Exercise 3E

1 a

$$5\begin{pmatrix}2.5\\0\end{pmatrix}+3\begin{pmatrix}5\\1.5\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+\begin{pmatrix}3\\2.5\end{pmatrix}+3\begin{pmatrix}1.5\\2\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

There are other ways of splitting the lamina up (*see below*).

$$\begin{pmatrix}43\\21\end{pmatrix}=16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\frac{43}{16}=\bar{x}$$

$$\frac{21}{16}=\bar{y}$$

Equate **i** and **j** components.

Centre of mass is $\left(\frac{43}{16},\frac{21}{16}\right)$

b

$$3\begin{pmatrix}3.5\\0\end{pmatrix}+3\begin{pmatrix}5\\1.5\end{pmatrix}+2\begin{pmatrix}4\\3\end{pmatrix}+2\begin{pmatrix}3\\4\end{pmatrix}+\begin{pmatrix}2.5\\5\end{pmatrix}+2\begin{pmatrix}2\\4\end{pmatrix}+\begin{pmatrix}1.5\\3\end{pmatrix}+2\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1.5\\1\end{pmatrix}+\begin{pmatrix}2\\0.5\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$$\begin{pmatrix}53\\40\end{pmatrix}=18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Collect terms.

$$\frac{53}{18}=\bar{x};\ \frac{40}{18}=\bar{y}$$

Centre of mass is $\left(\frac{53}{18},\frac{20}{9}\right)$

You could use decimals.

c

$$\begin{pmatrix}0.5\\2\end{pmatrix}+2\begin{pmatrix}1\\1\end{pmatrix}+2\begin{pmatrix}2\\0\end{pmatrix}+2\begin{pmatrix}3\\1\end{pmatrix}+3\begin{pmatrix}4.5\\2\end{pmatrix}+5\begin{pmatrix}4\\3.5\end{pmatrix}+\sqrt{13}\begin{pmatrix}1\\3.5\end{pmatrix}=\left(15+\sqrt{13}\right)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Use $\sum m_i\mathbf{r}_i=\bar{\mathbf{r}}\sum m_i$

$$\begin{pmatrix}49.61\\42.12\end{pmatrix}=18.61\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

Centre of mass is $(2.67, 2.26)$

d By symmetry, $\bar{y}=3$

x-coordinate of the centre of mass satisfies

$$\sqrt{29}\times3.5+\sqrt{20}\times4+\sqrt{29}\times3.5+\sqrt{20}\times4=2\left(\sqrt{20}+\sqrt{29}\right)\times\bar{x}$$

Find the mean of the x-coordinates of the vertices.

$$7\sqrt{29}+8\sqrt{20}=2\left(\sqrt{20}+\sqrt{29}\right)\bar{x}$$

$$3.73=\bar{x}$$

Centre of mass is (3.27, 3)

2 $AB = (12 + 2\pi) - 12 = 2\pi$

Let $A\hat{O}B = \theta \ (= 2\alpha)$, where θ is in radians

Then

$\theta = \frac{2\pi}{6} = \frac{\pi}{3} = 60^\circ$

Distance of G from O

$= \bar{x}$ say.

Then,

$(6 \times 3\cos 30^\circ) \times 2 + 2\pi \times \frac{6\sin\frac{\pi}{6}}{\frac{\pi}{6}} = \bar{x}(12 + 2\pi)$

$18\sqrt{3} + 36 = \bar{x}(12 + 2\pi)$

$\therefore \quad \bar{x} = \frac{18\sqrt{3} + 36}{12 + 2\pi}$

$= \frac{9(\sqrt{3} + 2)}{6 + \pi}$

Centre of mass is on line of symmetry through O,

and a distance of $\frac{9(\sqrt{3}+2)}{6+\pi}$ from O.

> COM of arc: $\frac{r\sin\alpha}{\alpha}$; α in RADIANS.

> Use $S = r\theta$.

> from the formula booklet.

> Use $\sum m_i x_i = \bar{x}\sum m_i$

> $\cos 30^\circ = \frac{\sqrt{3}}{2}$
> $\sin\frac{\pi}{6} = \frac{1}{2}$

> Simplify by dividing top and bottom by 2.

> State your answer.

