Chapter Review

1 a $v = 15 - 3t$
P is at rest when $v = 0$.
$0 = 15 - 3t$
$t = 5$

b $s = \int_0^5 v \, dt$

$= \int_0^5 (15 - 3t) \, dt$

$= \left[15t - \frac{3t^2}{2}\right]_0^5$

$= \left(15t \times 5 - \frac{3 \times 5^2}{2}\right) - 0$

$= 37.5$

The distance travelled by P is 37.5 m.

2 a $v = 6t + \frac{1}{2}t^3$

$a = \frac{dv}{dt}$

$= 6 + \frac{3}{2}t^2$

When $t = 4$, $a = 6 + \frac{3}{2} \times 4^2$
$= 30$

The acceleration of P when $t = 4$ is 30 m s^{-2}.

b $x = \int v \, dt$

$= \int \left(6t + \frac{1}{2}t^3\right) dt$

$= 3t^2 + \frac{t^4}{8} + k$, where k is a constant of integration.

When $t = 0$, $x = -5$
$-5 = 0 + 0 + k \Rightarrow k = -5$

$x = 3t^2 + \frac{t^4}{8} - 5$

When $t = 4$, $x = 3 \times 4^2 + \frac{4^4}{8} - 5$
$= 75$

$OP = 75$ m

3 a $v = \int a \, dt$

$= \int (2 - 2t) \, dt$

$= 2t - t^2 + c$, where c is a constant of integration.

When $t = 0$, $v = 8$

$8 = 0 - 0 + c \Rightarrow c = 8$

$v = 2t - t^2 + 8$

Let s m be the displacement from A at time t seconds.

$s = \int v \, dt$

$= \int (2t - t^2 + 8) \, dt$

$= t^2 - \frac{t^3}{3} + 8t + k$, where k is a constant of integration.

When $t = 0$, $s = 0$

$0 = 0 - 0 + 0 + k \Rightarrow k = 0$

Displacement of P from A at time t seconds $= t^2 - \frac{t^3}{3} + 8t$

b The greatest positive displacement of P occurs when $\frac{ds}{dt} = v = 0$:

$2t - t^2 + 8 = 0$

$t^2 - 2t - 8 = 0$

$(t + 2)(t - 4) = 0$

$t > 0$, so $t = 4$

When $t = 4$, $s = 4^2 - \frac{4^3}{3} + 8 \times 4$

$= 26\frac{2}{3} < 30$

Hence, P does not reach B.

c P returns to A when $s = 0$.

$t^2 - \frac{t^3}{3} + 8t = 0$

$t^3 - 3t^2 - 24t = 0$

$t(t^2 - 3t - 24) = 0$

$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-24)}}{2(1)}$

$= \frac{3 \pm \sqrt{105}}{2}$

$t > 0$, so $t = 6.62$

P returns to A when $t = 6.62$.

3 d Distance between two instants when P passes through A = 2 × maximum distance found in part **b**

$$= 2 \times \tfrac{80}{3}$$
$$= \tfrac{160}{3}$$

Total distance travelled by P between the two instants when it passes through A is $\tfrac{160}{3}$ m.

4 a $a = \dfrac{dv}{dt}$ so speed has maximum value when $a = 0$.

$0 = 8 - 2t^2$
$2t^2 = 8$
$t^2 = 4$
$t > 0$, so $t = 2$

$v = \int a \, dt$
$= \int (8 - 2t^2) dt$
$= 8t - \dfrac{2t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $v = 0$
$0 = 0 - 0 + c \Rightarrow c = 0$
$v = 8t - \dfrac{2t^3}{3}$

When $t = 2$, $v = (8 \times 2) - \dfrac{2 \times 2^3}{3}$
$= 16 - \dfrac{16}{3}$
$= \tfrac{32}{3}$

The greatest positive speed of the particle is $\tfrac{32}{3}$ m s^{-1}.

b $s = \int v \, dt$
$= \int \left(8t - \dfrac{2t^3}{3}\right) dt$
$= 4t^2 - \dfrac{t^4}{6}$

When $t = 2$, $s = 4 \times 2^2 - \dfrac{2^4}{6}$
$= 16 - \dfrac{16}{6}$
$= \tfrac{40}{3}$

The distance covered by the particle during the first two seconds is $\tfrac{40}{3}$ m.

5 a $s = -t^3 + 11t^2 - 24t$

$v = \frac{ds}{dt}$

$= -3t^2 + 22t - 24 \text{ m s}^{-1}$

b P is at rest when $v = 0$.

