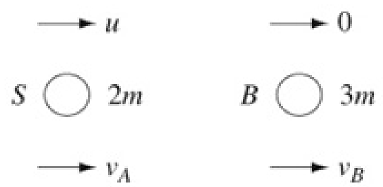
b Show that, for all possible values of e, the speed of A immediately after the collision is not greater than $\frac{2}{5}u$.

Given that the magnitude of the impulse exerted by B on A is $\frac{11}{5}mu$,

c find the value of e.

Solution:

a



$$2mu = 2mv_A + 3mv_B$$

Conservation of momentum.

$$2u = 2u_A + 3v_B \qquad (1)$$

$$eu = v_B - v_A \quad (2)$$

Newton's Law of Restitution.

$(1) + 2\times(2)$

$$2u + 2eu = 3v_B + 2v_B$$

$$v_B = \frac{2u}{5}(1+e)$$

In (1)

$$2u = 2v_A + \frac{6u}{5}(1+e)$$

$$v_A = u - \frac{3u}{5}(1+e)$$

$$v_A = \frac{u}{5}(2-3e)$$

Solve (1) and (2) simultaneously.

b

$e = 0 \; v_A = \frac{2u}{5}$

Using the smallest possible value for e gives the maximum velocity of A.

$e = 1 \; v_A = -\frac{u}{5}$

The maximum speed of A is $\frac{2u}{5}$.

Using the largest possible value for e gives the minimum velocity of A. This minimum must be considered as it may give a greater *speed*.

c For B:

$$\frac{11mu}{5} = 3m \times \frac{2u}{5}(1+e)$$

Impulse = change of momentum

$$11 = 6 + 6e$$

$$e = \frac{5}{6}$$

Work with *one* of A and B. It is easier to work with B as B's initial momentum is zero.

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2 Review Exercise
Exercise A, Question 11

Question:

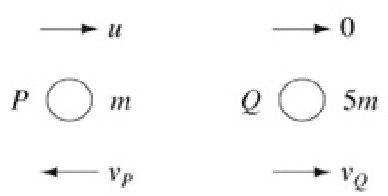
A sphere P, of mass m, is moving in a straight line with speed u on the surface of a smooth horizontal table. Another sphere Q, of mass $5m$ and having the same radius as P, is initially at rest on the table. The sphere P strikes the sphere Q directly, and the direction of motion of P is reversed by the impact. The coefficient of restitution between P and Q is e.

a Find an expression, in terms of u and e, for the speed of P after the impact.

b Find the set of possible values of e.

Solution:

a



The direction of motion of P is reversed by the impact.

$mu = 5mv_Q - mv_P$ ← Conservation of momentum

$u = 5v_Q - v_P$ (1)

$eu = v_P + v_Q$ (2) ← Newton's Law of Restitution

Eliminate v_Q ← v_Q is not needed!

$u = 5(eu - v_P) - v_P$

$6v_P = 5eu - u$

$v_P = \frac{u}{6}(5e - 1)$

b

$v_P > 0$ ← You know the direction for v_P is correct.

$\therefore 5e - 1 > 0 \quad e > \frac{1}{5}$

But

$0 \le e \le 1$ ← This is *always* true.

$\therefore \frac{1}{5} < e \le 1$

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2 Review Exercise
Exercise A, Question 12

Question:

A smooth sphere A of mass m is moving with speed u on a smooth horizontal table when it collides directly with another smooth sphere B of mass $3m$, which is at rest on the table. The coefficient of restitution between A and B is e. The spheres have the same radius and are modelled as particles.

a Show that the speed of B immediately after the collision is $\frac{1}{4}(1+e)u$.

b Find the speed of A immediately after the collision.

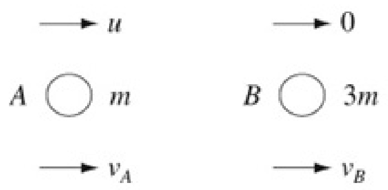
Immediately after the collision the total kinetic energy of the spheres is $\frac{1}{6}mu^2$.

c Find the value of e.

d Hence show that A is at rest after the collision.

Solution:

a



$$mu = mv_A + 3mv_B$$
$$u = v_A + 3v_B \quad (1)$$

Conservation of momentum.

$$eu = v_B - v_A \quad (2)$$

Newton's Law of Restitution.

$$(1)+(2) \quad u + eu = 4v_B$$
$$v_B = \frac{1}{4}(1+e)u$$

b Using (2):

$$v_A = v_B - eu$$
$$= \frac{1}{4}(1+e)u - eu$$
$$= \frac{1}{4}(1-3e)u$$

c K.E. after impact

$$= \frac{1}{2}mv_A^2 + \frac{1}{2}\times 3mv_B^2$$
$$= \frac{1}{2}m\left(\frac{1}{4}(1-3e)u\right)^2 + \frac{3}{2}m\left(\frac{1}{4}(1+e)u\right)^2$$
$$= \frac{1}{2}m\frac{u^2}{16}(1-6e+9e^2) + \frac{3}{2}m\frac{u^2}{16}\left(1+2e+e^2\right)$$
$$= \frac{mu^2}{32}\left(1-6e+9e^2+3+6e+3e^2\right)$$
$$= \frac{mu^2}{32}\left(4+12e^2\right)$$
$$= \frac{mu^2}{8}(1+3e^2)$$

$$\text{K.E. after impact} = \frac{1}{6}mu^2$$

From question.

$$\therefore \frac{1}{8}(1+3e^2) = \frac{1}{6}$$
$$6 + 18e^2 = 8$$
$$18e^2 = 2$$
$$e^2 = \frac{1}{9}$$
$$e = \frac{1}{3} \quad (e > 0)$$

d $v_A = \frac{u}{4}(1-3e)$

$$= \frac{u}{4}(1 - 3\times\frac{1}{3})$$
$$= 0$$

$\therefore$ A is at rest.

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2 Review Exercise
Exercise A, Question 13

Question:

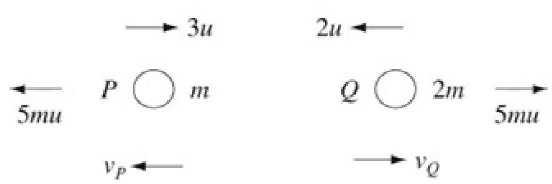
A particle P of mass m is moving with speed $3u$ in a straight line on a smooth horizontal plane. It collides with another particle Q of mass $2m$ which is moving with speed $2u$ along the same straight line but in the opposite direction. The coefficient of restitution between P and Q is e. The magnitude of the impulse given to each particle during the collision is $5mu$, and both P and Q have their directions of motion reversed by the collision.

a Show that $e = \frac{1}{2}$.

b Calculate the loss of kinetic energy due to the collision.

Solution:

a



For P:

$$5mu = mv_P - m(-3u)$$
$$v_P = 2u$$

Impulse = change of momentum, but watch signs.

For Q:

$$5mu = 2mv_Q - 2m(-2u)$$
$$v_Q = \frac{1}{2}u$$
$$e(3u + 2u) = v_P + v_Q$$
$$5eu = 2u + \frac{1}{2}u$$
$$e = \frac{2\frac{1}{2}}{5} = \frac{1}{2}$$

Newton's Law of Restitution.

b Loss of K.E.

$$= \frac{1}{2}m(3u)^2 + \frac{1}{2} \times 2m \times (2u)^2 - \left(\frac{1}{2}m(2u)^2 + \frac{1}{2} \times 2m \times \left(\frac{u}{2}\right)^2\right)$$
$$= \frac{9}{2}mu^2 + 4mu^2 - \left(2mu^2 + \frac{mu^2}{4}\right)$$
$$= \frac{25}{4}mu^2$$

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2 Review Exercise
Exercise A, Question 14

Question:

A smooth uniform sphere S of mass m is moving on a smooth horizontal plane with speed u. The sphere collides directly with another smooth uniform sphere T, of the same radius as S and a mass $2m$, which is at rest on the plane. The coefficient of restitution between the spheres is e.

a Show that the speed of T after the collision is $\frac{1}{3}u(1+e)$.

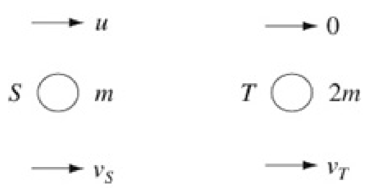
Given that $e > \frac{1}{2}$,

b **i** find the speed of S after the collision,

ii determine whether the direction of motion of S is reversed by the collision.

Solution:

a



$$mu = mv_S + 2mv_T$$
$$u = v_S + 2v_T \quad (1)$$
$$eu = v_T - v_S \quad (2)$$

Conservation of momentum.

Newton's Law of Restitution.

$$(1)+(2) \quad u + eu = 3v_T$$
$$v_T = \frac{1}{3}u(1+e)$$

b **i** from (2)

$$eu = \frac{1}{3}u(1+e) - v_S$$
$$v_S = \frac{1}{3}u(1+e) - eu$$
$$v_S = \frac{1}{3}u(1-2e)$$

but $e > \frac{1}{2} \Rightarrow 1-2e < 0$

$\therefore$ Speed of S is $\frac{1}{3}u(2e-1)$.

Speed must be positive.

ii The arrow in the diagram was the wrong way round, as shown in **b (i)**, so the direction of motion was reversed.

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2 Review Exercise
Exercise A, Question 15

Question:

A particle P of mass $3m$ is moving with speed $2u$ in a straight line on a smooth horizontal table. The particle P collides with a particle Q of mass $2m$ moving with speed u in the opposite direction to P. The coefficient of restitution between P and Q is e.

a Show that the speed of Q after the collision is $\frac{1}{5}u(9e+4)$.

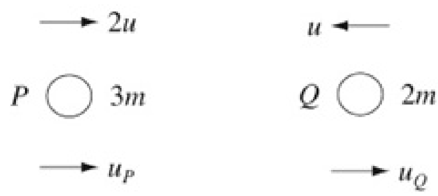
As a result of the collision, the direction of motion of P is reversed.

b Find the range of possible values of e.

c Given that the magnitude of the impulse of P on Q is $\frac{32}{5}mu$, find the value of e.

