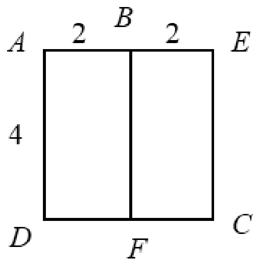
Exercise 3H

1



Let A be the origin and let AE lie on the positive x-axis.

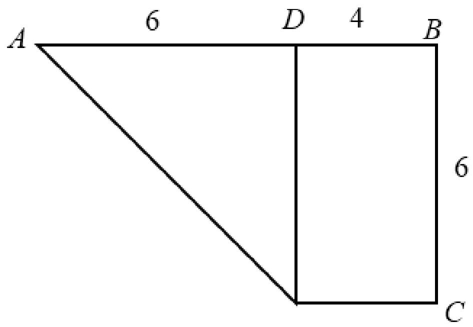
$$24\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=8\begin{pmatrix}1\\-2\end{pmatrix}+16\begin{pmatrix}3\\-2\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{24}\begin{pmatrix}56\\-48\end{pmatrix}$$

$$=\begin{pmatrix}\frac{7}{3}\\-2\end{pmatrix}$$

$\frac{7}{3}$ cm horizontally to the right of AD and 2 cm vertically below AB

2



Let A be the origin and let AB lie on the positive x-axis.

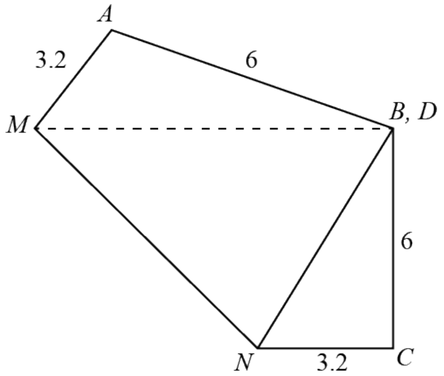
$$60\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=36\begin{pmatrix}4\\-2\end{pmatrix}+24\begin{pmatrix}8\\-3\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}=\frac{1}{60}\begin{pmatrix}336\\-144\end{pmatrix}$$

$$=\begin{pmatrix}5.6\\-2.4\end{pmatrix}$$

5.6 cm horizontally to the right of original AD and 2.4 cm vertically below AB

3



In triangle BCN,

$x^2 + 6^2 = (10 - x)^2$

$x = 3.2$

Let the point C be the origin and let CN lie on the negative x-axis.
The coordinates of the points on the shape are:
B is the point (0, 6)
C is the point (0, 0)
N is the point (−3.2, 0)
$BM = 6.8$
Therefore M is the point $(-6.8, 6)$
Let $\angle AMB$ be θ

$\tan\theta = \frac{6}{3.2} \Rightarrow \theta = 61.9275...$

Let P be the point directly below A that lies on MB

$\sin\theta = \frac{AP}{3.2} \Rightarrow AP = 2.8235...$

$\cos\theta = \frac{PM}{3.2} \Rightarrow PM = 1.5058...$

Therefore A is the point $(-(6.8 - 1.5058\ldots), 6 + 2.8235\ldots) = (-5.2942\ldots, 8.8285\ldots)$

The centre of mass of a triangle lies at $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

For triangle NCB:

$$\left(\frac{-3.2 + 0 + 0}{3}, \frac{0 + 0 + 6}{3}\right) = (-1.0666, 2)$$

NCB has area 9.6 cm^2
For triangle AMB:

$$\left(\frac{-5.2942.. + -6.8 + 0}{3}, \frac{8.8235.. + 6 + 6}{3}\right) = (-4.0314..., 6.9411...)$$

AMB has area 9.6 cm^2
For triangle MNB:

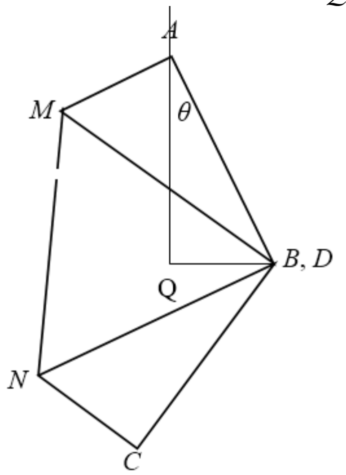
$$\left(\frac{-6.8 - 3.2 + 0}{3}, \frac{6 + 0 + 6}{3}\right) = (-3.333..., 4)$$

MNB has area 20.4 cm^2

$$60\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 40.8\begin{pmatrix}-3.333...\\ 4\end{pmatrix} + 9.6\begin{pmatrix}-4.0314..\\ 6.9411...\end{pmatrix} + 9.6\begin{pmatrix}-5.2942...\\ 8.8285...\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-3.7587..\\ 5.2431...\end{pmatrix}$$