$-3t^2 + 22t - 24 = 0$

$3t^2 - 22t + 24 = 0$

$(3t - 4)(t - 6) = 0$

$t = \frac{4}{3}$ or $t = 6$

P is at rest when $t = \frac{4}{3}$ and $t = 6$.

c $a = \frac{dv}{dt}$

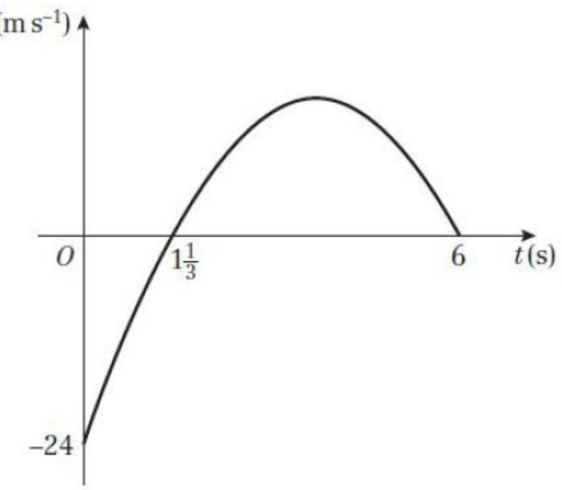
$= -6t + 22$

$a = 0$ when $0 = -6t + 22$

$t = \frac{11}{3}$

The acceleration is zero when $t = \frac{11}{3}$.

d



5 e The speed of P is 16 when $v = 16$ and $v = -16$.

When $v = 16$, $-3t^2 + 22t - 24 = 16$
$3t^2 - 22t + 40 = 0$
$(3t - 10)(t - 4) = 0$
$t = \frac{10}{3}$ or $t = 4$
When $v = -16$, $-3t^2 + 22t - 24 = -16$
$3t^2 - 22t + 8 = 0$

$$t = \frac{22 \pm \sqrt{22^2 - 4 \times 3 \times 8}}{2 \times 3}$$
$$= \frac{22 \pm \sqrt{388}}{6}$$
$$= 0.384 \text{ or } 6.95$$

From the graph in part **d**, the required values are $0 \le t < 0.384$, $\frac{10}{3} < t < 4$.

6 a The body is at rest when $v = 0$.
$3t^2 - 11t + 10 = 0$
$(3t - 5)(t - 2) = 0$
$t = \frac{5}{3}$ or $t = 2$

The body is at rest when $t = \frac{5}{3}$ and $t = 2$.

b $a = \frac{dv}{dt}$
$= 6t - 11$

When $t = 4$, $a = (6 \times 4) - 11$
$= 24 - 11$
$= 13$

When $t = 4$, the acceleration is 13 m s^{-2}.

6 c From part **a**, the body changes direction when $t = \frac{5}{3}$ and t = 2.

s_1 = displacement for $0 \le t \le \frac{5}{3}$

s_2 = displacement for $\frac{5}{3} \le t \le 2$

s_3 = displacement for $2 \le t \le 4$

$$s_1 = \int_0^{\frac{5}{3}} \left(3t^2 - 11t + 10\right) dt$$

$$= \left[t^3 - \frac{11t^2}{2} + 10t\right]_0^{\frac{5}{3}}$$

$$= \left(\left(\frac{5}{3}\right)^3 - \frac{11\times\left(\frac{5}{3}\right)^2}{2} + 10\times\frac{5}{3}\right) - 0$$

$$= \frac{125}{27} - \frac{275}{18} + \frac{50}{3}$$

$$= \frac{325}{54}$$

$$s_2 = \int_{\frac{5}{3}}^{2} \left(3t^2 - 11t + 10\right) dt$$

$$= \left[t^3 - \frac{11t^2}{2} + 10t\right]_{\frac{5}{3}}^{2}$$

$$= \left(2^3 - \frac{11\times 2^2}{2} + 10\times 2\right) - \frac{325}{54}$$

$$= 6 - \frac{325}{54}$$

$$= -\frac{1}{54}$$

$$s_3 = \int_2^4 \left(3t^2 - 11t + 10\right) dt$$

$$= \left[t^3 - \frac{11t^2}{2} + 10t\right]_2^4$$

$$= 4^3 - \frac{11\times 4^2}{2} + 10\times 4 - 6$$

$$= 64 - 88 + 40 - 6$$

$$= 10$$

Sign to be ignored when calculating distance:

Total distance = s1 + s2 + s3

$$= \frac{325}{54} + \frac{1}{54} + 10$$

$$= \frac{433}{27}$$

The total distance travelled is 16.0 m (1 d.p.).

7 a $v = \int a \, \mathrm{d}t$

$= \int (2t^3 - 8t) \, \mathrm{d}t$

$= \frac{t^4}{2} - 4t^2 + c$, where c is a constant of integration.