3 We choose coordinates with the midpoint of AB as the origin and AB lying on the x-axis.
Then the distance from AB to the centre of mass is just the modulus of the y-coordinate of the centre of mass since, by symmetry, the centre of mass lies on the y-axis.
Let the centre of mass have coordinates $(0, y)$ then since the total mass of the system is

$3a + a + a + a + a + 3a + 3a + a = 14a$ we have

$14ay = 3a \times 0 + a \times \left(-\tfrac{1}{2}a\right) + a \times \left(-\tfrac{1}{2}a\right)$

$+ a \times (-a) + a \times (-a)$

$+ 3a \times \left(-\tfrac{5}{2}a\right) + 3a \times \left(-\tfrac{5}{2}a\right)$

$+ a \times (-4a)$

Simplifying gives

$14ay = -a^2 - 2a^2 - 15a^2 - 4a^2 = -22a^2$

Hence

$y = -\frac{11a}{7}$

So the distance to AB is $\frac{11a}{7}$

4 a We choose coordinates so that the origin is at O and AB is parallel to the x-axis. Then the centre of mass of the straight piece of wire is O and its mass is 30.

For the circular piece, the mass is 15π and its centre of mass is at

$$\left(0, \frac{15\sin\frac{\pi}{2}}{\left(\frac{\pi}{2}\right)}\right) = \left(0, \frac{30}{\pi}\right)$$

Now by symmetry the centre of mass of the system lies on the y-axis and the y-coordinate will satisfy

$$(30+15\pi)y = 30\times 0 + 15\pi \times \frac{30}{\pi}$$

$$(30+15\pi)y = 450$$

$$y = \frac{450}{30+15\pi} = 5.83 \text{ to 3 s.f.}$$

And by our choice of coordinates this is precisely the distance to AB.

b i The total mass is $M = 100 + 100 + 8(30+15\pi) = 440 + 120\pi = 817$ g to 3 s. f.

ii We keep the same coordinate system as before, so that the centre of mass still lies on the y-axis by symmetry hence the new y-coordinate satisfies

$$(440+120\pi)y = 120\pi \times \frac{30}{\pi} = 3600$$

$$y = \frac{3600}{440+120\pi} = 4.41 \text{ to 3 s.f.}$$

and as before this is precisely the distance to AB.

5 We choose coordinates such that the origin lies at A and AB lies on the x-axis.
We start by finding the coordinates of the centre of mass of the unloaded framework. Its coordinates will satisfy

$$15\begin{pmatrix}x\\y\end{pmatrix}=\frac{3\times15}{12}\begin{pmatrix}\frac{3}{2}\\0\end{pmatrix}+\frac{4\times15}{12}\begin{pmatrix}3\\-2\end{pmatrix}$$
$$+\frac{5\times15}{12}\begin{pmatrix}\frac{3}{2}\\-2\end{pmatrix}$$

That is $\begin{pmatrix}x\\y\end{pmatrix}=\frac{3}{12}\begin{pmatrix}\frac{3}{2}\\0\end{pmatrix}+\frac{4}{12}\begin{pmatrix}3\\-2\end{pmatrix}+\frac{5}{12}\begin{pmatrix}\frac{3}{2}\\-2\end{pmatrix}$

So $\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{12}\begin{pmatrix}\frac{9}{2}+12+\frac{15}{2}\\-8-10\end{pmatrix}=\frac{1}{12}\begin{pmatrix}24\\-18\end{pmatrix}=\begin{pmatrix}2\\-\frac{3}{2}\end{pmatrix}$

Now we consider the whole system. The total mass is $M=15+10+20+30=75$
Hence the coordinates of the centre of mass satisfy

$$75\begin{pmatrix}x\\y\end{pmatrix}=10\begin{pmatrix}0\\0\end{pmatrix}+20\begin{pmatrix}3\\0\end{pmatrix}+30\begin{pmatrix}3\\-4\end{pmatrix}+15\begin{pmatrix}2\\-\frac{3}{2}\end{pmatrix}$$

Which simplifies to $75\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}180\\\frac{-285}{2}\end{pmatrix}$

So $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{12}{5}\\-\frac{19}{10}\end{pmatrix}$

6 We choose coordinates such that the origin is at O and the x-axis is parallel to AB.

The centre of mass of the top left semicircle is $\left(-1.5,\frac{3}{\pi}\right)$ and by symmetry the centre of mass of the top right semicircle is $\left(1.5,\frac{3}{\pi}\right)$

Finally, the centre of mass of the larger semicircle is $\left(0,-\frac{6}{\pi}\right)$.