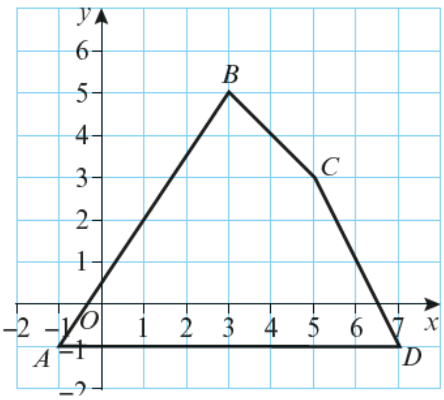
Let the centre of mass be Q



Triangle AQB has vertices at (−5.2942…, 8.8285…), (−3.7587…, 5.2431…) and (0, 6)

$$|AQ| = \sqrt{(-5.2942... - (-3.7587...))^2 + (8.8285... - 5.2431...)^2}$$
$$= 3.9003...$$
$$|QB| = \sqrt{(-3.7587.... - 0)^2 + (5.2431... - 6)^2}$$
$$= 3.8341...$$
$$|AB| = 6$$
$$\cos\theta = \frac{3.9003...^2 + 6^2 - 3.8341...^2}{2(3.9003...)(6)}$$
$$= 0.7801...$$
$$\theta = 38.7° \text{ (3 s.f.)}$$

4

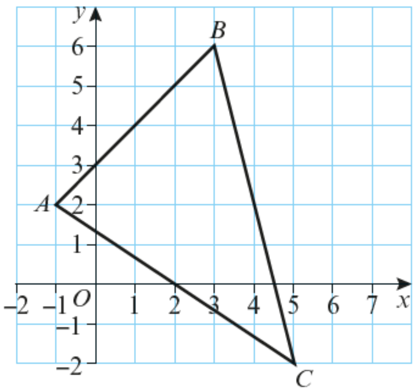


$$11M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2M\begin{pmatrix}1\\ 2\end{pmatrix} + 3M\begin{pmatrix}4\\ 4\end{pmatrix} + M\begin{pmatrix}6\\ 1\end{pmatrix} + 5M\begin{pmatrix}3\\ -1\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{35}{11}\\ \frac{12}{11}\end{pmatrix}$$

Therefore the centre of mass lies at $\left(\frac{35}{11}, \frac{12}{11}\right)$

5



AB has length $4\sqrt{2}$
BC has length $2\sqrt{17}$
AC has length $2\sqrt{13}$
Let the unit mass of AB be m, therefore AB has mass $4\sqrt{2}m$
Since BC and AC are twice as thick as AB, they have 4 times the unit mass i.e. $4m$
Therefore:
BC has mass $8\sqrt{17}m$
AC has mass $8\sqrt{13}m$

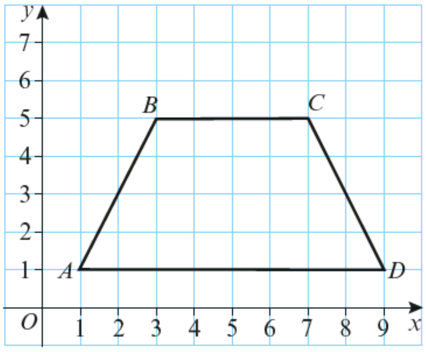
$$\left(4\sqrt{2}+8\sqrt{17}+8\sqrt{13}\right)m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=4\sqrt{2}m\begin{pmatrix}1\\4\end{pmatrix}+8\sqrt{17}m\begin{pmatrix}4\\2\end{pmatrix}+8\sqrt{13}m\begin{pmatrix}2\\0\end{pmatrix}$$

$$67.486\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}195.2850..\\ 88.5971...\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}2.893...\\ 1.312...\end{pmatrix}$$

Therefore the centre of mass lies at (2.89, 1.31) (3 s.f.)

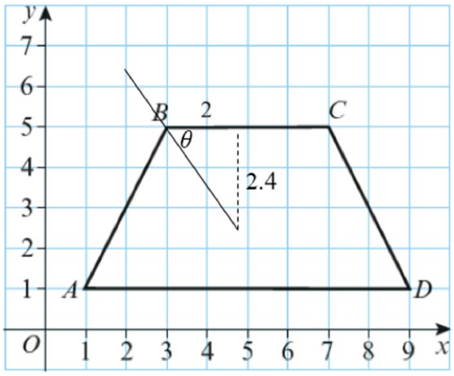
6



AD has length 8 and mass M so the mass per unit is $0.125M$
Therefore BC has mass $0.5M$

$$2.5M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 0.5M\begin{pmatrix}2\\3\end{pmatrix} + 0.5M\begin{pmatrix}5\\5\end{pmatrix} + 0.5M\begin{pmatrix}8\\3\end{pmatrix} + M\begin{pmatrix}5\\1\end{pmatrix}$$

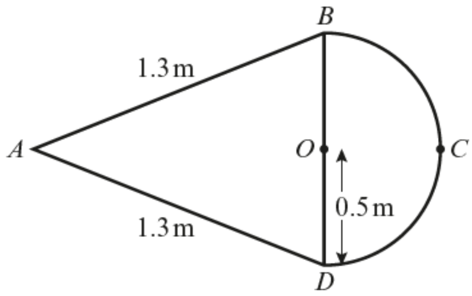
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}5\\2.6\end{pmatrix}$$



$$\tan\theta = \frac{2.4}{2}$$

$\theta = 50.2°$ (3 s.f.)