When $t = 0$, $v = 6$

$6 = 0 - 0 + c \Rightarrow c = 6$

$6 = \frac{t^4}{2} - 4t^2 + 6$

b $s = \int v \, \mathrm{d}t$

$= \int \left(\frac{t^4}{2} - 4t^2 + 6 \right) \mathrm{d}t$

$= \frac{t^5}{10} - \frac{4t^3}{3} + 6t + k$, where k is a constant of integration.

When $t = 0$, $s = 0$

$0 = 0 - 0 + 0 + k \Rightarrow k = 0$

$s = \frac{t^5}{10} - \frac{4t^3}{3} + 6t$

c Particle is at rest when $v = 0$.

$\frac{t^4}{2} - 4t^2 + 6 = 0$

$t^4 - 8t^2 + 12 = 0$

$(t^2 - 2)(t^2 - 6) = 0$

$t \geq 0$, so $t = \sqrt{2}$ or $t = \sqrt{6}$

The particle is at rest when $t = \sqrt{2}$ and $t = \sqrt{6}$.

8 $x = \frac{t^4 - 12t^3 + 28t^2 + 400}{50}$

$\frac{dx}{dt} = \frac{4t^3 - 36t^2 + 56t}{50}$

Maxima and minima occur when $\frac{dx}{dt} = 0$.

$\frac{4t^3 - 36t^2 + 56t}{50} = 0$

$t^3 - 9t^2 + 14t = 0$

$t(t^2 - 9t + 14) = 0$

$t(t - 2)(t - 7) = 0$

So turning points are at $t = 0$, $t = 2$ and $t = 7$.

From the sketch graph, the drone is at a greater height when $t = 2$ than when $t = 0$, and $t = 7$ corresponds to the minimum height over the given interval.

When $t = 2$, $x = \frac{2^4 - (12 \times 2^3) + (28 \times 2^2) + 400}{50}$

$= \frac{16 - 96 + 112 + 400}{50}$

$= 8.64$

When $t = 7$, $x = \frac{7^4 - (12 \times 7^3) + (28 \times 7^2) + 400}{50}$

$= \frac{2401 - 4116 + 1372 + 400}{50}$

$= 1.14$

The maximum height reached by the drone is 8.64 m, and the minimum height is 1.14 m.

9 When $t = 0$, $v = u = 800$, $s = 1500$
When $t = 25$, $v = 0$

Using $v = u + at$:
$0 = 800 + 25a$
$a = \frac{-800}{25} = -32$

$v = \int a \, dt$
$= \int -32 dt$
$= -32t + c$, where c is a constant of integration.

When $t = 0$, $v = 800$
$800 = 0 + c \Rightarrow c = 800$
$v = 800 - 32t$

$s = \int v \, dt$
$= \int (800 - 32t) dt$
$= 800t - 16t^2 + k$, where k is a constant of integration.

When $t = 0$, $s = 1500$
$1500 = 0 - 0 + k \Rightarrow k = 1500$
$s = 800t - 16t^2 + 1500$
So $a = 1500$, $b = 800$, $c = -16$

10 a $v = \int a \, dt$
$= \int (20 - 6t) dt$
$= 20t - 3t^2 + c$, where c is a constant of integration.

When $t = 0$, $v = 7$
$7 = 0 - 0 + c \Rightarrow c = 7$
$v = 20t - 3t^2 + 7$
$= 7 + 20t - 3t^2$

10 b When $t = 7$, $v = 7 + 20 \times 7 - 3 \times 7^2$
$= 7 + 140 - 147$
$= 0$

P's maximum speed in the interval $0 \leq t \leq 7$ is when $\frac{dv}{dt} = 0$

$\frac{dv}{dt} = 20 - 6t$
$0 = 20 - 6t$
$t = \frac{10}{3}$

When $t = \frac{10}{3}$, $v = 7 + 20 \times \frac{10}{3} - 3 \times \left(\frac{10}{3}\right)^2$
$= \frac{121}{3}$

The greatest speed of P in the interval $0 \leq t \leq 7$ is $40\frac{1}{3}$ m s^{-1}.

c $s = \int_0^7 \left(7 + 20t - 3t^2\right) dt$
$= \left[7t + 10t^2 - t^3\right]_0^7$
$= 7 \times 7 + 10 \times 7^2 - 7^3 - 0$
$= 196$

The distance travelled by P in the interval $0 \leq t \leq 7$ is 196 m.

