Exercise 2E

1 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int (6t^2\,\mathbf{i} + (8-4t^3)\,\mathbf{j})\mathrm{d}t$

$= 2t^3\,\mathbf{i} + (8t - t^4)\mathbf{j} + C$

When $t = 0$, $\mathbf{v} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$\mathbf{v} = 2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j}$

When $t = 2$

$\mathbf{v} = 16\,\mathbf{i} + (8\times 2 - 2^4)\,\mathbf{j} = 16\,\mathbf{i}$

The velocity of P when $t = 2$ is $16\,\mathbf{i}$ m s^{-1}

b $\mathbf{r} = \int \mathbf{v}\mathrm{d}t = \int (2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j})\mathrm{d}t$

$= \frac{1}{2}t^4\,\mathbf{i} + \left(4t^2 - \frac{1}{5}t^5\right)\mathbf{j} + D$

When $t = 0$, $\mathbf{r} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + D \Rightarrow D = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$\mathbf{r} = \frac{t^4}{2}\mathbf{i} + \left(4t^2 - \frac{t^5}{5}\right)\mathbf{j}$

When $t = 4$

$\mathbf{r} = \frac{4^4}{2}\mathbf{i} + \left(4\times 4^2 - \frac{4^5}{5}\right)\mathbf{j} = 128\,\mathbf{i} - 140.8\,\mathbf{j}$

The position vector of P when $t = 4$ is $(128\,\mathbf{i} - 140.8\,\mathbf{j})$ m

2 a $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int ((3t^2 + 2)\,\mathbf{i} + (6t - 4)\,\mathbf{j})\,\mathrm{d}t$

$= (t^3 + 2t)\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j} + A$

When $t = 2$, $\mathbf{v} = 9\,\mathbf{j}$

$9\,\mathbf{j} = 12\,\mathbf{i} + 4\,\mathbf{j} + A \Rightarrow A = -12\,\mathbf{i} + 5\,\mathbf{j}$

Hence

$\mathbf{r} = \left(t^3 + 2t - 12\right)\mathbf{i} + \left(3t^2 - 4t + 5\right)\mathbf{j}$

When $t = 0$

$\mathbf{r} = -12\,\mathbf{i} + 5\,\mathbf{j}$

$|\,\mathbf{r}\,|^2 = (-12)^2 + 5^2 = 169 \Rightarrow |\,\mathbf{r}\,| = \sqrt{169} = 13$

The distance of P from O when $t = 0$ is 13 m.

2 b When P is moving parallel to $\mathbf{i}$, $\mathbf{v}$ has no $\mathbf{j}$ component.

$\Rightarrow 6t - 4 = 0$

$\Rightarrow t = \frac{2}{3}$

$\mathbf{a} = \dot{\mathbf{v}} = 6t\,\mathbf{i} + 6\,\mathbf{j}$

When $t = \frac{2}{3}$ s, $\mathbf{a} = 4\,\mathbf{i} + 6\,\mathbf{j}$

The acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$ is $(4\,\mathbf{i} + 6\,\mathbf{j})\,\mathrm{m\,s^{-2}}$

3 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left((2t-4)\,\mathbf{i} + 6\sin t\,\mathbf{j}\right)\,\mathrm{d}t$

$= (t^2 - 4t)\,\mathbf{i} - 6\cos t\,\mathbf{j} + \mathbf{c}$

When $t = \frac{\pi}{2}$ s, $\mathbf{v} = 0\ \mathrm{m\,s^{-1}}$, so

$0 = \left(\frac{\pi^2}{4} - \frac{4\pi}{2}\right)\mathbf{i} - 0\,\mathbf{j} + \mathbf{c}$

$\mathbf{c} = \left(2\pi - \frac{\pi^2}{4}\right)\mathbf{i}$

The velocity of the particle is given by $\left[\left(t^2 - 4t + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 6\cos t\,\mathbf{j}\right]\ \mathrm{m\,s^{-1}}$

b When $t = \frac{3\pi}{2}$ s,

$\mathbf{v} = \left(\frac{9\pi^2}{4} - \frac{12\pi}{2} + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 0\,\mathbf{j}$

$\mathbf{v} = \left(\frac{8\pi^2}{4} - 6\pi + 2\pi\right)\mathbf{i}$

$\mathbf{v} = (2\pi^2 - 4\pi)\,\mathbf{i}$

Since the velocity only has an $\mathbf{i}$ component when $t = \frac{3\pi}{2}$ s, this is also the speed.

