Chapter Review

1 a $\frac{\pi\times2^2}{2}\times\left(\frac{4\times2}{3\pi}\right)+2\times\frac{3}{2}\times\left(-\frac{1}{2}\right)=\left(\frac{\pi\times2^2}{2}+2\times\frac{3}{2}\right)\bar{x}$

$$\frac{16}{3}-\frac{3}{2}=(2\pi+3)\bar{x}$$

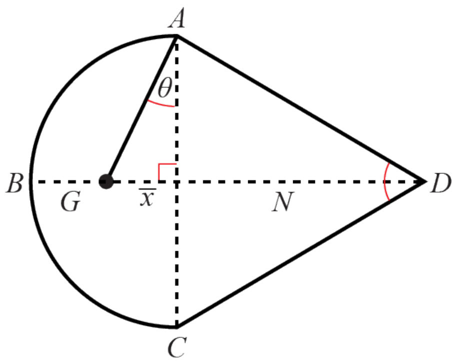
$$\frac{23}{6(2\pi+3)}=\bar{x}$$

0.413 m (3 s.f.)

Use $\sum m_i x_i = \bar{x}\sum m_i$ taking AC as the y-axis.

A decimal answer is acceptable.

b



G is the centre of mass

θ is the required angle

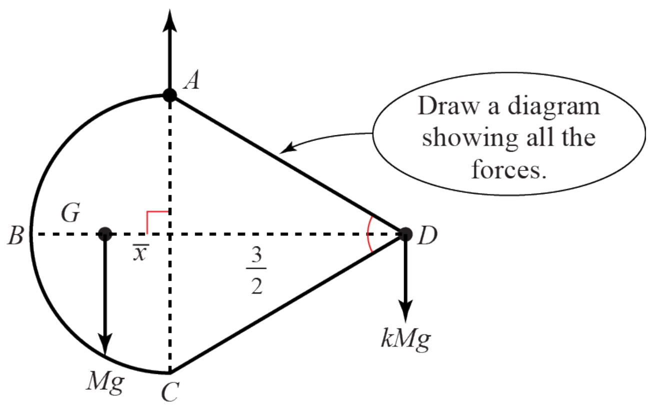
G will be on the line of symmetry.

In equilibrium, AG will be vertical.

$\tan\theta=\frac{x}{2}=\frac{23}{12(2\pi+3)}$

$\theta=12°$

c



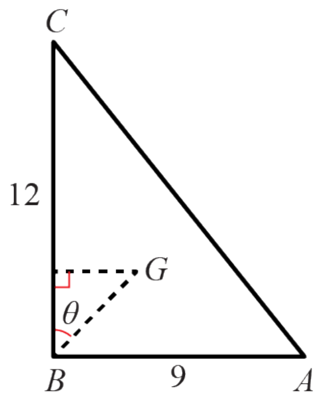
M(A),

$Mg\bar{x}=kMg\times\frac{3}{2}$

$\Rightarrow k=\frac{2}{3}\times\frac{23}{6(2\pi+3)}$

$=\frac{23}{9(2\pi+3)}=0.275$

Taking moments about A means we don't need to know the force A.

2 

A is (9, 0)
B is (0, 0)
C is (0, 12)
then G is (3, 4)

Take BA and BC as axes.

Take the mean of the 3 points.

G will be vertically below B.

In equilibrium, BG will be vertical.

Hence required angle is $G\hat{B}C = \theta$.

$\tan\theta = \frac{3}{4} \Rightarrow \theta = 36.9°$.

3 $3\begin{pmatrix}1\\6\end{pmatrix}+5\begin{pmatrix}-1\\5\end{pmatrix}+2\begin{pmatrix}2\\-3\end{pmatrix}+4\begin{pmatrix}-1\\-4\end{pmatrix}=(3+5+2+4)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Use $\sum m_i\mathbf{r}_i = \bar{\mathbf{r}}\sum m_i$

$\begin{pmatrix}3\\18\end{pmatrix}+\begin{pmatrix}-5\\25\end{pmatrix}+\begin{pmatrix}4\\-6\end{pmatrix}+\begin{pmatrix}-4\\-16\end{pmatrix}=14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Simplify.

$\begin{pmatrix}-2\\21\end{pmatrix}=14\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Hence, coordinates of the centre of mass are $\left(-\frac{1}{7}, \frac{3}{2}\right)$.