11 a $a \propto (7 - t^2)$

So $a = k(7 - t^2)$

$= 7k - kt^2$

$$v = \int a \, \mathrm{d}t$$

$$= \int \left(7k - kt^2\right) \mathrm{d}t$$

$$= 7kt - \frac{kt^3}{3} + c, \text{ where } c \text{ is a constant of integration.}$$

When $t = 0$, $v = 0$

$0 = 0 - 0 + c \Rightarrow c = 0$

$$v = 7kt - \frac{kt^3}{3}$$

When $t = 3$, $v = 6$

$6 = 21k - 9k$

$12k = 6$

$k = \frac{1}{2}$

$$v = \frac{7t}{2} - \frac{t^3}{6}$$

$$s = \int v \, \mathrm{d}t$$

$$= \int \left(\frac{7t}{2} - \frac{t^3}{6}\right) \mathrm{d}t$$

$$= \frac{7t}{4} - \frac{t^4}{24} + d, \text{ where } d \text{ is a constant of integration.}$$

When $t = 0$, $v = 0$

$0 = 0 - 0 + d \Rightarrow d = 0$

$s = \frac{1}{24} t^2 (42 - t^2)$

12 a Time cannot be negative so $t \geq 0$.

When $t = 0$, $s = 0$

When $t = 5$, $s = 54 - 10 \times 53 + 25 \times 52$

$= 625 - 1250 + 625$

$= 0$

So when $t = 5$, the mouse is again at a distance of zero from the hole: it has returned.

12 b $s = t^4 - 10t^3 + 25t^2$

When mouse is at the greatest distance, $\frac{ds}{dt} = 0$

$\frac{ds}{dt} = 4t^3 - 30t^2 + 50t$

When $\frac{ds}{dt} = 0$, $4t^3 - 30t^2 + 50t = 0$

$$2t(2t^2 - 15t + 25) = 0$$
$$2t(2t - 5)(t - 5) = 0$$

$s = 0$ when $t = 0$ and $t = 5$, so maximum is when $t = 2.5$.

When $t = 2.5$, $s = 2.54 - 10 \times 2.53 + 25 \times 2.52$
$= 39.1$

The greatest distance of the mouse from the hole is 39.1 m.

13 a Any two from:
As the rocket rises, it burns large amounts of fuel, reducing mass and therefore allowing the same force to produce greater acceleration. (Hence positive terms in the equation.)

While the rocket remains in the atmosphere, the air resistance forces on it will be changing in a complex way: the increasing speed will cause them to increase, but reduced density of the atmosphere at greater heights will reduce their effect.

At greater heights, the gravitational pull of the Earth is less, which increases the resultant force on the rocket and increases the acceleration. (In practice, this effect is small compared to that of the mass reduction.)

As the fuel from each tank in the booster rockets is used up, they may become less efficient, reducing the thrust they produce. (The fuel feed mechanisms are designed to prevent this and ensure smooth transitions between each stage, but any astronaut can tell you that there is no such thing as a smooth journey into space!)

b $v = \int a \, dt$

$= \int\left((6.7\times10^{-7})t^3 - (3.98\times10^{-4})t^2 + 0.105t + 0.859\right)dt$

$= (1.68 \times 10^{-7})t^4 - (1.33 \times 10^{-4})t^3 + 0.0525t^2 + 0.859t + c$, where c is a constant of integration.

When $t = 124$, $v = 974$
$974 = (1.68 \times 10^{-7})(124)^4 - (1.33 \times 10^{-4})(124)^3 + 0.0525(124)^2 + 0.859(124) + c$

$\Rightarrow c = 266$
$v = (1.68 \times 10^{-7})t^4 - (1.33 \times 10^{-4})t^3 + 0.0525t^2 + 0.859t + 274$

c When $t = 446$, $v = (1.68 \times 10^{-7})(446)^4 - (1.33 \times 10^{-4})(446)^3 + 0.0525(446)^2 + 0.859(446) + 274$
$= 5950$

When $t = 446$ s, the velocity of the space shuttle is 5950 m s^{-1} (5.95 km s^{-1}).

13 d For this section of the flight:
$a = 28.6$, $u = 5950$, $v = 7850 \text{ m s}^{-1}$
$v = u + at$

$7850 = 5950 + 28.6t$
$$t = \frac{7850 - 5950}{28.6}$$
$$= 66.4$$

Total time to reach escape velocity $= 446 + 66.4$
$= 512$ (3 s.f.)

The rocket cuts its main engines 512 s after launch.

14 a $\mathbf{r}_P = (3t^2 + 4)\mathbf{i} + \left(2t - \frac{1}{2}\right)\mathbf{j}$
$\mathbf{v}_P = (6t\mathbf{i} + 2\mathbf{j}) \text{ m s}^{-1}$
$\mathbf{r}_Q = (t + 6)\mathbf{i} + \frac{3}{2}t^2\mathbf{j}$
$\mathbf{v}_Q = (\mathbf{i} + 3t\mathbf{j}) \text{ m s}^{-1}$

b Substituting $t = 2$ into $\mathbf{v}_P = 6t\mathbf{i} + 2\mathbf{j}$ gives:
$\mathbf{v}_P = 6(2)\mathbf{i} + 2\mathbf{j}$
$= 12\mathbf{i} + 2\mathbf{j}$
$|\mathbf{v}_P| = \sqrt{12^2 + 2^2}$
$= 2\sqrt{37}$
$\mathbf{v} = 12.2 \text{ m s}^{-1}$ (3 s.f.)