Hence, the centre of mass of the framework is given by

$$6\pi\begin{pmatrix}x\\y\end{pmatrix}=3\pi\begin{pmatrix}0\\-\frac{6}{\pi}\end{pmatrix}+1.5\pi\begin{pmatrix}-1.5\\\frac{3}{\pi}\end{pmatrix}+1.5\pi\begin{pmatrix}1.5\\\frac{3}{\pi}\end{pmatrix}$$

So
$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0\\-\frac{6}{\pi}\end{pmatrix}+\frac{1}{4}\begin{pmatrix}-1.5\\\frac{3}{\pi}\end{pmatrix}+\frac{1}{4}\begin{pmatrix}1.5\\\frac{3}{\pi}\end{pmatrix}$$

So
$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{4\pi}\begin{pmatrix}0\\-6\end{pmatrix}=\frac{1}{2\pi}\begin{pmatrix}0\\-3\end{pmatrix}$$

7 a The ladder has a line of symmetry that is parallel to the rungs and between the 3rd and 4th rungs hence the height of the centre of mass is $3\times50+25=175\,\text{cm}=1.75\,\text{m}$.

7 **b** Choose coordinates so that the origin is at the centre of mass found in part **a** *before* the bottom rung is removed, and the rungs are parallel to the x-axis.

Before the bottom rung is removed, the ladder contains 20 lengths of wire, each of which is 50 cm long.

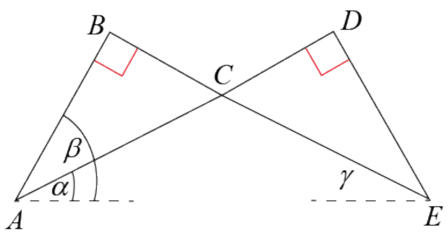
The bottom rung is 50 cm long and its centre of mass is at $\begin{pmatrix} 0 \\ -125 \end{pmatrix}$, so when the bottom rung is removed, the new centre of mass satisfies

$$19\begin{pmatrix} x \\ y \end{pmatrix} = 20\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -125 \end{pmatrix}$$

Hence

$$y = \frac{125}{19}\text{ cm} = \frac{5}{76}\text{ m}$$

Challenge



Let $\angle EAD$ be α

$$\tan\alpha = \frac{4}{8} \Rightarrow \sin\alpha = \frac{4}{\sqrt{80}}$$

Let $\angle EAB$ be β

$$\tan\beta = \frac{8}{4} \Rightarrow \sin\alpha = \frac{8}{\sqrt{80}}$$

Let the height of the midpoint of AC above AE be h then:

$$\sin\alpha = \frac{h}{2.5}$$

$$h = 2.5\left(\frac{4}{\sqrt{80}}\right)$$

$$= \frac{10}{\sqrt{80}}$$

Let the height of the midpoint of AB above AE be l then:

$$\sin\beta = \frac{l}{2}$$

$$l = 2\left(\frac{8}{\sqrt{80}}\right)$$

$$= \frac{16}{\sqrt{80}}$$

Let the angle $\angle AEB$ be γ then $\alpha = \gamma$

Let the height of the midpoint of BC above AE be m then:

Challenge (continued)

$\sin\alpha = \dfrac{m}{6.5}$

$m = 6.5\left(\dfrac{4}{\sqrt{80}}\right)$

$= \dfrac{26}{\sqrt{80}}$

$12\bar{y} = 5\times\dfrac{10}{\sqrt{80}} + 4\times\dfrac{16}{\sqrt{80}} + 3\times\dfrac{26}{\sqrt{80}}$

$= \dfrac{192}{\sqrt{80}}$

$\bar{y} = \dfrac{16}{\sqrt{80}}$

Let the height of C above AE be n

$\sin\alpha = \dfrac{n}{5}$

$n = 5\left(\dfrac{4}{\sqrt{80}}\right)$

$= \dfrac{20}{\sqrt{80}}$

Therefore the centre of mass lies:

$\dfrac{20}{\sqrt{80}} - \dfrac{16}{\sqrt{80}} = \dfrac{4}{\sqrt{80}}$

$\dfrac{\sqrt{5}}{5}$ cm below C