The speed of P at $\frac{3\pi}{2}$ s is $(2\pi^2 - 4\pi)\ \mathrm{m\,s^{-1}}$

4 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int ((5t-3)\,\mathbf{i} + (8-t)\,\mathbf{j})\,\mathrm{d}t$

$= \left(\frac{5}{2}t^2 - 3t\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2\right)\mathbf{j} + C$

When $t = 0,\ \mathbf{v} = 2\,\mathbf{i} - 5\,\mathbf{j}$

$2\,\mathbf{i} - 5\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 2\,\mathbf{i} - 5\,\mathbf{j}$

Hence

$\mathbf{v} = \left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}$

The velocity of P after t seconds is $\left(\left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}\right)$ m s^{-1}

b P is moving parallel to $\mathbf{i} - \mathbf{j}$ when, in the expression giving the velocity of P

$(\text{coefficient of } \mathbf{i} \text{ component}) = -1 \times (\text{coefficient of j component})$

$\left(\frac{5}{2}t^2 - 3t + 2\right) = -\left(8t - \frac{1}{2}t^2 - 5\right)$

$\frac{5}{2}t^2 - 3t + 2 = -8t + \frac{1}{2}t^2 + 5$

$2t^2 + 5t - 3 = 0$

$(2t-1)(t+3) = 0$

Hence,

$t = \frac{1}{2},\ -3$

As $t \geq 0,\ t = \frac{1}{2}$

c When $t = \frac{1}{2}$

$\mathbf{v} = \left(\frac{5}{8} - \frac{3}{2} + 2\right)\mathbf{i} + \left(4 - \frac{1}{8} - 5\right)\mathbf{j}$

$= \frac{9}{8}\mathbf{i} - \frac{9}{8}\mathbf{j}$

$|\mathbf{v}|^2 = \left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2 = 2 \times \left(\frac{9}{8}\right)^2$

$\Rightarrow |\mathbf{v}| = \frac{9\sqrt{2}}{8}$

The speed of P when it is moving parallel to $\mathbf{i} - \mathbf{j}$ is $\frac{9\sqrt{2}}{8}$ m s^{-1}

5 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int (2\mathbf{i} - 2t\,\mathbf{j})\,\mathrm{d}t$

$= 2t\,\mathbf{i} - t^2\mathbf{j} + A$

When $t = 0$, $\mathbf{v} = 2\mathbf{j}$

$2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + A \Rightarrow A = 2\mathbf{j}$

Hence

$\mathbf{v} = 2t\,\mathbf{i} + (2 - t^2)\,\mathbf{j}$

Let the position vector of P at time t seconds be $\mathbf{p}$ m.

$\mathbf{p} = \int \mathbf{v}\,\mathrm{d}t = \int 2t\,\mathbf{i} + (2 - t^2)\,\mathbf{j}$

$= t^2\,\mathbf{i} + \left(2t - \frac{1}{3}t^3\right)\mathbf{j} + B$

When $t = 0$, $\mathbf{v} = 6\mathbf{i}$

$6\mathbf{i} = 0\mathbf{i} + 0\mathbf{j} + B \Rightarrow B = 6\mathbf{i}$

Hence

$\mathbf{p} = (t^2 + 6)\,\mathbf{i} + \left(2t - \frac{1}{3}t^3\right)\mathbf{j}$

The position vector of P at time t seconds is $\left((t^2 + 6)\,\mathbf{i} + \left(2t - \frac{1}{3}t^3\right)\mathbf{j}\right)$ m

b Let the position vector of Q at time t seconds be $\mathbf{q}$ m.