Solution:

a



Speed must be positive.

$$3m \times 2u - 2mu = 3mu_P + 2mu_Q$$
$$\therefore 4u = 3u_P + 2u_Q \quad (1)$$

Conservation of momentum.

$$e(2u + u) = u_Q - u_P$$
$$3eu = u_Q - u_P \quad (2)$$

Newton's Law of Restitution.

Eliminate u_P between (1) and (2):

$$4u = 3(u_Q - 3eu) + 2u_Q$$
$$4u = 5u_Q - 9eu$$
$$u_Q = \frac{1}{5}u(9e + 4)$$

b Using (2)

$$u_P = u_Q - 3eu$$
$$= \frac{1}{5}u(9e + 4) - 3eu$$
$$= \frac{2}{5}u(2 - 3e)$$

But

$$u_P < 0$$
$$\therefore 2 - 3e < 0$$

Direction of motion of P is reversed.

$$e > \frac{2}{3}$$
$$\therefore \frac{2}{3} < e \leq 1$$

Use the general condition $0 \leq e \leq 1$.

c For P:

$$\frac{32}{5}mu = 2m \times \frac{1}{5}u(9e + 4) + 2mu$$
$$32 = 2(9e + 4) + 10$$
$$18e = 14$$
$$e = \frac{7}{9}$$

Impulse = change of momentum.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 16

Question:

A small smooth ball A of mass m is moving on a horizontal table with speed u when it collides directly with another small smooth ball B of mass $3m$ which is at rest on the table. The balls have the same radius and the coefficient of restitution between the balls is e. The direction of motion of A is reversed as a result of the collision.

a Find, in terms of e and u the speeds of A and B immediately after the collision.

In the subsequent motion B strikes a vertical wall, which is perpendicular to the direction of motion of B, and rebounds.

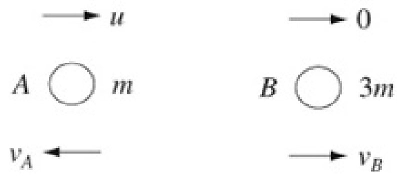
The coefficient of restitution between B and the wall is $\frac{3}{4}$.

Given that there is a second collision between A and B,

b find the range of values of e for which the motion described is possible.

Solution:

a



The direction of motion of A is reversed.

$mu = -mv_A + 3mv_B$ ← Conservation of momentum.

$u = 3v_B - v_A \quad (1)$

$eu = v_A + v_B \quad (2)$ ← Newton's Law of Restitution.

$(1)+(2) \quad u(1+e) = 4v_B$

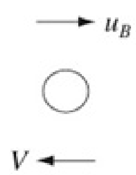
$$v_B = \frac{1}{4}u(1+e)$$

Using (2)

$eu = v_A + \frac{1}{4}u(1+e)$

$v_A = \frac{1}{4}u(3e-1)$

b



$V = \frac{3}{4} \times \frac{1}{4}u(1+e)$ ← Newton's Law of Restitution with $e = \frac{3}{4}$.

$= \frac{3}{16}u(1+e)$

For a second collision

$V > v_A$ ← B must move faster than A.

$\frac{3u}{16}(1+e) > \frac{1}{4}u(3e-1)$

$3 + 3e > 12e - 4$

$7 > 9e$

$e < \frac{7}{9}$

and

$v_A > 0$ ← The direction of motion of A is correct in the diagram.

$\therefore \quad e > \frac{1}{3}$

So $\frac{1}{3} < e < \frac{7}{9}$

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 13

Question:

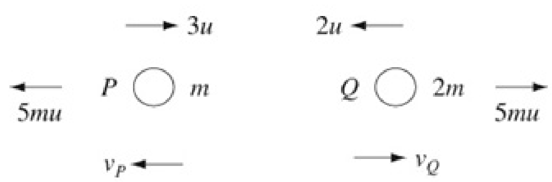
A particle P of mass m is moving with speed $3u$ in a straight line on a smooth horizontal plane. It collides with another particle Q of mass $2m$ which is moving with speed $2u$ along the same straight line but in the opposite direction. The coefficient of restitution between P and Q is e. The magnitude of the impulse given to each particle during the collision is $5mu$, and both P and Q have their directions of motion reversed by the collision.

a Show that $e = \frac{1}{2}$.

b Calculate the loss of kinetic energy due to the collision.

Solution:

a



For P:

$$5mu = mv_P - m(-3u)$$
$$v_P = 2u$$

Impulse = change of momentum, but watch signs.

For Q:

$$5mu = 2mv_Q - 2m(-2u)$$
$$v_Q = \frac{1}{2}u$$
$$e(3u + 2u) = v_P + v_Q$$
$$5eu = 2u + \frac{1}{2}u$$
$$e = \frac{2\frac{1}{2}}{5} = \frac{1}{2}$$

Newton's Law of Restitution.

b Loss of K.E.

$$= \frac{1}{2}m(3u)^2 + \frac{1}{2} \times 2m \times (2u)^2 - \left(\frac{1}{2}m(2u)^2 + \frac{1}{2} \times 2m \times \left(\frac{u}{2}\right)^2\right)$$
$$= \frac{9}{2}mu^2 + 4mu^2 - \left(2mu^2 + \frac{mu^2}{4}\right)$$
$$= \frac{25}{4}mu^2$$

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 18

Question:

A smooth sphere P of mass $2m$ is moving in a straight line with speed u on a smooth horizontal table. Another smooth sphere Q of mass m is at rest on the table. The sphere P collides directly with Q. The coefficient of restitution between P and Q is $\frac{1}{3}$. The spheres are modelled as particles.

a Show that, immediately after the collision, the speeds of P and Q are $\frac{5}{9}u$ and $\frac{8}{9}u$ respectively.

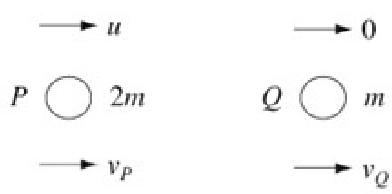
After the collision, Q strikes a fixed vertical wall which is perpendicular to the direction of motion of P and Q. The coefficient of restitution between Q and the wall is e. When P and Q collide again, P is brought to rest.

b Find the value of e.

c Explain why there must be a third collision between P and Q.

Solution:

a



$$2mu = 2mv_P + mv_Q$$

Conservation of momentum.

$$2u = 2v_P + v_Q \quad (1)$$

$$\frac{1}{3}u = v_Q - v_P \quad (2)$$

Newton's Law of Restitution.

$$(1) + 2 \times (2)\ \frac{8u}{3} = v_Q + 2v_Q$$

$$3v_Q = \frac{8u}{3}$$

$$v_Q = \frac{8u}{9}$$

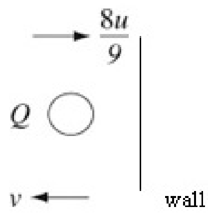
Using (2)

$$v_P = v_Q - \frac{1}{3}u$$

$$v_P = \frac{8u}{9} - \frac{1}{3}u$$

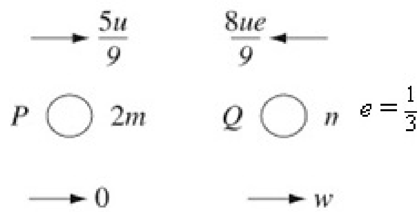
$$v_P = \frac{5u}{9}$$

b



$$v = \frac{8ue}{9}$$

Newton's Law of Restitution.



Here, $e = \frac{1}{3}$ again.

$$2m \times \frac{5u}{9} - m \times \frac{8ue}{9} = mw$$

Conservation of momentum.

$$10u - 8ue = 9w \qquad (3)$$

$$\frac{1}{3}\left(\frac{5u}{9} + \frac{8ue}{9}\right) = w$$

Newton's Law of Restitution.

$$5u + 8ue = 27w \qquad (4)$$

Eliminating w between (3) and (4):

$$3(10u - 8ue) = 5u + 8ue$$
$$30u - 24ue = 5u + 8ue$$
$$32ue = 25u$$
$$e = \frac{25}{32}$$

c Q is now moving towards the wall once more.
After Q hits the wall, it will return to collide with P once more.

Solutionbank M2

Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 19

Question:

Two small smooth spheres, P and Q of equal radius, have masses $2m$ and $3m$ respectively. The sphere P is moving with speed $5u$ on a smooth horizontal table when it collides directly with Q which is at rest on the table. The coefficient of restitution between P and Q is e.

a Show that the speed of Q immediately after the collision is $2(1+e)u$.

After the collision, Q hits a smooth vertical wall which is at the edge of the table and perpendicular to the direction of motion of Q. The coefficient of restitution between Q and the wall is f, $0 < f \leq 1$.

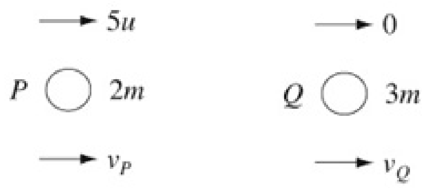
b Show that, when $e = 0.4$, there is a second collision between P and Q.

Given that $e = 0.8$ and there is a second collision between P and Q,

c find the range of possible values of f.

Solution:

a



$$2m \times 5u = 2mv_P + 3mv_Q$$

Conservation of momentum.

$$10u = 2v_P + 3v_Q \quad (1)$$

$$e \times 5u = v_Q - v_P \quad (2)$$

Newton's Law of Restitution.

$(1) + 3 \times (2)$:

$$10u + 15eu = 3v_Q + 2v_Q$$

$$10u + 15eu = 5v_Q$$

$$v_Q = 2u + 3eu = 2(1+e)u$$

b From (2)

$$v_P = v_Q - 5eu$$

$$= 2(1+e)u - 5eu$$

Find direction of motion for P, as if P is moving towards the wall there must be a second collision between P and Q.

$$v_P = 2 \times 1.4u - 5 \times 0.4u$$

$$= 0.8u$$

$e = 0.4$ in **b**.