7 a



AO has length 1.2 m
Therefore the centre of mass of the triangle lies 0.4 m to the left of O.
The centre of mass of the semi-circle lies at $\frac{2}{3\pi}$ to the right of O.
Let O be the origin and let OC lie on the positive x-axis.

$$20\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 4\begin{pmatrix}-0.4\\ 0\end{pmatrix} + 16\begin{pmatrix}\frac{2}{3\pi}\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.0897...\\ 0\end{pmatrix}$$

So 0.0898 m (3 s.f.) to the right of BD

b C, as the centre of mass of the composite lamina is between O and C.

c Taking moments about the centre of mass gives:

$0.0897... \times T_B = 0.4102.... \times T_C$

$T_B = 4.5700...T_C \quad (1)$

Res(↑) $T_B + T_C = 20g$

$T_C = 20g - T_B \quad (2)$

Substituting (2) into (1) gives:

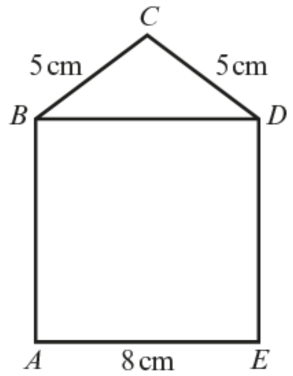
$T_B = 4.5700...(20g - T_B)$

$T_B = 16.4g$ N (3 s.f.)

Therefore:

$T_C = 3.59g$ N (3 s.f.)

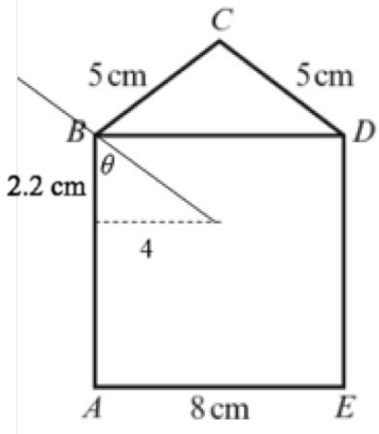
8 a



Let A be the origin and let AE lie on the positive x-axis.
The perpendicular height of the triangle is is 3 cm therefore the centre of mass of the triangle lies at (4, 9).
The triangle has an area of 12 cm^2 and a mass of 12 × 60 = 720 g
The square has an area of 64 cm^2 and a mass of 20 × 64 = 1280 g

$$2000\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=1280\begin{pmatrix}4\\4\end{pmatrix}+720\begin{pmatrix}4\\9\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}4\\5.8\end{pmatrix}$$

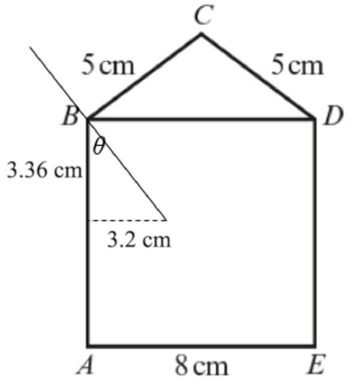


$$\tan\theta=\frac{4}{2.2}$$

$\theta = 61.2°$ (3 s.f.)

8 b $2500\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 1280\begin{pmatrix}4\\4\end{pmatrix} + 720\begin{pmatrix}4\\9\end{pmatrix} + 500\begin{pmatrix}0\\0\end{pmatrix}$

$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}3.2\\4.64\end{pmatrix}$



$\tan\theta = \dfrac{3.2}{3.36}$

$\theta = 43.6°$ (3 s.f.)

Challenge

Let the width be $2y$ and the length be $y + x$

The hypotenuse of the top right-angled triangle is $\sqrt{2}y$.

From the bottom right angles triangle:

$x^2 = y^2 + (x + y - \sqrt{2}y)^2$

$x^2 = y^2 + (x + (1 - \sqrt{2})y)^2$

$x^2 = y^2 + x^2 + 2(1 - \sqrt{2})xy + (3 - 2\sqrt{2})y^2$

$0 = 2(1 - \sqrt{2})xy + (4 - 2\sqrt{2})y$

$x = \dfrac{\sqrt{2} - 2}{1 - \sqrt{2}}y$

$\dfrac{2y}{y + x} = \dfrac{2y}{y + \dfrac{\sqrt{2} - 2}{1 - \sqrt{2}}y}$

$= \dfrac{2y}{(\sqrt{2} + 1)y}$

Ratio of sides $= 2 : \sqrt{2} + 1$