4 Taking AB and AD as axes:

$2a^2\begin{pmatrix}a\\\frac{1}{2}a\end{pmatrix}+2\times\frac{1}{2}a^2\begin{pmatrix}\frac{7a}{3}\\\frac{a}{3}\end{pmatrix}=3a^2\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

Centre of mass of the *two* triangles.

$\begin{pmatrix}2a\\a\end{pmatrix}+\begin{pmatrix}\frac{7a}{3}\\\frac{a}{3}\end{pmatrix}=3\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

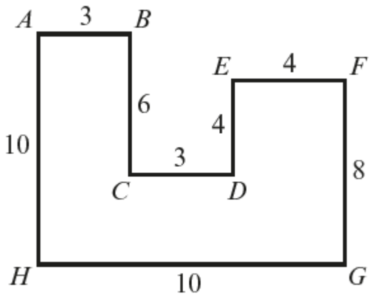
$\frac{1}{3}\left\{\begin{pmatrix}2a\\0\end{pmatrix}+\begin{pmatrix}3a\\0\end{pmatrix}+\begin{pmatrix}2a\\a\end{pmatrix}\right\}$

$\begin{pmatrix}\frac{13a}{9}\\\frac{4a}{9}\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$

$=\begin{pmatrix}\frac{7a}{3}\\\frac{a}{3}\end{pmatrix}$

a Distance from AD is $\frac{13a}{9}$ **b** Distance from AB is $\frac{4a}{9}$

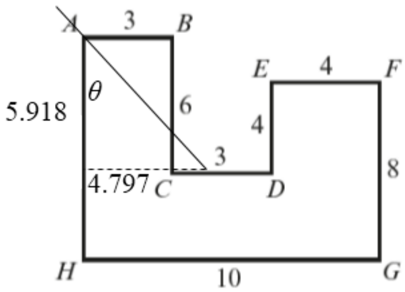
5 a



Let H be the origin and let HG lie on the positive x-axis.

$$74\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 30\begin{pmatrix}1.5\\ 5\end{pmatrix} + 12\begin{pmatrix}4.5\\ 2\end{pmatrix} + 32\begin{pmatrix}8\\ 4\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{355}{74}\\ \frac{151}{37}\end{pmatrix}$$



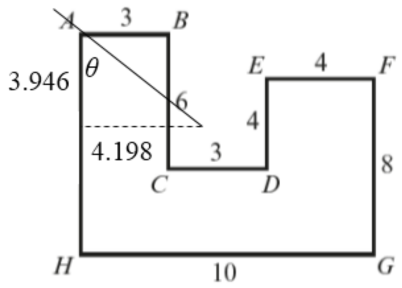
$$\tan\theta = \frac{\frac{355}{74}}{\frac{219}{37}}$$

$\theta = 39.0248...$

$\quad = 39.0°$ (3 s.f.)

5 b $15M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=10M\begin{pmatrix}\frac{355}{74}\\ \frac{151}{37}\end{pmatrix}+5M\begin{pmatrix}3\\ 10\end{pmatrix}$

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}\frac{466}{111}\\ \frac{224}{37}\end{pmatrix}$



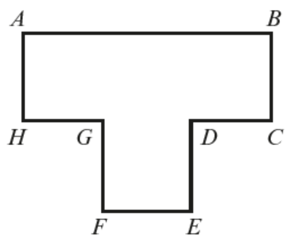
$\tan\theta=\dfrac{\frac{460}{111}}{\frac{146}{37}}$

$\theta=46.4034...°$

46.4034….. – 39.0248 = 7.38 (3 s.f)

Therefore the change in angle is 7.38°

6 a



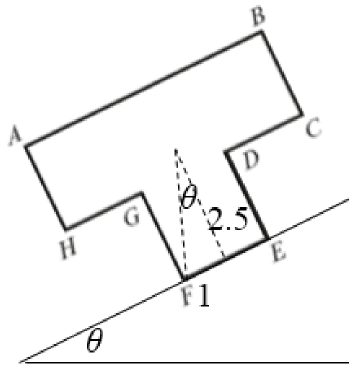
Let *F* be the origin and let *FE* lie on the positive *x*-axis.