$v_P > 0 \therefore P$ moves towards the wall and will collide with Q after Q rebounds from the wall.

c

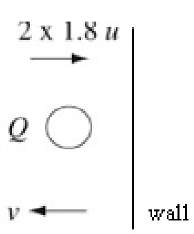
$$e = 0.8$$

$$v_P = 2 \times 1.8u - 5 \times 0.8u$$

$$= -0.4u$$

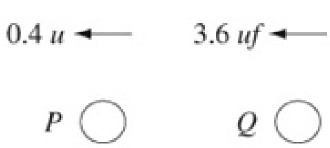
Q hits the wall:

$e = 0.8$



Newton's Law of Restitution.

$$v = 3.6uf$$



For a second collision

$$3.6uf > 0.4u$$

$$f > \frac{0.4}{3.6} = \frac{1}{9}$$

Range for f is

$$\frac{1}{9} < f \leq 1$$

All coefficients of restitution are less than or equal to 1.

Solutionbank M2
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 20

Question:

A particle A of mass $2m$, moving with speed $2u$ in a straight line on a smooth horizontal table, collides with a particle B of mass $3m$, moving with speed u in the same direction as A. The coefficient of restitution between A and B is e.

a Show that the speed of B after the collision is

$$\frac{1}{5}u(7+2e).$$

b Find the speed of A after the collision, in terms of u and e.

The speed of A after the collision is $\frac{11}{10}u$.

c Show that $e=\frac{1}{2}$.

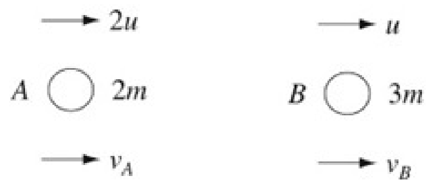
At the instant of collision, A and B are at a distance d from a vertical barrier fixed to the surface at right-angles to their direction of motion. Given that B hits the barrier, and that the coefficient of restitution between B and the barrier is $\frac{11}{16}$,

d find the distance of A from the barrier at the instant that B hits the barrier,

e show that, after B rebounds from the barrier, it collides with A again at a distance $\frac{5}{32}d$ from the barrier.

Solution:

a



$$2m \times 2u + 3m \times u = 2mv_A + 3mv_B$$

Conservation of momentum.

$$7u = 2v_A + 3v_B \quad (1)$$

$$e(2u - u) = v_B - v_A$$

Newton's Law of Restitution.

$$eu = v_B - v_A \quad (2)$$

$(1) + 2 \times (2)$

$$7u + 2eu = 3v_B + 2v_B$$

$$v_B = \frac{1}{5}u(7 + 2e)$$

b Using (2)

$$v_A = v_B - eu$$

$$= \frac{1}{5}u(7 + 2e) - eu$$

$$= \frac{1}{5}u(7 - 3e)$$

c $\frac{1}{5}u(7 - 3e) = \frac{11u}{10}$

$$14u - 6eu = 11u$$

$$6eu = 3u \quad e = \frac{1}{2}$$

d For B:

Distance to barrier $= d$

Speed $= \frac{1}{5}u(7+1) = \frac{8u}{5}$

Use $e = \frac{1}{2}$ now.

$\therefore$ Time to barrier $= d \div \frac{8u}{5} = \frac{5d}{8u}$

Distance moved by A in this time:

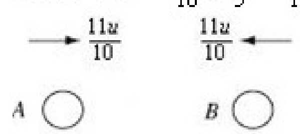
$= \frac{1}{5}u\left(7 - \frac{3}{2}\right) \times \frac{5d}{8u}$

$= \frac{11u}{5 \times 2} \times \frac{5d}{8u} = \frac{11d}{16}$

$\therefore$ A is $d - \frac{11d}{16} = \frac{5d}{16}$ from the barrier.

e After B hits the barrier:

Speed of $B = \frac{11}{16} \times \frac{8u}{5} = \frac{11u}{10}$



Equal speeds, opposite directions.

$\therefore$ A and B will collide at mid-point of the distance from A to the barrier at the instant B hits the barrier, i.e. they collide at distance $\frac{5d}{32}$ from the barrier.

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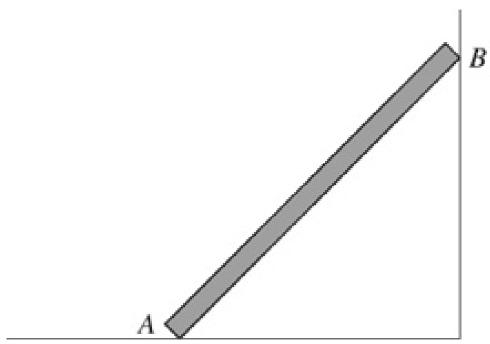
2 Review Exercise
Exercise A, Question 21

Question:



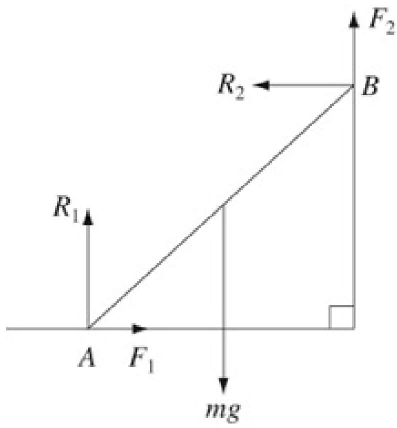
The diagram shows a uniform heavy plank of wood AB, of mass m, whose lower end A is resting on rough horizontal ground and whose upper end B is resting against a rough vertical wall. The coefficient of friction between the plank and the ground and between the plank and the wall is $\frac{2}{3}$.

The plank is about to slip at both ends.

a Suggest a suitable model for the plank so that the forces exerted on it by the ground and the wall can be found.

b Show that the horizontal component of the force exerted by the wall on the plank is $\frac{6mg}{13}$.

Solution:



a uniform rod

b

$$R(\rightarrow) \quad R_2 = F_1$$

At the ground $F_1 = \frac{2}{3}R_1$ — Friction is limiting.

$$\therefore R_2 = \frac{2}{3}R_1$$

$$R(\uparrow) \quad F_2 + R_1 = mg$$

At the wall $F_2 = \frac{2}{3}R_2$ — Friction is limiting.

$$\therefore \frac{2}{3}R_2 + \frac{3}{2}R_2 = mg$$

Substitute for F_2 and R_1 in $F_2 + R_1 = mg$.

$$\frac{13}{6}R_2 = mg$$

$$R_2 = \frac{6}{13}mg$$

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Edexcel AS and A Level Modular Mathematics

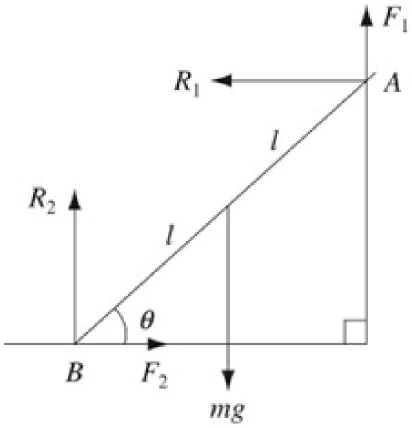
2 Review Exercise
Exercise A, Question 22

Question:

A uniform rod, of mass m, rests with one end A against a rough vertical wall and the other end B on a rough horizontal floor. The vertical plane through the rod is perpendicular to the wall. The coefficient of friction between the wall and the rod is μ_1. The coefficient of friction between the floor and the rod is μ_2. Given that θ is the inclination of the rod to the floor when the rod is on the point of slipping, show that

$$2\mu_2 \tan\theta = 1 - \mu_1\mu_2.$$

Solution:



Let $AB = 2l$

$F_1 = \mu_1 R_1 \quad F_2 = \mu_2 R_2$

The rod must be on the point of slipping at both ends.

M(B) $\quad mgl\cos\theta = F_1 \times 2l\cos\theta + R_1 \times 2l\sin\theta$

$mg\cos\theta = 2\mu_1 R_1 \cos\theta + 2R_1 \sin\theta \quad (1)$

R($\uparrow$) $\quad F_1 + R_2 = mg$

$\mu_1 R_1 + R_2 = mg \quad (2)$

R($\rightarrow$) $\quad F_2 = R_1$

$\therefore R_1 = \mu_2 R_2 \quad (3)$

Resolving will not use the angle θ.

From (2) and (3)

$R_1 = \mu_2(mg - \mu_1 R_1)$

$R_1(1 + \mu_1\mu_2) = \mu_2 mg$

R_1 is needed to substitute in equation (1).

$$R_1 = \frac{\mu_2 mg}{(1+\mu_1\mu_2)}$$

Substitute in (1)

$$mg\cos\theta = 2\mu_1\cos\theta \times \frac{\mu_2 mg}{(1+\mu_1\mu_2)} + 2\sin\theta \times \frac{\mu_2 mg}{(1+\mu_1\mu_2)}$$

$$1 = \frac{2\mu_1\mu_2}{(1+\mu_1\mu_2)} + \frac{2\tan\theta \times \mu_2}{(1+\mu_1\mu_2)}$$

$$1 + \mu_1\mu_2 = 2\mu_1\mu_2 + 2\mu_2\tan\theta$$

$$2\mu_2\tan\theta = 1 - \mu_1\mu_2$$

Cancel mg and divide by $\cos\theta$ using $\frac{\sin\theta}{\cos\theta} = \tan\theta$.
R_1 is needed to substitute in equation (1).

