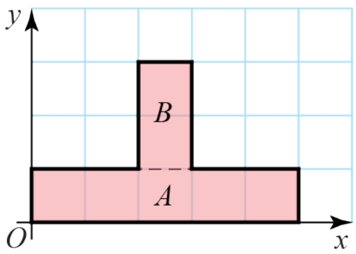
Exercise 3D

1 a Divide the shape into two rectangles.



Rectangle A has an area of 5 square units and its centre of mass lies at (2.5, 0.5).
Rectangle B has an area of 2 square units and its centre of mass lies at (2.5, 2).

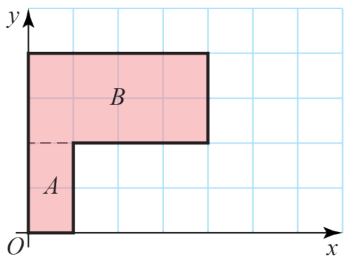
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$7\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 5\begin{pmatrix}2.5\\ 0.5\end{pmatrix} + 2\begin{pmatrix}2.5\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{7}\begin{pmatrix}17.5\\ 6.5\end{pmatrix}$$

$$= \begin{pmatrix}\frac{5}{2}\\ \frac{13}{14}\end{pmatrix}$$

b Divide the shape into two rectangles.



Rectangle A has an area of 2 square units and its centre of mass lies at (0.5, 1).
Rectangle B has an area of 8 square units and its centre of mass lies at (2, 3).

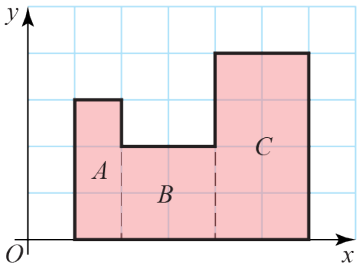
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$10\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2\begin{pmatrix}0.5\\ 1\end{pmatrix} + 8\begin{pmatrix}2\\ 3\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{10}\begin{pmatrix}17\\ 26\end{pmatrix}$$

$$= \begin{pmatrix}1.7\\ 2.6\end{pmatrix}$$

1 c Divide the shape into three rectangles.



Rectangle A has an area of 3 square units and its centre of mass lies at (1.5, 1.5).
Rectangle B has an area of 4 square units and its centre of mass lies at (3, 1).
Rectangle C has an area of 8 square units and its centre of mass lies at (5, 2).

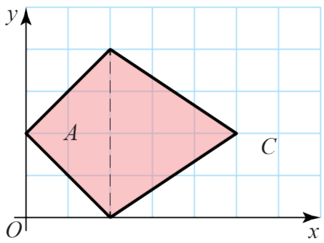
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$15\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3\begin{pmatrix}1.5\\ 1.5\end{pmatrix} + 4\begin{pmatrix}3\\ 1\end{pmatrix} + 8\begin{pmatrix}5\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{15}\begin{pmatrix}56.5\\ 24.5\end{pmatrix}$$

$$= \begin{pmatrix}\frac{113}{30}\\ \frac{49}{30}\end{pmatrix}$$

d Divide the shape into two triangles.



Triangle A has an area of 4 square units and its centre of mass lies at $\left(\frac{4}{3}, 2\right)$.

Triangle B has an area of 6 square units and its centre of mass lies at (3, 2).

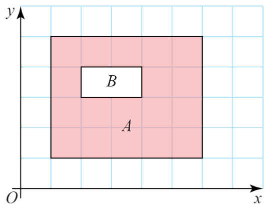
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$10\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 4\begin{pmatrix}\frac{4}{3}\\ 2\end{pmatrix} + 6\begin{pmatrix}3\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{10}\begin{pmatrix}\frac{70}{3}\\ 20\end{pmatrix}$$

$$= \begin{pmatrix}\frac{7}{3}\\ 2\end{pmatrix}$$

1 e Label the two rectangles A and B.



Rectangle A has an area of 20 square units and its centre of mass lies at (3.5, 3).
Rectangle B has an area of 2 square units and its centre of mass lies at (3, 3.5).

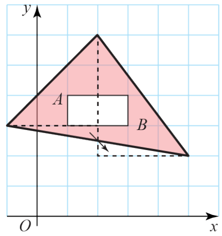
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$18\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = 20\begin{pmatrix}3.5\\3\end{pmatrix} - 2\begin{pmatrix}3\\3.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \frac{1}{18}\begin{pmatrix}64\\53\end{pmatrix}$$

$$= \begin{pmatrix}\frac{32}{9}\\\frac{53}{18}\end{pmatrix}$$

1 f Removing the small triangle from below triangle A and placing it below triangle B gives two right-angled triangles, one of area 4.5 square units, the other of area 6 square units. Therefore the total area of the original triangle is 10.5 square units.