$16\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=4\begin{pmatrix}1\\ 1\end{pmatrix}+12\begin{pmatrix}1\\ 3\end{pmatrix}$

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}1\\ 2.5\end{pmatrix}$

Therefore the centre of mass lies 2.5 cm above *FE*.

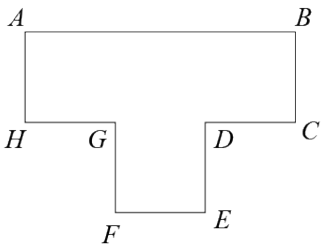
6 b



$$\tan\theta = \frac{1}{2.5}$$

$$\theta = \tan^{-1}\left(\frac{2}{5}\right) \quad \text{as required}$$

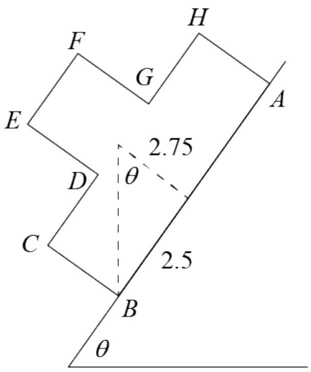
7



Let the mass of the lamina be M.
Let B be the origin and let BA lie on the positive x-axis.
From question 6 the centre of mass of the lamina lies at (3, 1.5)

$$2M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = M\begin{pmatrix}3\\ 1.5\end{pmatrix} + M\begin{pmatrix}2\\ 4\end{pmatrix}$$

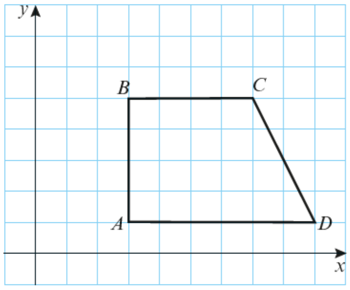
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}2.5\\ 2.75\end{pmatrix}$$



$$\tan\theta = \frac{2.5}{2.75}$$

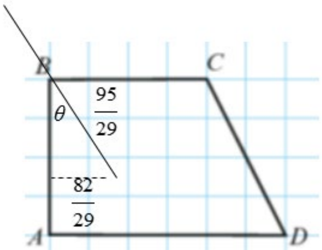
$$\theta = \tan^{-1}\left(\frac{10}{11}\right)$$

8



$$\frac{29}{12}M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 0.25M\begin{pmatrix}3\\3\end{pmatrix} + \frac{2}{3}M\begin{pmatrix}5\\5\end{pmatrix} + 0.5M\begin{pmatrix}8\\3\end{pmatrix} + M\begin{pmatrix}6\\1\end{pmatrix}$$

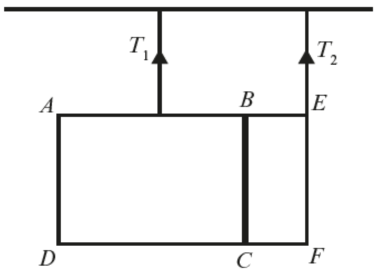
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{169}{29}\\ \frac{79}{29}\end{pmatrix}$$



$$\tan\theta = \frac{\frac{82}{29}}{\frac{66}{29}}$$

$$\theta = \tan^{-1}\left(\frac{41}{33}\right)$$

9 a



$ABCD$ has an area of 96 cm^2 and mass M
$BEFC$ has an area of 32 cm^2 and a mass of M.
Let D be the origin and let DF lie on the positive x-axis.

$$2M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = M\begin{pmatrix}6\\ 4\end{pmatrix} + M\begin{pmatrix}14\\ 4\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}10\\ 4\end{pmatrix}$$

Res(↑) $T_1 + T_2 = 2Mg$ **(1)**

Taking moments about the centre of mass gives:

$4 \times T_1 = 6 \times T_2$

$T_1 = 1.5T_2$ **(2)**

Substituting **(2)** into **(1)** gives:

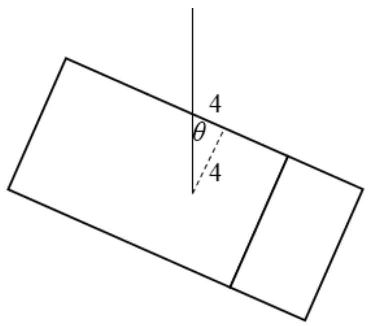
$1.5T_2 + T_2 = 2Mg$

$T_2 = 0.8Mg$

Therefore:

$T_1 = 1.2Mg$

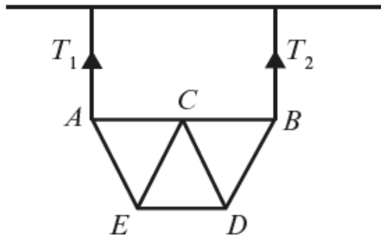
b



$$\tan\theta = \frac{4}{4}$$

$\theta = 45°$

10 a



Height of triangle ACE = $\sqrt{5^2 - 2.5^2} = \frac{5\sqrt{3}}{2}$

Let A be the origin and let AB lie on the positive x-axis.

ACE has mass M and its centre of mass of lies at $\left(\frac{5}{2}, -\frac{5\sqrt{3}}{6}\right)$

CBD has mass M and its centre of mass of lies at $\left(\frac{15}{2}, -\frac{5\sqrt{3}}{6}\right)$

CED has mass $2M$ and its centre of mass of lies at $\left(5, -\frac{5\sqrt{3}}{3}\right)$

2M lies at $\left(\frac{15}{2}, -\frac{5\sqrt{3}}{2}\right)$

$$6M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = M\begin{pmatrix}\frac{5}{2}\\ -\frac{5\sqrt{3}}{6}\end{pmatrix} + M\begin{pmatrix}\frac{15}{2}\\ -\frac{5\sqrt{3}}{6}\end{pmatrix} + 2M\begin{pmatrix}5\\ -\frac{5\sqrt{3}}{3}\end{pmatrix} + 2M\begin{pmatrix}\frac{15}{2}\\ -\frac{5\sqrt{3}}{2}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{35}{6}\\ -\frac{5\sqrt{3}}{3}\end{pmatrix}$$

Res(↑) $T_1 + T_2 = 6Mg$ **(1)**

Taking moments about the centre of mass gives:

$\frac{35}{6} \times T_1 = \frac{25}{6} \times T_2$

$T_1 = \frac{5}{7}T_2$ **(2)**

Substituting **(2)** into **(1)** gives:

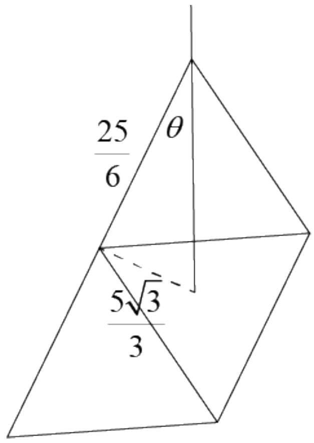
$\frac{5}{7}T_2 + T_2 = 6Mg$

$T_2 = \frac{7}{2}Mg$

Therefore:

$T_1 = \frac{5}{2}Mg$

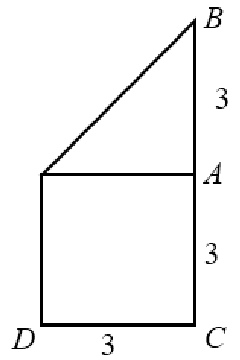
10 b



$$\tan\theta = \frac{\frac{5\sqrt{3}}{3}}{\frac{25}{6}}$$

$\theta = 34.715\ldots$
$= 34.7°$ (3 s.f.)

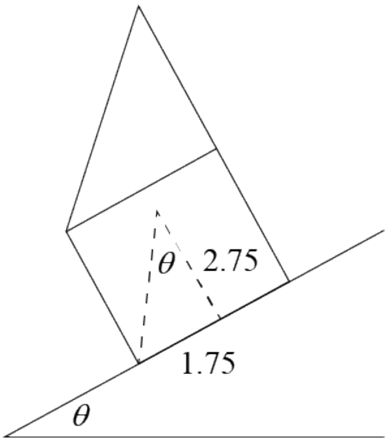
Challenge



Let D be the origin and let DC lie on the positive x-axis.

$$18M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=9M\begin{pmatrix}1.5\\ 1.5\end{pmatrix}+9M\begin{pmatrix}2\\ 4\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}1.75\\ 2.75\end{pmatrix}$$



$$\tan\theta=\frac{1.75}{2.75}$$

$\theta = 32.471\ldots$

$\quad = 32.5°$ (3 s.f.)

The lamina will not topple as the critical angle is 32.5°