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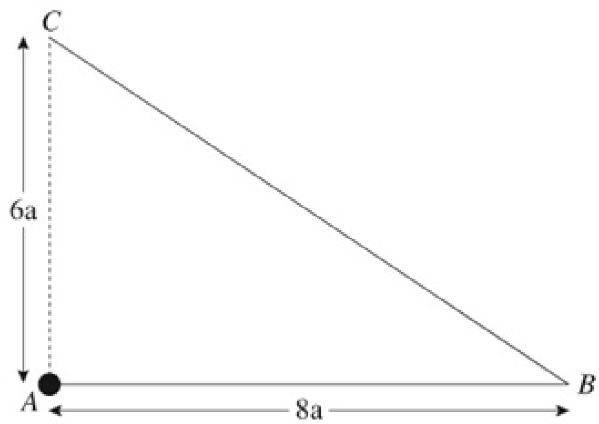
2 Review Exercise
Exercise A, Question 23

Question:

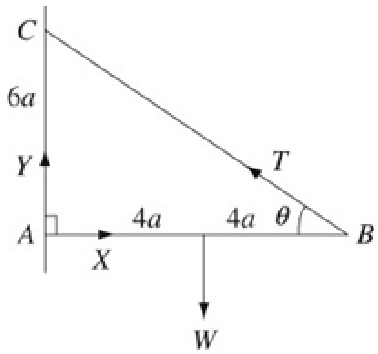


A uniform rod AB, of length $8a$ and weight W, is free to rotate in a vertical plane about a smooth pivot at A. One end of a light inextensible string is attached to B. The other end is attached to point C which is vertically above A, with $AC = 6a$. The rod is in equilibrium with AB horizontal, as shown in the diagram.

a By taking moments about A, or otherwise, show that the tension in the string is $\frac{5}{6}W$.

b Calculate the magnitude of the horizontal component of the force exerted by the pivot on the rod.

Solution:



a M(A) $T\sin\theta \times 8a = W \times 4a$

$BC = 10a \Rightarrow \sin\theta = \frac{3}{5}$ ← By Pythagoras or recognition of a (3, 4, 5) Δ.

$T \times \frac{3}{5} \times 8 = 4w$

$T = \frac{5w}{6}$

b R(→) $X = T\cos\theta$

$X = \frac{5w}{6} \times \frac{4}{5}$

$X = \frac{2}{3}W$

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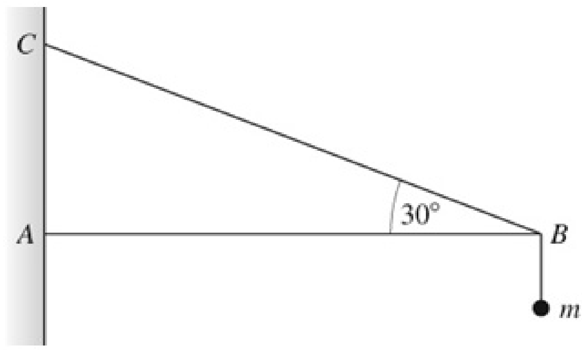
2 Review Exercise
Exercise A, Question 24

Question:

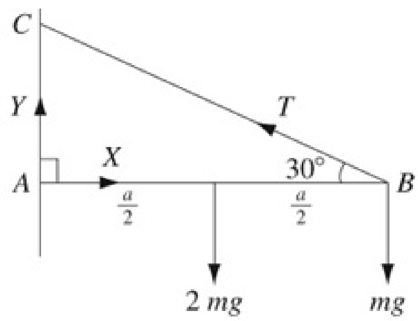


A uniform rod AB has mass $2m$ and length a. The end A is smoothly hinged at a fixed point. A particle of mass m is suspended from the rod at the end B. The loaded rod is held in equilibrium in a horizontal position by a light string, one end of which is attached to the rod at B, the other end being fixed to a point C vertically above A, as shown in the diagram. The string makes an angle of $30°$ with the horizontal.

a Show that the tension in the string is $4mg$.

b Find the magnitude of the force exerted by the hinge on the rod at A.

Solution:



a M(A) $2mg \times \frac{a}{2} + mga = Ta\sin 30°$

$2mg = T \times \frac{1}{2}$

$T = 4mg$

Taking moments about A gives an equation without X or Y.

b R(↑) $Y + T\sin 30° = 2mg + mg$

$Y = 3mg - 4mg\sin 30°$

$Y = 3mg - 4mg \times \frac{1}{2}$

$Y = mg$

R(→) $X = T\cos 30°$

$X = 4mg \times \frac{\sqrt{3}}{2}$

$X = 2mg\sqrt{3}$

Resultant $= \sqrt{[1 + (2\sqrt{3})^2]}\,mg$

$= mg\sqrt{13}$

Use the exact value of $\cos 30°$ for an exact final answer.

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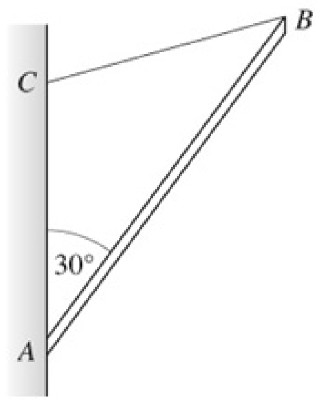
Edexcel AS and A Level Modular Mathematics

2 Review Exercise
Exercise A, Question 25

Question:



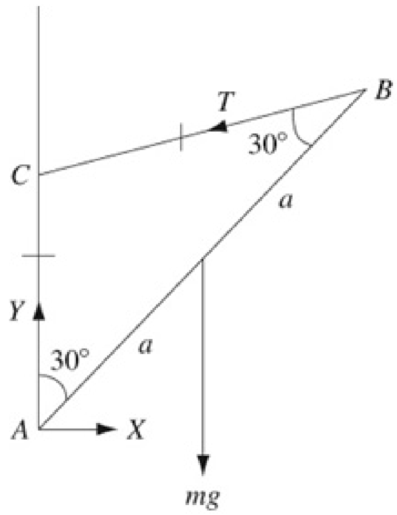
A uniform rod AB of mass m and length $2a$ is smoothly hinged to a vertical wall at A and is supported in equilibrium by a rope which is modelled as a light string. One end of the rope is attached to the end B of the rod and the other end is attached to a point C of the wall, where C is vertically above A, $AC = CB$, and $\angle CAB = 30°$, as shown in the diagram.

a Show that the tension in the rope is $0.5mg$.

b Find the magnitude of the vertical component of the force acting on the rod at A.

c If the rope were not modelled as a light string, state how this would affect the tension throughout the rope.

Solution:



ΔABC is isoceles.

a M(A) $mga \sin 30° = T \times 2a \sin 30°$

$2T = mg$

$T = 0.5mg$

Taking moments about A gives an equation in which the only unknown is T.

b R(↑) $Y = mg + T\cos 60°$

$Y = mg + \frac{1}{2}mg \times \frac{1}{2}$

$Y = \frac{5mg}{4}$

c The tension would not be constant throughout the length of the string.

Solutionbank M2

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2 Review Exercise
Exercise A, Question 26

Question:

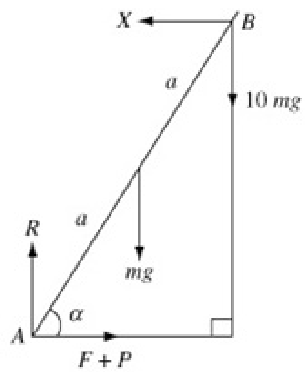
A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The coefficient of friction between the ladder and the ground is 0.6. The other end B of the ladder rests against a smooth vertical wall.

A builder of mass $10m$ stands at the top of the ladder. To prevent the ladder from slipping, the builder's friend pushes the bottom of the ladder horizontally towards the wall with a force of magnitude P. This force acts in a direction perpendicular to the wall. The ladder rests in equilibrium in a vertical plane perpendicular to the wall and makes an angle α with the horizontal, where $\tan\alpha = \frac{3}{2}$.

a Show that the reaction of the wall on the ladder has magnitude $7mg$.

b Find, in terms of m and g, the range of values of P for which the ladder remains in equilibrium.

Solution:



a M(A) $X \times 2a \sin\alpha = 10mg \times 2a\cos\alpha + mg \times a\cos\alpha$

$$2X\tan\alpha = 20mg + mg$$

Divide by $\cos\alpha$ as you know the value of $\tan\alpha$.

$$2X \times \frac{3}{2} = 20mg + mg$$

$$3X = 21mg$$

$$X = 7mg$$

b R(↑) $R = 10mg + mg = 11mg$

R(→) $F + P = X$

$$P = X - F$$

P is minimum when F acts towards the wall and has its maximum magnitude.

$$F = \mu R = 0.6 \times 11mg = 6.6mg$$

$$\therefore P_{\min} = 7mg - 6.6mg$$

$$= 0.4mg$$

P is maximum when F acts away from the wall and has its maximum magnitude.

$$P_{\max} = 7mg - (-6.6mg)$$

$$= 13.6mg$$

$$\therefore 0.4mg \le P \le 13.6mg$$

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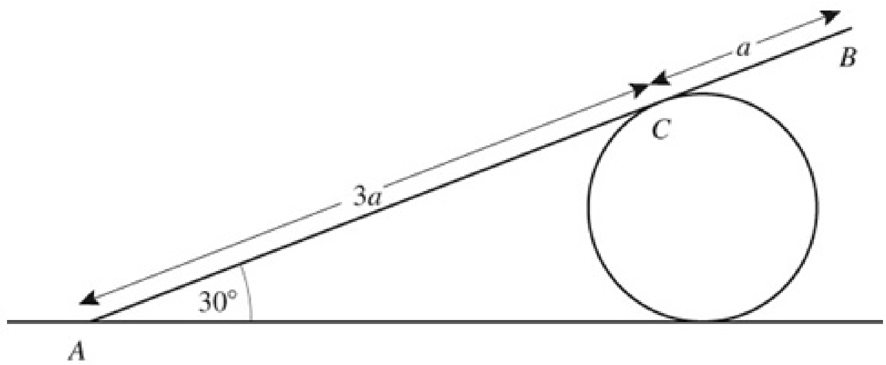
2 Review Exercise
Exercise A, Question 27

Question:



A piece of equipment used in an acrobatic show consists of a smooth cylinder which is fixed, with its axis horizontal, to a rough horizontal plane. A plank, which is modelled as a uniform rod AB of mass m and length $4a$, rests in equilibrium on the cylinder at the point C, where $AC = 3a$.