The centre of mass of the original triangle is given by

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) \text{ where}$$

$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of the triangle, so

$$\left(\frac{-1+2+5}{3}, \frac{3+6+2}{3}\right) = \left(2, \tfrac{11}{3}\right)$$

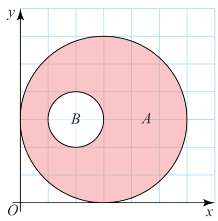
The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$8.5\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 10.5\begin{pmatrix}2\\ \frac{11}{3}\end{pmatrix} - 2\begin{pmatrix}2\\ 3.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{8.5}\begin{pmatrix}17\\ 31.5\end{pmatrix}$$

$$= \begin{pmatrix}2\\ \frac{63}{17}\end{pmatrix}$$

g Label the two circles A and B.



Circle A has an area of 9π square units and its centre of mass lies at (3, 3).
Circle B has an area of π square units and its centre of mass lies at (2, 3).

$$8\pi\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 9\pi\begin{pmatrix}3\\ 3\end{pmatrix} - \pi\begin{pmatrix}2\\ 3\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{8}\begin{pmatrix}25\\ 24\end{pmatrix}$$

$$= \begin{pmatrix}\frac{25}{8}\\ 3\end{pmatrix}$$

2 Q is the point $(2a, a)$.
Divide $PQRST$ into a rectangle and a triangle. Let T be the origin and let TS lie on the x-axis.
The rectangle has an area of $8a^2$ square units and its centre of mass lies at $(2a, a)$.
The triangle has an area of $2a^2$ square units and its centre of mass lies at $\left(2a, \frac{5a}{3}\right)$.

The centre of mass of the figure $(\bar{x}, \bar{y})$ is given by

$$6a^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 8a^2\begin{pmatrix}2a\\ a\end{pmatrix} - 2a^2\begin{pmatrix}2a\\ \frac{5a}{3}\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \frac{1}{6}\begin{pmatrix}12a\\ \frac{14a}{3}\end{pmatrix}$$

$$= \begin{pmatrix}2a\\ \frac{7a}{9}\end{pmatrix}$$

The centre of mass of $PQRST$ is $\frac{2a}{9}$ units from Q.

3 We choose axes with origin at A and x-axis parallel to AC, so that C has coordinates $(5a,0)$ and B has coordinates $\left(\frac{5a}{2},\frac{5\sqrt{3}a}{2}\right)$ so that the centre of mass of the complete triangle ABC is

$$\frac{1}{3}\begin{pmatrix}0\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}5a\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}\frac{5a}{2}\\\frac{5\sqrt{3}a}{2}\end{pmatrix}=\begin{pmatrix}\frac{5a}{2}\\\frac{5\sqrt{3}a}{6}\end{pmatrix}$$

The triangle has mass proportional to its area, which is

$$\frac{1}{2}\times 5a\times\frac{5\sqrt{3}a}{2}=\frac{25\sqrt{3}a^2}{4}$$

Now consider the square $DEFG$. By symmetry, its centre of mass is $\left(\frac{5a}{2},\frac{3a}{2}\right)$ and it has mass a^2.

Hence the centre of mass (x,y) of the lamina satisfies

$$\left(\frac{25\sqrt{3}a^2}{4}-a^2\right)\begin{pmatrix}x\\y\end{pmatrix}=\frac{25\sqrt{3}a^2}{4}\begin{pmatrix}\frac{5a}{2}\\\frac{5\sqrt{3}a}{6}\end{pmatrix}-a^2\begin{pmatrix}\frac{5a}{2}\\\frac{3a}{2}\end{pmatrix}$$

By considering the y-component,

$$\frac{\left(25\sqrt{3}-4\right)a^2y}{4}=\frac{25\sqrt{3}a^2}{4}\frac{5\sqrt{3}a}{6}-\frac{3a^3}{2}$$

So

$$\frac{\left(25\sqrt{3}-4\right)y}{4}=\frac{375a}{24}-\frac{3a}{2}=\frac{113a}{8}$$

So

$$y=\frac{113a}{2\left(25\sqrt{3}-4\right)}$$

Now in this coordinate system the distance from B to the centre of mass is

$$\frac{5\sqrt{3}a}{2}-y$$

Hence the distance is

$$\frac{5\sqrt{3}a}{2}-\frac{113a}{2\left(25\sqrt{3}-4\right)}\simeq 2.89a$$

4 a We choose coordinates with the origin at A and the x-axis parallel to AC, hence C has coordinates (24, 0) and B has coordinates $(0,18)$ hence the coordinates of the centre of mass is given by the average, hence

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{3}\begin{pmatrix}0\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}24\\0\end{pmatrix}+\frac{1}{3}\begin{pmatrix}0\\18\end{pmatrix}=\begin{pmatrix}8\\6\end{pmatrix}$$