$\mathbf{q} = \int \mathbf{v}\mathrm{d}t = \int ((3t^2 - 4)\mathbf{i} - 2t\,\mathbf{j})\mathrm{d}t$

$= (t^3 - 4t)\,\mathbf{i} - t^2\,\mathbf{j} + C$

From part **a**, when $t = 3$

$\mathbf{p} = (3^2 + 6)\,\mathbf{i} + \left(2 \times 3 - \frac{3^3}{3}\right)\mathbf{j} = 15\mathbf{i} - 3\mathbf{j}$

As the particles collide when $t = 3$, $\mathbf{q}(3) = \mathbf{p}(3)$

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$15\mathbf{i} - 3\mathbf{j} = (3^3 - 4 \times 3)\,\mathbf{i} - 3^2\,\mathbf{j} + C$

$15\mathbf{i} - 3\mathbf{j} = 15\mathbf{i} - 9\mathbf{j} + C$

$C = 6\mathbf{j}$

Hence,

$\mathbf{q} = (t^3 - 4t)\,\mathbf{i} + (6 - t^2)\mathbf{j}$

When $t = 0$, $\mathbf{q} = 6\mathbf{j}$

The position vector of Q at time $t = 0$ is $6\mathbf{j}$ m.

6 a $\mathbf{v} = \int \mathbf{a}\,dt = \int ((4t-3)\mathbf{i} - 6t^2 t\,\mathbf{j})dt$

$= (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j} + A$

When $t = 0$, $\mathbf{v} = 0$

$0 = 0\mathbf{i} + 0\mathbf{j} + A \Rightarrow A = 0$

$\mathbf{v} = (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j}$

When $t = \frac{1}{2}$

$$\mathbf{v} = \left(2\left(\frac{1}{2}\right)^2 - 3\times\frac{1}{2}\right)\mathbf{i} - 2\left(\frac{1}{2}\right)^3\mathbf{j}$$

$$= -\mathbf{i} - \frac{1}{4}\mathbf{j}$$

The velocity of P when $t = \frac{1}{2}$ is $\left(-\mathbf{i} - \frac{1}{4}\mathbf{j}\right)\text{m s}^{-1}$

b $\mathbf{r} = \int \mathbf{v}\,dt = \int ((2t^2 - 3t)\mathbf{i} - 2t^3\,\mathbf{j})dt$

$= \left(\frac{2}{3}t^3 - \frac{3}{2}t^2\right)\mathbf{i} - \frac{1}{2}t^4\,\mathbf{j} + B$

When $t = 0$, $\mathbf{r} = 4\mathbf{i} - 6\mathbf{j}$

$4\mathbf{i} - 6\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + B \Rightarrow B = 4\mathbf{i} - 6\mathbf{j}$

$\mathbf{r} = \left(\frac{2}{3}t^3 - \frac{3}{2}t^2 + 4\right)\mathbf{i} - \left(\frac{1}{2}t^4 + 6\right)\mathbf{j}$

When $t = 6$

$\mathbf{r} = (144 - 54 + 4)\mathbf{i} - (648 + 6)\mathbf{j} = 94\mathbf{i} - 654\mathbf{j}$

The position vector of P when $t = 6$ is $(94\mathbf{i} - 654\mathbf{j})$ m

7 a $\mathbf{v} = \int \mathbf{a}\,dt = \int ((8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j})dt$

$= (2t^4 - 3t^2)\mathbf{i} + (4t^2 - 3t)\mathbf{j} + C$

When $t = 2$, $\mathbf{v} = 16\mathbf{i} + 3\mathbf{j}$

$16\mathbf{i} + 3\mathbf{j} = 20\mathbf{i} + 10\mathbf{j} + C \Rightarrow C = -4\mathbf{i} - 7\mathbf{j}$

$\mathbf{v} = (2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}$

The velocity of P after t seconds is $\left((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}\right)\text{ ms}^{-1}$

b When P is moving parallel to **i**, the **j** component of the velocity is zero.