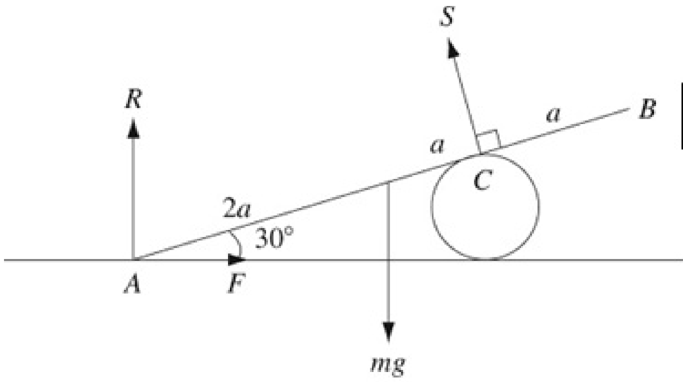
The end A of the plank rests on the plane and AB makes an angle of $30°$ with the horizontal, as shown in the diagram. The points A, B and C lie in a vertical plane which is perpendicular to the axis of the cylinder.

a Find the magnitude of the force exerted on the plank by the cylinder at the point C.

Given that the plank is in limiting equilibrium and that the coefficient of friction between the plank and the plane is μ,

b show that $\mu = \frac{1}{3}\sqrt{3}$.

Solution:



There is no frictional force at C.

a $M(A)$ $S \times 3a = mg \times 2a\cos 30°$

$3S = 2mg \times \frac{\sqrt{3}}{2}$

$S = mg\frac{\sqrt{3}}{2}$

Take moments about A for an equation without R and F.

b $F = \mu R$

$R(\uparrow)$ $R = mg - S\cos 30°$

$R = mg - mg\frac{\sqrt{3}}{3} \times \frac{\sqrt{3}}{2}$

$R = mg - \frac{1}{2}mg = \frac{1}{2}mg$

$R(\rightarrow)$ $F = S\cos 60°$

$F = mg\frac{\sqrt{3}}{3} \times \frac{1}{2}$

Friction is limiting.

$\mu = \frac{F}{R} = \frac{\sqrt{3}}{6} \div \frac{1}{2} = \frac{\sqrt{3}}{3}$

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2 Review Exercise
Exercise A, Question 28

Question:

A non-uniform ladder AB, of length $4a$ and weight W, has its centre of mass at a distance a from B. The ladder rests with A against a rough vertical wall and with its lower end B on smooth horizontal ground.

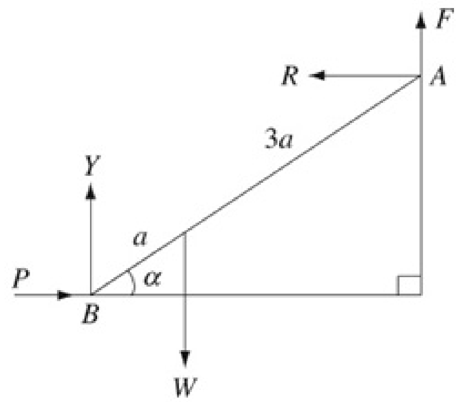
The coefficient of friction between the wall and the ladder is μ. The ladder is in a vertical plane perpendicular to the wall and makes an angle α with the horizontal where $\tan \alpha = 2$. A man can just prevent the ladder from slipping down by applying a horizontal force of magnitude P, perpendicular to the wall, at B. The ladder is modelled as a non-uniform rod.

a Draw a diagram showing all the forces acting on the ladder.

b Find an expression for P in terms of W and μ.

Solution:

a



b

$$M(A)\ W \times 3a\cos\alpha + P \times 4a\sin\alpha = Y \times 4a\cos\alpha$$

$$3W + 4P\tan\alpha = 4Y$$

Divide by $\cos\alpha$ and use $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$.

$$3W + 8P = 4Y \quad (1)$$

$$R(\rightarrow)\quad P = R \quad (2)$$

$$R(\uparrow)\quad F + Y = W \quad (3)$$

$$F = \mu R \quad (4)$$

Friction is limiting.

Using (3) and (4)

$$Y = W - \mu R$$

Using this in (1)

$$3W + 8P = 4(W - \mu R)$$

and since $P = R$ (from (2))

$$3W + 8P = 4(W - \mu P)$$

$$3W + 8P = 4W - 4\mu P$$

$$P(4\mu + 8) = W$$

$$P = \frac{W}{4\mu + 8}$$

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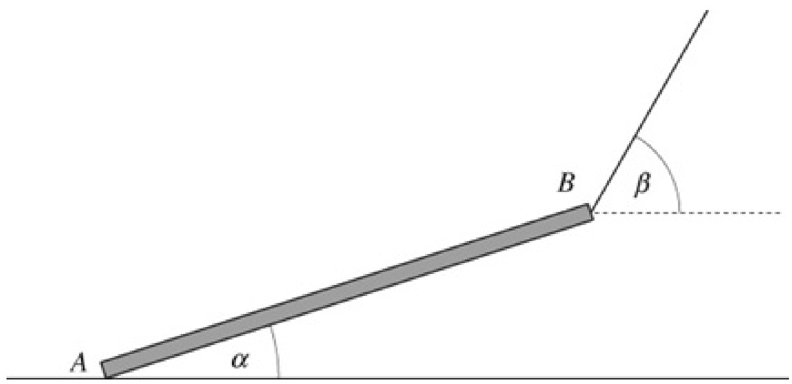
2 Review Exercise
Exercise A, Question 29

Question:



A straight log AB has weight W and length $2a$. A cable is attached to one end B of the log. The cable lifts the end B off the ground. The end A remains in contact with the ground, which is rough and horizontal. The log is in limiting equilibrium. The log makes an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$. The cable makes an angle β to the horizontal, as shown in the diagram. The coefficient of friction between the log and the ground is 0.6. The log is modelled as a uniform rod and the cable as light.

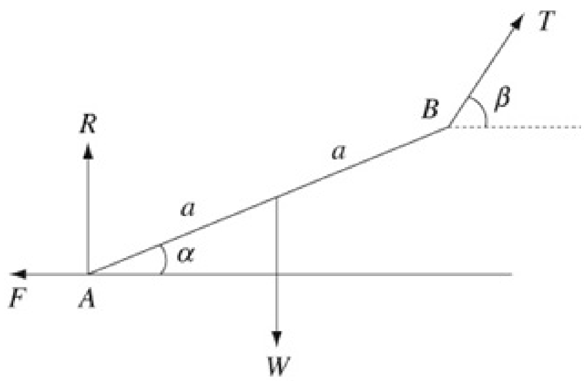
a Show that the normal reaction on the log at A is $\frac{2}{5}W$.

b Find the value of β.

The tension in the cable is kW.

c Find the value of k.

Solution:



a

$$\text{M}(B)\ R\times 2a\cos\alpha + F\times 2a\sin\alpha = W\times a\cos\alpha$$

T is not wanted.

$$2R+2F\tan\alpha = W$$

Divide by $\cos\alpha$ as the value of $\tan\alpha$ is known.

$$2R+2F\times\tfrac{5}{12} = W$$

$$2R+\tfrac{5}{6}F = W$$

$$F=\mu R = 0.6R$$

Friction is limiting.

$$\therefore 2R+\tfrac{5}{6}\times 0.6R = W$$

$$2.5R = W$$

$$R = \frac{W}{2.5}=\frac{2W}{5}$$

b

$\text{R}(\rightarrow)\quad T\cos\beta = F$

As $F=0.6R$ and $R=\frac{2}{5}W$

$$T\cos\beta = 0.6\times\tfrac{2}{5}W$$

$$T\cos\beta = \tfrac{6}{25}W \quad (1)$$

$$\text{R}(\uparrow)\quad T\sin\beta+R = W$$

$$T\sin\beta = W-\tfrac{2}{5}W$$

$$T\sin\beta = \tfrac{3}{5}W \quad (2)$$

$$(2)\div(1)\quad \frac{T\sin\beta}{T\cos\beta} = \frac{3}{5}W\times\frac{25}{6W}$$

$$\tan\beta = \tfrac{5}{2}$$

$$\beta = 68.2^\circ$$

c

$$T = \frac{3}{5}W\div\sin 68.2$$

$$= 0.646W$$

$$k = 0.646$$

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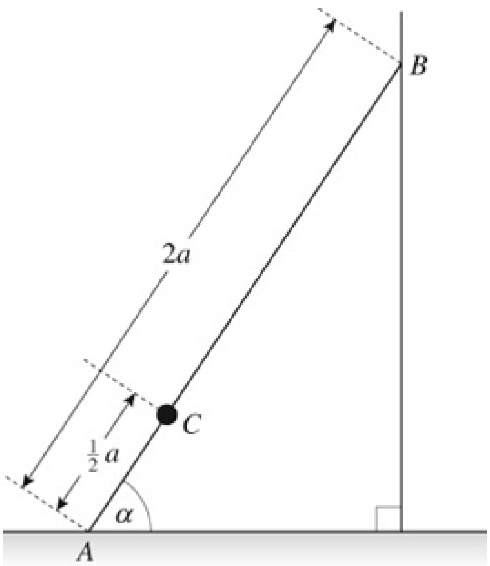
2 Review Exercise
Exercise A, Question 30

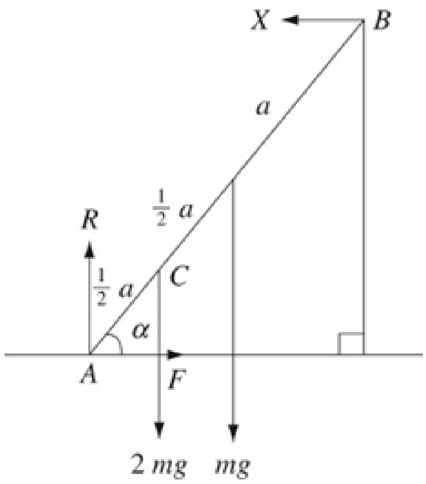
Question:



A uniform ladder AB, of mass m and length $2a$, has one end A on rough horizontal ground. The other end B rests against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall. The ladder makes an angle α with the horizontal, where $\tan\alpha = \frac{4}{3}$. A child of mass $2m$ stands on the ladder at C where $AC = \frac{1}{2}a$, as shown in the diagram. The ladder and the child are in equilibrium.

By modelling the ladder as a rod and the child as a particle, calculate the least possible value of the coefficient of friction between the ladder and the ground.