14 c When the particles are moving parallel to each other:

$m(6t\mathbf{i}+2\mathbf{j})=n(\mathbf{i}+3t\mathbf{j})$

Comparing coefficients:

For i:

$6mt=n \quad (1)$

For j:

$2m=3nt \Rightarrow m=\frac{3}{2}nt \quad (2)$

Substituting (2) into (1) gives:

$6\left(\frac{3}{2}nt\right)t=n$

$9t^2=1$

$t^2=\frac{1}{9}$

$t=\pm\frac{1}{3}$

Since $t\neq\frac{1}{3}$, the particles are moving parallel to each other at $t=\frac{1}{3}$ s

14 d If the particles collide then $\mathbf{r}_P = \mathbf{r}_Q$

$$\left(3t^2+4\right)\mathbf{i}+\left(2t-\frac{1}{2}\right)\mathbf{j}=\left(t+6\right)\mathbf{i}+\frac{3}{2}t^2\mathbf{j}$$

Comparing coefficients:

For i:

$$3t^2+4=t+6$$
$$3t^2-t-2=0$$
$$\left(3t+2\right)\left(t-1\right)=0$$
$$t=1 \text{ or } t=-\frac{2}{3}$$

For j:

$$2t-\frac{1}{2}=\frac{3}{2}t^2$$
$$4t-1=3t^2$$
$$3t^2-4t+1=0$$
$$\left(3t-1\right)\left(t-1\right)=0$$
$$t=1 \text{ or } t=\frac{1}{3}$$

Since both particles are in the same position at $t = 1$ they must collide.

Substituting $t = 1$ into $\mathbf{r}_P=\left(3t^2+4\right)\mathbf{i}+\left(2t-\frac{1}{2}\right)\mathbf{j}$ gives:

$$\mathbf{r}_P=\left(3\left(1\right)^2+4\right)\mathbf{i}+\left(2\left(1\right)-\frac{1}{2}\right)\mathbf{j}$$
$$=\left(7\mathbf{i}+\frac{3}{2}\mathbf{j}\right)\text{ m}$$

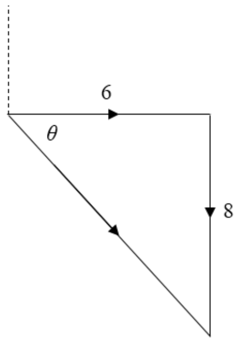
15 a $\mathbf{r}=\left(3t^2-4\right)\mathbf{i}+\left(8-4t^2\right)\mathbf{j}$

$\mathbf{v}=6t\mathbf{i}-8t\mathbf{j}$

$\mathbf{a}=6\mathbf{i}-8\mathbf{j}$ which is constant

15 b $|\mathbf{a}| = \sqrt{6^2 + (-8)^2}$

$= 10 \text{ m s}^{-2}$



$\tan\theta = \dfrac{8}{6}$

$\theta = 53.130...$

$= 53.1° \text{ (3 s.f.)}$

Therefore the particle makes an angle of 143.1° with the j vector

16 a $\mathbf{r} = 2\cos 3t\mathbf{i} - 2\sin 3t\mathbf{j}$

$\mathbf{v} = -6\sin 3t\mathbf{i} - 6\cos 3t\mathbf{j}$

When $t = \dfrac{\pi}{6}$

$\mathbf{v} = -6\sin 3\left(\dfrac{\pi}{6}\right)\mathbf{i} - 6\cos 3\left(\dfrac{\pi}{6}\right)\mathbf{j}$

$= -6\sin\left(\dfrac{\pi}{2}\right)\mathbf{i} - 6\cos\left(\dfrac{\pi}{2}\right)\mathbf{j}$

$= -6\mathbf{i} \text{ m s}^{-1}$

b $\mathbf{v} = -6\sin 3t\mathbf{i} - 6\cos 3t\mathbf{j}$

$\mathbf{a} = -18\cos 3t\mathbf{i} + 18\sin 3t\mathbf{j}$

$|\mathbf{a}| = \sqrt{(-18\cos 3t)^2 + (18\sin 3t)^2}$

$= \sqrt{324\cos^2 3t + 324\sin^2 3t}$

$= \sqrt{324(\cos^2 3t + \sin^2 3t)}$

Since $\cos^2 3t + \sin^2 3t = 1$

$|\mathbf{a}| = 18 \text{ m s}^{-2}$ therefore constant

17 a $\mathbf{v} = (4ct - 6)\mathbf{i} + (7 - c)t^2\mathbf{j}$

$\mathbf{a} = 4c\mathbf{i} + 2t(7 - c)\mathbf{j}$

Using $\mathbf{F} = m\mathbf{a}$ gives:

$\mathbf{F} = 0.5(4c\mathbf{i} + 2t(7 - c)\mathbf{j})$

$= 2c\mathbf{i} + t(7 - c)\mathbf{j}$ N as required

17 b When $t = 5$:

$$\mathbf{F} = 2c\mathbf{i} + (5)(7-c)\mathbf{j}$$

$$|\mathbf{F}| = \sqrt{(2c)^2 + (5)^2(7-c)^2}$$

$$= \sqrt{4c^2 + 25(49 - 14c + c^2)}$$

$$= \sqrt{29c^2 - 350c + 1225}$$

Since $\mathbf{F} = 17$

$$\sqrt{29c^2 - 350c + 1225} = 17$$

$$29c^2 - 350c + 1225 = 289$$

$$29c^2 - 350c + 936 = 0$$

$$c = \frac{350 \pm \sqrt{350^2 - 4(29)(936)}}{2(29)}$$

$$= \frac{350 \pm \sqrt{13\,924}}{58}$$

$$= \frac{350 \pm 118}{58}$$

$$c = \frac{234}{29} \text{ or } c = 4$$

18 a $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$$

$$\mathbf{v} = \int \left((8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}\right)\mathrm{d}t$$

$$= (2t^4 - 3t^2)\mathbf{i} + (4t^2 - 3t)\mathbf{j} + c$$

When $t = 2$, $\mathbf{v} = 16\mathbf{i} + 3\mathbf{j}$, therefore:

$$16\mathbf{i} + 3\mathbf{j} = \left(2(2)^4 - 3(2)^2\right)\mathbf{i} + \left(4(2)^2 - 3(2)\right)\mathbf{j} + c$$

$$= 20\mathbf{i} + 10\mathbf{j} + c$$

$$c = -4\mathbf{i} - 7\mathbf{j}$$

$$\mathbf{v} = \left((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}\right) \text{ m s}^{-1}$$

b When P is moving parallel to i:

$$4t^2 - 3t - 7 = 0$$

$$(4t - 7)(t + 1) = 0$$

$$t = \frac{7}{4} \text{ or } t = -1$$

t > 0 therefore $t = \frac{7}{4}$

19 a $\mathbf{a} = 4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j},\ t \geq 1$

$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$

$\mathbf{v} = \int \left(4t\mathbf{i} + 5t^{-\frac{1}{2}}\mathbf{j}\right)\mathrm{d}t$

$= 2t^2\mathbf{i} + 10t^{\frac{1}{2}}\mathbf{j} + c$

When $t = 1$, $\mathbf{v} = 4\mathbf{i} + 10\mathbf{j}$, therefore:

$4\mathbf{i} + 10\mathbf{j} = 2(1)^2\,\mathbf{i} + 10(1)^{\frac{1}{2}}\,\mathbf{j} + c$

$4\mathbf{i} + 10\mathbf{j} = 2\mathbf{i} + 10\mathbf{j} + c$

$c = 2\mathbf{i}$

$\mathbf{v} = 2(t^2 + 1)\mathbf{i} + 10t^{\frac{1}{2}}\mathbf{j}$

When $t = 5$:

$\mathbf{v} = 2\left((5)^2 + 1\right)\mathbf{i} + 10(5)^{\frac{1}{2}}\,\mathbf{j}$

$= 52\mathbf{i} + 10\sqrt{5}\mathbf{j}$

$|\mathbf{v}| = \sqrt{52^2 + \left(10\sqrt{5}\right)^2}$

$= \sqrt{3204}$

$= 6\sqrt{89}\ \mathrm{m\ s^{-1}}$

$= 56.6\ \mathrm{ms^{-1}}$ (3 s.f)

20 a $\mathbf{a} = 2t\mathbf{i} + 3\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t$

$\mathbf{v} = \int (2t\mathbf{i} + 3\mathbf{j})\,\mathrm{d}t$

$\mathbf{v} = t^2\mathbf{i} + 3t\mathbf{j} + c$

When $t = 0$, $\mathbf{v} = 3\mathbf{i} + 13\mathbf{j}$, therefore:

$3\mathbf{i} + 13\mathbf{j} = (0)^2\,\mathbf{i} + 3(0)\,\mathbf{j} + c$

$c = 3\mathbf{i} + 13\mathbf{j}$

$\mathbf{v} = (t^2 + 3)\mathbf{i} + (3t + 13)\,\mathbf{j}\,\mathrm{ms^{-1}}$

b When the train is moving north-east:

$t^2 + 3 = 3t + 13$

$t^2 - 3t - 10 = 0$

$(t - 5)(t + 2) = 0$

$t = 5$ or $t = -2$

Therefore $t = 5$

Challenge

1 $v = \int a\,\mathrm{d}t$

$= \int (3t^2 - 18t + 20)\,\mathrm{d}t$

$= t^3 - 9t^2 + 20t + c$, where c is a constant of integration.