$4t^2 - 3t - 7 = 0$

$(t+1)(4t-7) = 0$

$t \geq 0 \Rightarrow t = \frac{7}{4}$ s

8 a $\mathbf{r}_P = \int \mathbf{v}_P \, \mathrm{d}t = \int \left((4t-3)\mathbf{i} + 4\mathbf{j} \right) \mathrm{d}t$

$= (2t^2 - 3t)\mathbf{i} + 4t\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r}_P = (\mathbf{i} + 2\mathbf{j})$ m

$\mathbf{i} + 2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = \mathbf{i} + 2\mathbf{j}$

The position of P at time t is given by $\left((2t^2 - 3t + 1)\mathbf{i} + (4t + 2)\mathbf{j} \right)$ m.

b i $\mathbf{r}_Q = \int \mathbf{v}_Q \, \mathrm{d}t = \int 5\mathbf{i} + k\mathbf{j} \, \mathrm{d}t$

$= 5t\mathbf{i} + kt\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r} = (11\mathbf{i} + 5\mathbf{j})$ m

$11\mathbf{i} + 5\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = 11\mathbf{i} + 5\mathbf{j}$

$\mathbf{r}_Q = (5t + 11)\mathbf{i} + (kt + 5)\mathbf{j}$

When the particles collide, their position vectors are identical, so:

$$\mathbf{r}_P = \mathbf{r}_Q$$

$(2t^2 - 3t + 1)\mathbf{i} + (4t + 2)\mathbf{j} = (5t + 11)\mathbf{i} + (kt + 5)\mathbf{j}$

Considering the coefficients of **i**:

$2t^2 - 3t + 1 = 5t + 11$

$2t^2 - 8t - 10 = 0$

$t^2 - 4t - 5 = 0$

$(t - 5)(t + 1) = 0$

The negative root can be ignored, so the particles collide when $t = 5$ s

Equating the coefficients of **j** when $t = 5$ s:

$20 + 2 = 5k + 5$

$k = \dfrac{22 - 5}{5} = 3.4$

The value of k is 3.4

ii Substituting $k = 3.4$ and $t = 5$ into equation for $\mathbf{r}_Q$:

$\mathbf{r}_Q = (25 + 11)\mathbf{i} + \left((5 \times 3.4) + 5 \right)\mathbf{j}$

$= 36\mathbf{i} + 22\mathbf{j}$

The position vector of the points where the particles meet is $(36\mathbf{i} + 22\mathbf{j})$ m.

Challenge

$\mathbf{v} = (3t\cos t\mathbf{i} + 5t\mathbf{j})\ \text{m s}^{-1}$, $\mathbf{r}_0 = (4\mathbf{i} + \mathbf{j})\,\text{m}$, $t = 0$ s

$\mathbf{r} = \int \mathbf{v}\,dt = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,dt$

To evaluate $\int t\cos t\,dt$, let $u = t$ and $\dfrac{dv}{dt} = \cos t$

Then $\dfrac{du}{dt} = 1$ and $v = \sin t$

Using integration by parts, $\int t\cos t\,dt = t\sin t - \int \sin t\,dt$

$= t\sin t + \cos t \qquad \textbf{(1)}$

$\mathbf{r} = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,dt$

$= \left(3\int t\cos t\,dt\right)\mathbf{i} + \left(5\int t\,dt\right)\mathbf{j}$

$= 3(t\sin t + \cos t)\mathbf{i} + \dfrac{5t^2}{2}\mathbf{j} + c \qquad$ (using **(1)**)

When $t = 0$ s, $\mathbf{r} = (4\mathbf{i} + \mathbf{j})\,\text{m}$

$4\mathbf{i} + \mathbf{j} = 3(0+1)\mathbf{i} + 0\mathbf{j} + c$

$c = \mathbf{i} + \mathbf{j}$

Hence, $\mathbf{r} = (3(t\sin t + \cos t) + 1)\mathbf{i} + \left(\dfrac{5t^2}{2} + 1\right)\mathbf{j}$

When $t = \dfrac{\pi}{2}$,

$$\mathbf{r} = \left(3\left(\frac{\pi}{2}\sin\frac{\pi}{2} + 0\right) + 1\right)\mathbf{i} + \left(\frac{5\pi^2}{2\times 4} + 1\right)\mathbf{j}$$

$$\mathbf{r} = \left(\frac{3\pi}{2} + 1\right)\mathbf{i} + \left(\frac{5\pi^2}{8} + 1\right)\mathbf{j}$$

The position of P at time $t = \dfrac{\pi}{2}$ s is $\left(\left(\dfrac{3\pi}{2} + 1\right)\mathbf{i} + \left(\dfrac{5\pi^2}{8} + 1\right)\mathbf{j}\right)$ m relative to O.