Solution:



$\tan\alpha = \frac{4}{3}$

M(A)

$$2mg \times \frac{1}{2}a\cos\alpha + mg \times a\cos\alpha = X \times 2a\sin\alpha$$

$$mg + mg = 2X\tan\alpha$$

Divide by $\cos\alpha$ and use $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$.

$$2mg = 2X \times \frac{4}{3}$$

$$X = \frac{3}{4}mg$$

R(↑) $R = 2mg + mg$

$R = 3mg$

R(→) $F = X$

$\therefore F = \frac{3}{4}mg$

$$F \le \mu R$$

$$\frac{3}{4}mg \le \mu \times 3mg$$

$$\frac{1}{4} \le \mu$$

The least value of μ is $\frac{1}{4}$.

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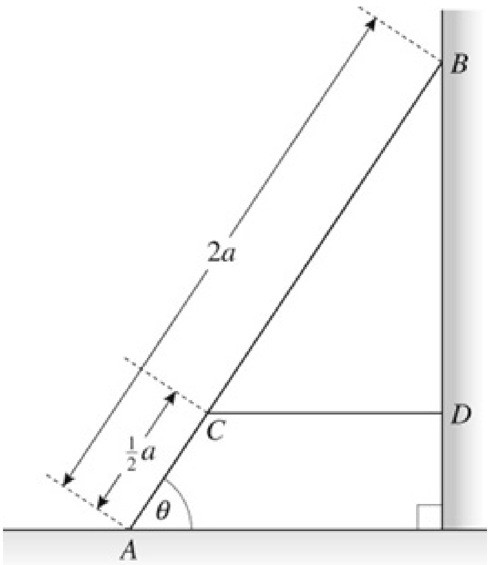
2 Review Exercise
Exercise A, Question 31

Question:



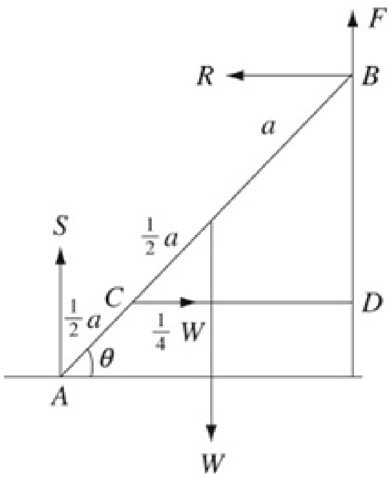
A uniform ladder, of weight W and length $2a$, rests in equilibrium with one end A on a smooth horizontal floor and the other end B on a rough vertical wall. The ladder is in a vertical plane perpendicular to the wall. The coefficient of friction between the wall and the ladder is μ. The ladder makes an angle θ with the floor, where $\tan\theta = 2$. A horizontal light inextensible string CD is attached to the ladder at the point C, where $AC = \frac{1}{2}a$. The string is attached to the wall at the point D, with BD vertical, as shown in the diagram. The tension in the string is $\frac{1}{4}W$. By modelling the ladder as a rod,

a find the magnitude of the force of the floor on the ladder,

b show that $\mu \geq \frac{1}{2}$.

c State how you have used the modelling assumption that the ladder is a rod.

Solution:



$\tan\theta = 2$

a $M(B) \quad S \times 2a\cos\theta$

$$= \frac{1}{4}W \times \frac{3}{2}a\sin\theta + W \times a\cos\theta$$

Taking moments about B gives an equation with S the only unknown quantity.

$$2S = \frac{3}{8}W\tan\theta + W$$

Divide by $\cos\theta$ and use $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

$$2S = \frac{3W}{4} + W$$

$\tan\theta = 2$

$$S = \frac{7W}{8}$$

b $R(\rightarrow) \quad R = \frac{1}{4}W$

$R(\uparrow) \quad F + S = W$

$$F = W - \frac{7W}{8} = \frac{1}{8}W$$

$$F \le \mu R$$

$$\frac{1}{8}W \le \frac{1}{4}W \times \mu$$

$$\therefore \mu \ge \frac{1}{2}$$

c The ladder will be straight.

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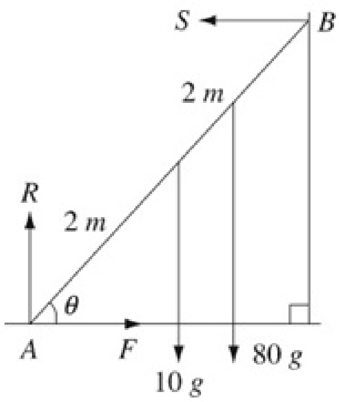
2 Review Exercise
Exercise A, Question 32

Question:

A uniform ladder AB, of mass 10 kg and length 4 m, rests in equilibrium with the end A on rough horizontal ground. The end B of the ladder rests against a smooth vertical wall, the ladder being in a vertical plane perpendicular to the wall. The coefficient of friction between the ladder and the ground is $\frac{1}{3}$. The ladder is inclined at an angle θ to the horizontal, where $\tan\theta = 2$. A man of mass 80 g stands on the ladder at a point which is a distance x metres from A.

Find the range of possible values of x.

Solution:



$\mu = \frac{1}{3}$

$\tan\theta = 2$

$M(A) \quad 10g \times 2\cos\theta + 80gx\cos\theta = S \times 4\sin\theta$

$20g + 80gx = 4S\tan\theta$

Divide by $\cos\theta$ and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$20g + 80gx = 8S$

$\tan\theta = 2$

$R(\uparrow) \quad R = 80g + 10g$

$R = 90g$

$R(\rightarrow) \quad F = S$

$\therefore F = \frac{1}{8}(20g + 80gx)$

For equilibrium

$F \leq \mu R$

$F \leq \frac{1}{3}R$

$\therefore \frac{1}{8}(20g + 80gx) \leq \frac{1}{3} \times 90g$

$\frac{1}{4} + x \leq 3$

$x \leq \frac{11}{4}$

x must be positive (or zero)

$\therefore 0 \leq x \leq \frac{11}{4}$

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2 Review Exercise
Exercise A, Question 33

Question:

A uniform ladder, of mass M and length 5 m, has one end on rough horizontal ground with the other end placed against a smooth vertical wall. The coefficient of friction between the ladder and the ground is 0.3. The highest point of the wall is higher than the highest point on the ladder. Given that the top of the ladder is 4 m vertically above the level of the ground,

a show that the ladder cannot remain in equilibrium in this position.

A brick is placed on the bottom rung of the ladder in order to enable it to stay in equilibrium in the position described above. Assuming that the brick is at the very bottom of the ladder and does not touch the ground,

b show that the horizontal frictional force exerted on the ladder by the ground is independent of the mass of the brick.

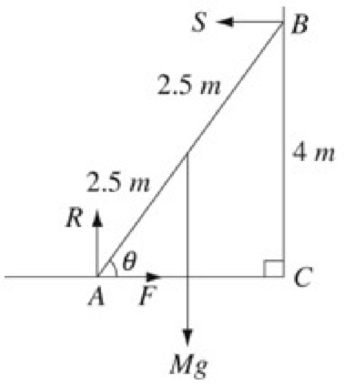
c Find, in terms of M, the smallest mass of the brick which will enable the ladder to remain in equilibrium.

The ladder, without the brick, is now extended so that the top of the ladder is higher than the top of the wall.

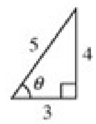
d Draw a diagram showing the forces acting on the ladder in this situation.

Solution:

a



$\mu = 0.3$



ΔABC is a (3,4,5) Δ.

Assume equilibrium:

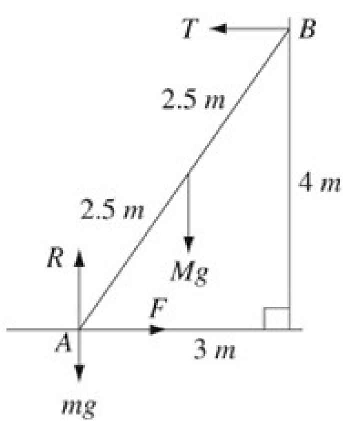
$$\begin{aligned} M(B) \quad R\times 3 &= Mg\times 1.5 + F\times 4 \\ \text{R}(\uparrow) \quad R &= Mg \\ \therefore 4F &= 3Mg - 1.5Mg \\ F &= \tfrac{3}{8}Mg = 0.375Mg \end{aligned}$$

but $\mu = 0.3$

$\therefore$ maximum possible friction force
$= 0.3R = 0.3Mg$

$\therefore$ Equilibrium is not possible.

b



$$\begin{aligned} \text{M}(A) \quad 4T &= 1.5Mg \\ \text{R}(\rightarrow) \quad T &= F \\ \therefore F &= \tfrac{1.5}{4}Mg = \tfrac{3}{8}Mg \end{aligned}$$

$\therefore$ independent of mass of brick.

a

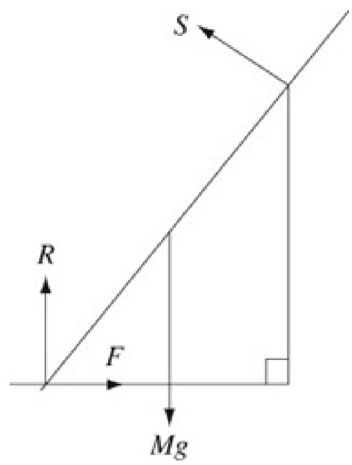
c R(↑) $R = Mg + mg$

For equilibrium

$$\begin{aligned} F &\le \mu R \\ \tfrac{3}{8}Mg &\le 0.3(Mg + mg) \\ 0.375M &\le 0.3M + 0.3m \\ 0.075M &\le 0.3m \\ m &\ge \frac{0.075M}{0.3} = 0.25M \end{aligned}$$

The smallest mass of the brick is $\frac{1}{4}M$.

d



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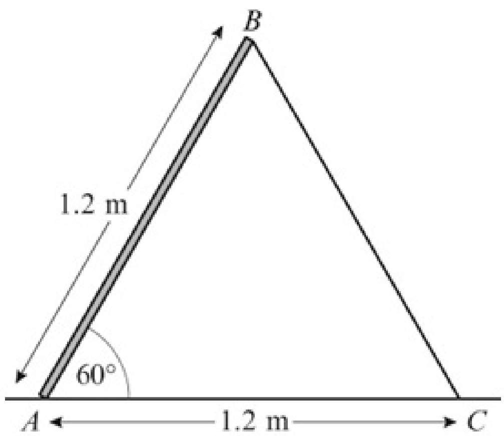
2 Review Exercise
Exercise A, Question 34

Question:



A trap door is propped open at $60°$ to the horizontal by a pole. The trap door is modelled as a uniform rod AB, of mass 10 kg and length 1.2 m, smoothly hinged at A. The pole is modelled as a light rod BC, smoothly hinged to AB at B. The points A and C are at the same horizontal level, $AC = 1.2$ m and the plane ABC is vertical, as shown in the diagram.