When $t = 0$, $v = 0$

$0 = 0 - 0 + 0 + c \Rightarrow c = 0$

$v = t^3 - 9t^2 + 20t$

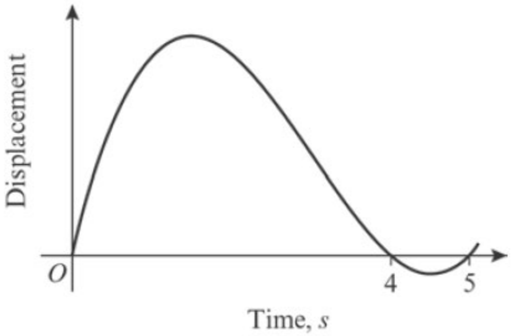
Checking for crossing points to find if the velocity becomes negative during first 5 s:

$t^3 - 9t^2 + 20t = 0$

$t(t^2 - 9t + 20) = 0$

$t(t - 4)(t - 5) = 0$

This means that displacement is positive for the first four seconds and negative in the fifth second (see sketch graph), so need to find distances separately.



s_1 = distance travelled in first 4 s

s_2 = distance travelled in fifth second

$$s_1 = \int_0^4 (t^3 - 9t^2 + 20t)\,\mathrm{d}t$$

$$= \left[\frac{t^4}{4} - 3t^3 + 10t^2\right]_0^4$$

$$= \left(\frac{4^4}{4} - 3 \times 4^3 + 10 \times 4^2\right) - 0$$

$= 64 - 192 + 160$

$= 32$

$$s_2 = \int_4^5 (t^3 - 9t^2 + 20t)\,\mathrm{d}t$$

$$= \left[\frac{t^4}{4} - 3t^3 + 10t^2\right]_4^5$$

$$= \left(\frac{5^4}{4} - 3 \times 5^3 + 10 \times 5^2\right) - 32$$

$$= \frac{625}{4} - 375 + 250 - 32$$

$= -0.75$

Total distance $= 32 + 0.75 = 32.75$

The particle covers 32.75 m in the first 5 s of its motion.

2 $v = \int a \, dt$

$= \int (6t + 2) dt$

$= 3t^2 + 2t + c$, where c is a constant of integration.

Assuming that velocity does not change direction during this time, distance travelled between $t = 3$ and $t = 4$

$v = \int_3^4 (3t^2 + 2t + c) dt$

$= \left[t^3 + t^2 + ct\right]_3^4$

$= (4^3 + 4^2 + 4c) - (3^3 + 3^2 + 3c)$

$= 64 + 16 + 4c - 27 - 9 - 3c$

$= 44 + c$

So $50 = 44 + c \Rightarrow c = 6$

$v = 3t^2 + 2t + 6$

When $t = 5$, $v = 3 \times 5^2 + 2 \times 5 + 6$

$= 75 + 10 + 6$

$= 91$

At 5 s, the velocity is 91 m s^{-1}.

3 a $s = (20 - t^2)\sqrt{t+1}, \ t \geq 0$

At $t = 0$:

$s = (20 - (0)^2)\sqrt{(0)+1}$

$= 20$ m

3 b $s = (20 - t^2)(t+1)^{\frac{1}{2}}$

$$v = \frac{1}{2}(20 - t^2)(t+1)^{-\frac{1}{2}} - 2t(t+1)^{\frac{1}{2}}$$

$$= \frac{(20 - t^2) - 4t(t+1)}{2(t+1)^{\frac{1}{2}}}$$

$$= \frac{20 - t^2 - 4t^2 - 4t}{2(t+1)^{\frac{1}{2}}}$$

$$= \frac{-5t^2 - 4t + 20}{2(t+1)^{\frac{1}{2}}}$$

When $v = 0$:

$$5t^2 + 4t - 20 = 0$$

$$t = \frac{-4 \pm \sqrt{4^2 - 4(5)(-20)}}{10}$$

$$= \frac{-4 \pm 4\sqrt{26}}{10}$$

$$= \frac{-2 \pm 2\sqrt{26}}{5}$$

$t = 1.64$ or $t = -2.44$ (3 s.f.), t = 1.64 only, so the particle changes direction once