Hence the distance from A to the centre of mass is

$$\sqrt{8^2+6^2}=10$$

4 b We can treat the lamina as a single particle of mass 15 kg located at the centre of mass, so that the centre of mass of the new system satisfies

$$20\begin{pmatrix} x \\ y \end{pmatrix} = 15\begin{pmatrix} 8 \\ 6 \end{pmatrix} + 5\begin{pmatrix} 24 \\ 0 \end{pmatrix}$$

So

$$20\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 240 \\ 90 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 4.5 \end{pmatrix}$$

5 a We choose coordinates so that O is the origin and that the x-axis is parallel to PQ then by considering the lamina as two rectangles joined together the centre of mass (x, y) satisfies

$$48\begin{pmatrix} x \\ y \end{pmatrix} = 36\begin{pmatrix} 3 \\ 3 \end{pmatrix} + 12\begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

$$48\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 108+108 \\ 108+60 \end{pmatrix} = \begin{pmatrix} 216 \\ 168 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4.5 \\ 3.5 \end{pmatrix}$$

b The total mass of the lamina is $48 \times 30 = 1440$, so the centre of mass of the new system satisfies

$$2140\begin{pmatrix} x \\ y \end{pmatrix} = 1440\begin{pmatrix} 4.5 \\ 3.5 \end{pmatrix} + 200\begin{pmatrix} 0 \\ 6 \end{pmatrix} + 500\begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

So

$$214\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1248 \\ 924 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{624}{107} \\ \frac{462}{107} \end{pmatrix} = \begin{pmatrix} 5.83 \\ 4.32 \end{pmatrix}$$

6 a By decomposing the lamina into rectangles of dimensions 10 × 4, 2 × 2 and 4 × 2 and choosing coordinates with origin at A and AH lies on the x-axis we have the centre of mass (x, y) will satisfy

$$(40+4+8)\begin{pmatrix} x \\ y \end{pmatrix} = 40\begin{pmatrix} 5 \\ 4 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 8\begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

So

$$52\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200+4+72 \\ 160+4 \end{pmatrix}$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{52}\begin{pmatrix} 276 \\ 164 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 69 \\ 41 \end{pmatrix}$$

6 b Since $\bar{x} = \frac{69}{13}$ and $\bar{y} = \frac{41}{13}$ for the original plate, the holes must be symmetrically placed about the line $x = \frac{69}{13}$ and $y = \frac{41}{13}$

c Since the point must be symmetrical around the line found in **b,** a $= \frac{107}{26}$

7 Choose coordinates such that the origin is at O and the line AB lies on the x-axis then we have that the x-coordinate of the centre of mass is $-\frac{a}{8}$ on the other hand it should satisfy

$$-\left(9\pi a^2 - \pi x^2\right)\frac{a}{8} = \left(9\pi a^2 \times 0\right) + \pi x^2 \times (-x)$$

So $\quad \left(\pi x^2 - 9\pi a^2\right)\frac{a}{8} = -\pi x^3$

So $\quad x^3 + \frac{a}{8}x^2 - \frac{9}{8}a^3 = 0$

Note that $x = a$ solves this, now factorising gives

$\left(x-a\right)\left(x^2 + \frac{9}{8}ax + \frac{9}{8}a^2\right)$

And noting that the quadratic factor has negative discriminant we see that $x = a$ is the only solution.

Challenge

We choose coordinates so that the origin is M, the centre of the hexagon and the line BE lies on the x-axis, by symmetry the centre of mass of the pentagon lies on BE as well, so it suffices to look at the x-coordinate which in modulus is equal to the distance from M to N.
Also, it is clear that N will lie to the left of M, so let this distance be d.

Now by considering the hexagon as composed of 6 equilateral triangles, its
Area is

$$6x^2 \sin\frac{\pi}{3} = 3\sqrt{3}x^2$$

Now considering the triangle removed to make the pentagon, the area of the pentagon is given by

$$\frac{3\sqrt{3}}{2}x^2 - x^2\cos\frac{\pi}{6}\sin\frac{\pi}{6}$$

$$= \frac{3\sqrt{3}}{2}x^2 - \frac{\sqrt{3}}{4}x^2 = \frac{5\sqrt{3}}{4}x^2$$

And the area of the triangle removed to make
the pentagon is

$$\frac{\sqrt{3}}{4}x^2$$

Hence d satisfies

$$-d \times \frac{5\sqrt{3}}{4}x^2 = \left(\frac{1}{2} + \frac{1}{3}\sin\frac{\pi}{3}\right)x \times -\frac{\sqrt{3}}{4}x^2$$

$$\frac{5\sqrt{3}}{4}x^2 d = \frac{2\sqrt{3}}{12}x^3$$

So

$$d = \frac{2\times 4}{12\times 5}x = \frac{2}{15}x$$