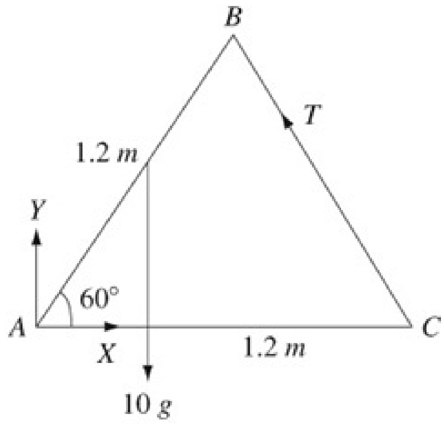
Find, to 3 significant figures,

a the thrust in BC,

b the magnitude of the force acting on the rod AB at A.

Solution:

a



$$M(A) \quad T \times 1.2\cos 30° = 10g \times 0.6\cos 60°$$

ΔABC is equilateral.

$$T = \frac{10 \times 9.8 \times 0.6\cos 60°}{1.2\cos 30°}$$

$$T = 28.29\ldots$$

$$T = 28.3 \text{ N}$$

Keep a record of T to at least 4 significant figures for use later.

b $R(\uparrow) \quad Y + T\cos 30° = 10g$

$$Y = 10 \times 9.8 - 28.29\cos 30°$$

$$Y = 73.50$$

$$R(\rightarrow) \quad X = T\cos 60°$$

$$= 14.145$$

Magnitude of resultant

$$= \sqrt{(73.50^2 + 14.145^2)}$$

$$= 74.84$$

$$= 74.8 \text{ N}$$

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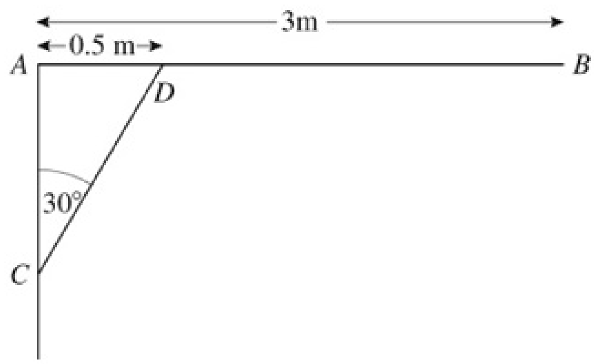
2 Review Exercise
Exercise A, Question 35

Question:



A uniform pole AB, of mass 30 kg and length 3 m, is smoothly hinged to a vertical wall at one end A. The pole is held in equilibrium in a horizontal position by a light rod CD. One end C of the rod is fixed to the wall vertically below A. The other end D is freely jointed to the pole so that $\angle ACD = 30°$ and $AD = 0.5$ m, as shown in the diagram. Find

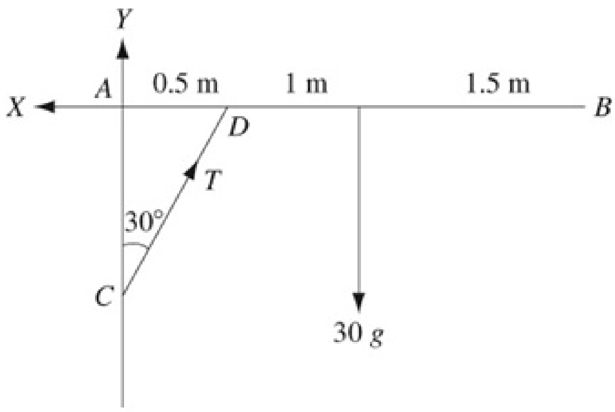
a the thrust in the rod CD,

b the magnitude of the force exerted by the wall on the pole at A.

The rod CD is removed and replaced by a longer light rod CM, where M is the midpoint of AB. The rod is freely jointed to the pole at M. The pole AB remains in equilibrium in a horizontal position.

c Show that the force exerted by the wall on the pole at A now acts horizontally.

Solution:



a M(A)

$$T \times 0.5 \cos 30° = 30g \times 1.5$$
$$T = \frac{30 \times 9.8 \times 1.5}{0.5 \cos 30°}$$
$$T = 1018$$
$$T = 1020 \text{ N}$$

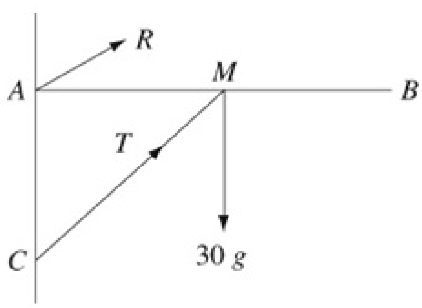
Take moments about A to get an equation with T as the only unknown.

b R(↑) $Y + T\cos 30° = 30g$
$$Y = 30g - T\cos 30°$$
$$Y = -587.6$$
R(→) $X = T\cos 60°$
$$= 509.2$$

Magnitude of resultant

$$= \sqrt{(509.2^2 + (-587.6)^2)}$$
$$= 777.5 = 778 \text{ N}$$

c



Consider the moments of the forces about M.

Moment of T = 0

Moment of $30g$ = 0

So, for equilibrium, the moment of R must be zero.
Hence R must pass through M and so is horizontal.

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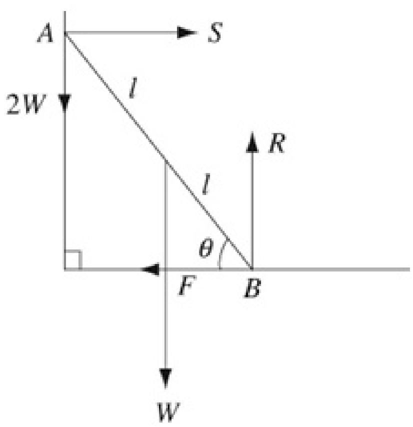
2 Review Exercise
Exercise A, Question 36

Question:

A uniform ladder rests with its lower end on a rough horizontal path and its upper end against a smooth vertical wall. The ladder rests in a vertical plane perpendicular to the wall. A woman stands on the top of this ladder, and the ladder is in limiting equilibrium. The weight of the woman is twice the weight of the ladder, and the coefficient of friction between the path and the ladder is $\frac{5}{12}$. By modelling the ladder as a uniform rod and the woman as a particle, find, to the nearest degree, the angle between the ladder and the horizontal.

Solution:



$\mu = \frac{5}{12}$

Ladder is uniform, so let its length be $2l$.

$$\text{R}(\uparrow) \quad R = 2W + W = 3W$$

$$F = \mu R = \frac{5}{12} \times 3W$$

Friction is limiting.

$$F = \frac{5}{4} W$$

M(A)

$$W \times l\cos\theta + F \times 2l\sin\theta = R \times 2l\cos\theta$$

S is not known and not needed.

$$W\cos\theta + \frac{5}{4}W \times 2\sin\theta = 3W \times 2\cos\theta$$

$$W + \frac{5}{2}W\tan\theta = 6W$$

Divide by $\cos\theta$ and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

$$\frac{5}{2}\tan\theta = 5$$

$$\tan\theta = 2$$

$$\theta = 63.4\ldots$$

$$\theta = 63° \text{ (nearest degree)}$$

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2 Review Exercise
Exercise A, Question 37

Question:

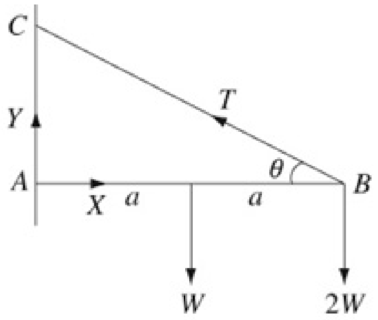
A uniform rod AB, of length $2a$ and weight W, is hinged to a vertical post at A and is supported in a horizontal position by a string attached to B and to a point C vertically above A, where $\angle ABC = \theta$.

A load of weight $2W$ is hung from B.

Find the tension in the string and the horizontal and vertical resolved parts of the force exerted by the hinge on the rod.

Show that, if the reaction of the hinge at A is at right angles to BC, then $AC = 2a\sqrt{5}$.

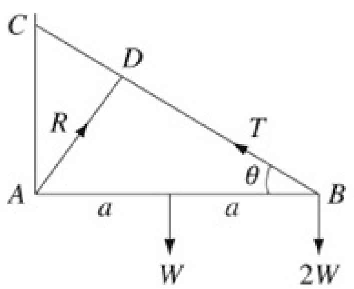
Solution:



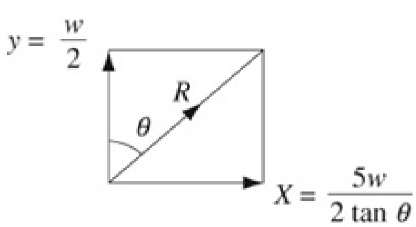
$$\text{M}(A) \quad T \times 2a \sin\theta = W \times a + 2W \times 2a$$
$$2T \sin\theta = 5W$$
$$T = \frac{5W}{2\sin\theta}$$

W and θ were both given in the question and are therefore acceptable in the answer.

$$\text{R}(\uparrow) \quad Y + T\sin\theta = W + 2W$$
$$Y = 3W - \frac{5W}{2}$$
$$Y = \frac{W}{2}$$
$$\text{R}(\rightarrow) \quad X = T\cos\theta$$
$$X = \frac{5W}{2\sin\theta} \times \cos\theta$$
$$X = \frac{5W}{2\tan\theta}$$



When R is perpendicular to BC
$\angle CAD = \theta$.



$$\tan\theta = \frac{X}{Y} = \frac{\frac{5W}{2\tan\theta}}{\frac{W}{2}} = \frac{5}{\tan\theta}$$

Using the components of R to find $\angle CAD$.

$$\tan^2\theta = 5$$
$$\tan\theta = \sqrt{5}$$
$$AC = 2a\tan\theta$$
$$\therefore AC = 2a\sqrt{5}$$

Using ΔACB.