3 c $s=\left(20-t^2\right)(t+1)^{\frac{1}{2}}$

When $s = 0$:

$\left(20-t^2\right)(t+1)^{\frac{1}{2}}=0$

$t^2=20$ or $t=-1$

Therefore $t=\pm 2\sqrt{5}$

And since t must be positive, $t=2\sqrt{5}$

Substituting $t=2\sqrt{5}$ into $v=\dfrac{-5t^2-4t+20}{2(t+1)^{\frac{1}{2}}}$ gives:

$$v=\frac{-5\left(2\sqrt{5}\right)^2-4\left(2\sqrt{5}\right)+20}{2\left(2\sqrt{5}+1\right)^{\frac{1}{2}}}$$

$$=\frac{-100-8\sqrt{5}+20}{2\left(2\sqrt{5}+1\right)^{\frac{1}{2}}}$$

$$=\frac{-80-8\sqrt{5}}{2\left(2\sqrt{5}+1\right)^{\frac{1}{2}}}$$

$$=\frac{-8\left(10+\sqrt{5}\right)}{2\left(2\sqrt{5}+1\right)^{\frac{1}{2}}}$$

So the speed of the particle is $\dfrac{4\left(10+\sqrt{5}\right)}{\left(2\sqrt{5}+1\right)^{\frac{1}{2}}}$ m s^{-1}

4 a $\mathbf{r} = 6\sin\omega t\mathbf{i} + 4\cos\omega t\mathbf{j}$

$\dot{\mathbf{r}} = 6\omega\cos\omega t\mathbf{i} - 4\omega\sin\omega t\mathbf{j}$

$|\dot{\mathbf{r}}| = \sqrt{(6\omega\cos\omega t)^2 + (-4\omega\sin\omega t)^2}$

$v = |\dot{\mathbf{r}}|$ therefore:

$$\begin{aligned} v^2 &= (6\omega\cos\omega t)^2 + (-4\omega\sin\omega t)^2 \\ &= 36\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t \\ &= 20\omega^2\cos^2\omega t + 16\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t \\ &= 20\omega^2\cos^2\omega t + 16\omega^2(\cos^2\omega t + \sin^2\omega t) \end{aligned}$$

Since $\cos^2\omega t + \sin^2\omega t = 1$

$$\begin{aligned} v^2 &= 20\omega^2\cos^2\omega t + 16\omega^2 \\ &= 2\omega^2(10\cos^2\omega t + 8) \end{aligned}$$

Since $\cos^2\omega t = \frac{1}{2}(1 + \cos 2\omega t)$

$$v^2 = 2\omega^2\left(10\left(\frac{1}{2}(1 + \cos 2\omega t)\right) + 8\right)$$

$= 2\omega^2(13 + 5\cos 2\omega t)$ as required

4 **b** $v = \sqrt{2\omega^2(13 + 5\cos 2\omega t)}$

$-1 \leq \cos 2\omega t \leq 1$

Therefore:

$\sqrt{2\omega^2(13-5)} \leq v \leq \sqrt{2\omega^2(13+5)}$

$\sqrt{16\omega^2} \leq v \leq \sqrt{36\omega^2}$

$4\omega \leq v \leq 6\omega$ as required

c When $t = \dfrac{\pi}{3\omega}$:

$\mathbf{r} = 6\sin\omega\left(\dfrac{\pi}{3\omega}\right)\mathbf{i} + 4\cos\omega\left(\dfrac{\pi}{3\omega}\right)\mathbf{j}$

$= 6\sin\left(\dfrac{\pi}{3}\right)\mathbf{i} + 4\cos\left(\dfrac{\pi}{3}\right)\mathbf{j}$

$= 3\sqrt{3}\mathbf{i} + 2\mathbf{j}$

$\tan\theta = \dfrac{2}{3\sqrt{3}}$

$\theta = 21.051...$

$\dot{\mathbf{r}} = 6\omega\cos\omega\left(\dfrac{\pi}{3\omega}\right)\mathbf{i} - 4\omega\sin\omega\left(\dfrac{\pi}{3\omega}\right)\mathbf{j}$

$= 6\omega\cos\left(\dfrac{\pi}{3}\right)\mathbf{i} - 4\omega\sin\left(\dfrac{\pi}{3}\right)\mathbf{j}$

$= 3\omega\mathbf{i} - 2\sqrt{3}\omega\mathbf{j}$

$\tan\theta = -\dfrac{2\sqrt{3}\omega}{3\omega}$

$= -\dfrac{2\sqrt{3}}{3}$

$\theta = -49.106...$

Therefore the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ is:

$21.051\ldots + 49.106\ldots = 70.158\ldots$

$= 70.2°$ (1 d.p.)