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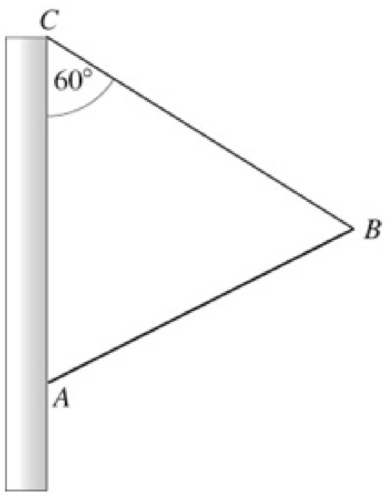
2 Review Exercise
Exercise A, Question 38

Question:

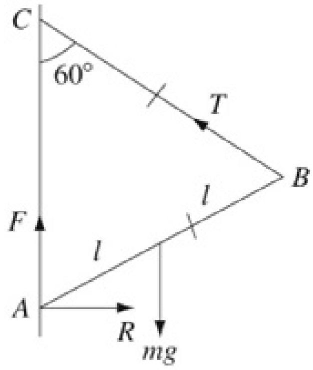


A uniform rod AB of mass m rests in equilibrium with A in contact with a rough vertical wall. The coefficient of friction between the rod and the wall is μ. A light string is attached to B and to a point C of the wall, where C is vertically above A. The plane ABC is perpendicular to the wall, $BC = BA$ and $\angle ACB = 60°$, as shown in the diagram.

a Show that the tension in the string is $\frac{1}{2}mg$.

b Show that $\mu \geq \sqrt{3}$.

Solution:



$\triangle ABC$ is equilateral.

a $\quad M(A) \quad T \times 2l \sin 60° = mg \times l \sin 60°$

$$2T = mg$$

$$T = \frac{1}{2}mg$$

b $\quad R(\uparrow) \quad F + T\cos 60° = mg$

$$F = mg - \frac{1}{2}mg\cos 60°$$

$$F = mg - \frac{1}{2}mg \times \frac{1}{2}$$

$$F = \frac{3}{4}mg$$

$R(\rightarrow) \quad R = T\cos 30°$

$$R = \frac{1}{2}mg \times \frac{\sqrt{3}}{2}$$

Use the exact value for an exact final answer.

$$R = mg\frac{\sqrt{3}}{4}$$

$$F \leq \mu R$$

Equilibrium condition.

$$\frac{3}{4}mg \leq \mu mg\frac{\sqrt{3}}{4}$$

$$\mu \geq \frac{3}{\sqrt{3}}$$

$$\mu \geq \sqrt{3}$$

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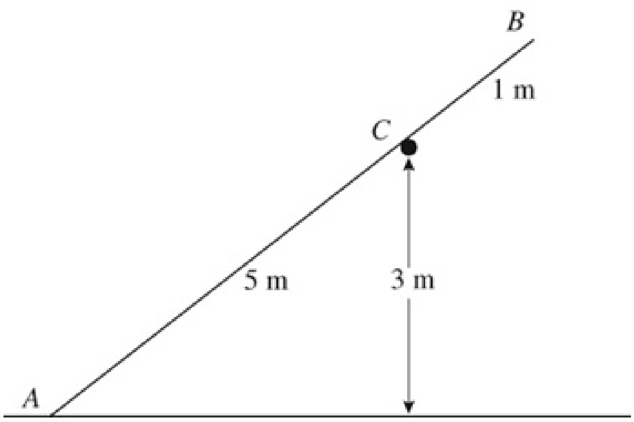
2 Review Exercise
Exercise A, Question 39

Question:



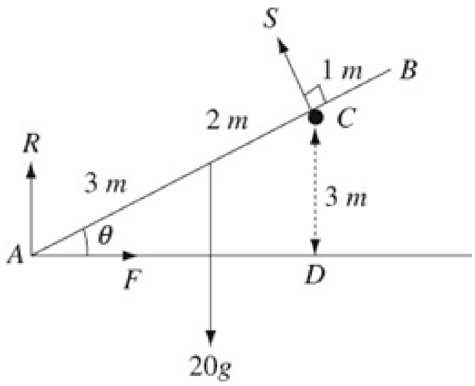
A smooth horizontal rail is fixed at a height of 3 m above a horizontal playground whose surface is rough. A straight uniform pole AB, of mass 20 kg and length 6 m, is placed to rest at a point C on the rail with the end A on the playground. The vertical plane containing the pole is at right angles to the rail. The distance AC is 5 m and the pole rests in limiting equilibrium as shown in the diagram. Calculate

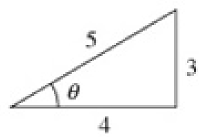
a the magnitude of the force exerted by the rail on the pole, giving your answer to the nearest N,

b the coefficient of friction between the pole and the playground, giving your answer to 2 decimal places,

c the magnitude of the force exerted by the playground on the pole giving your answer to the nearest N.

Solution:





ΔACD is a (3, 4, 5) Δ.

a M(A)

$$5S = 20g \times 3\cos\theta$$
$$5S = 20g \times 3 \times \frac{4}{5}$$
$$5S = 48g$$
$$S = \frac{48 \times 9.8}{5}$$
$$S = 94.08$$
$$S = 94 \text{ N}$$

b R($\rightarrow$)

$$F = S\sin\theta$$
$$F = 94.08 \times \frac{3}{5}$$
$$F = 56.448$$

Keep extra figures for intermediate answers (or use the memory function of your calculator).

R($\uparrow$)

$$R + S\cos\theta = 20g$$
$$R = 20 \times 9.8 - 94.08 \times \frac{4}{5}$$
$$R = 120.74$$

$$F = \mu R$$

Friction is limiting.

$$\mu = \frac{F}{R} = \frac{56.448}{120.74}$$
$$\mu = 0.467\ldots$$
$$\mu = 0.47 \text{ (2 d.p.)}$$

c Magnitude of force

$$= \sqrt{(120.74^2 + 56.448^2)}$$
$$= 133.2$$
$$= 133 \text{ N (nearest N)}$$

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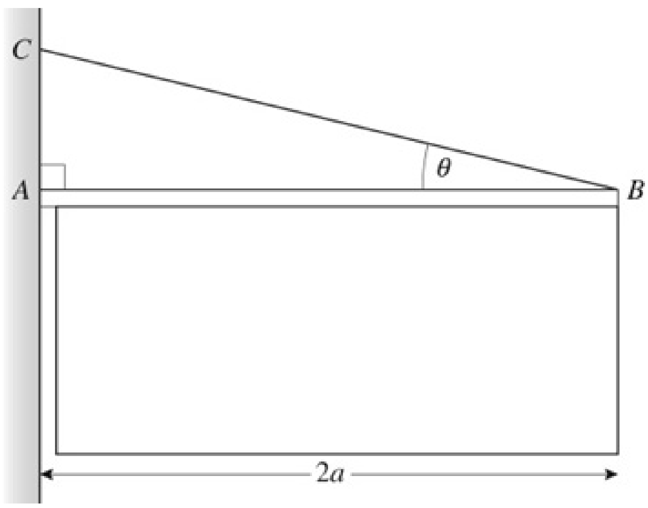
2 Review Exercise
Exercise A, Question 40

Question:

A pole of mass m and length $2a$ is used to display a light banner. The pole is modelled as a uniform rod AB, freely hinged to a vertical wall at the point A. It is held in a horizontal position by a light wire. One end of the wire is attached to the end B of the rod and the other end is attached to the wall at a point C which is vertically above A such that $\angle ABC$ is θ, where $\tan\theta = \frac{1}{2}$, as shown in the diagram.

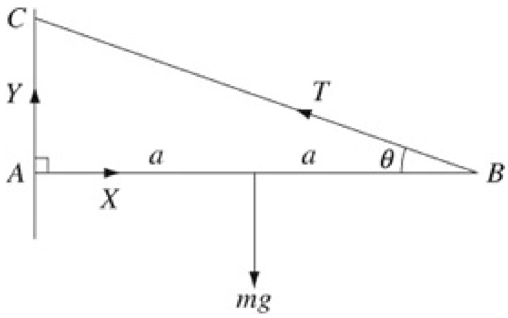


a Show that the tension in the wire is $\dfrac{mg}{2\sin\theta}$.

b Find, in terms of m and g, the magnitude of the force exerted by the wall on the rod at A.

c State, briefly, where in your calculation you have used the modelling assumption that the pole is a rod.

Solution:



$\tan\theta = \frac{1}{2}$

a $M(A)$ $T \times 2a\sin\theta = mg \times a$

$T = \frac{mg}{2\sin\theta}$

Take moments about A for an equation with T as the only unknown quantity.

b $R(\uparrow)$ $Y + T\sin\theta = mg$

$Y = mg - \frac{mg}{2\sin\theta} \times \sin\theta$

$Y = \frac{1}{2}mg$

$R(\rightarrow)$ $X = T\cos\theta$

$X = \frac{mg}{2\sin\theta} \times \cos\theta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$X = \frac{mg}{2\tan\theta} = mg$

$\tan\theta = \frac{1}{2}$

Magnitude of resultant

$= \sqrt{\left[(\frac{1}{2}mg)^2 + (mg)^2\right]}$

$= \sqrt{\left(\frac{1}{4}+1\right)}mg = mg\frac{\sqrt{5}}{4}$

c In the moments equation.