Exercise 2D

1 **a** $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = 3\mathbf{i} + (3t^2 - 4)\mathbf{j}$

When $t = 3$,

$\mathbf{v} = 3\mathbf{i} + 23\mathbf{j}$

The velocity of P when $t = 3$ is $(3\mathbf{i} + 23\mathbf{j})\,\mathrm{m\,s^{-1}}$

b $\mathbf{a} = \dot{\mathbf{v}} = 6t\,\mathbf{j}$

When $t = 3$,

$\mathbf{a} = 18\,\mathbf{j}$

The acceleration of P when $t = 3$ is $18\,\mathbf{j}\,\mathrm{m\,s^{-2}}$

2 $m = 3\text{ g} = 0.003\text{ kg}$, $\mathbf{v} = (t^2\mathbf{i} + (2t - 3)\mathbf{j})\text{ m s}^{-1}$, $t = 4\text{ s}$, $\mathbf{F} = ?$

$\mathbf{a} = \dot{\mathbf{v}}$

$\mathbf{a} = 2t\mathbf{i} + 2\mathbf{j}$

When $t = 4$ s, $\mathbf{a} = 8\mathbf{i} + 2\mathbf{j}$

$\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = 0.003 \times (8\mathbf{i} + 2\mathbf{j})$

$\quad = 0.024\mathbf{i} + 0.006\mathbf{j}$

The force $\mathbf{F}$ is $(0.024\mathbf{i} + 0.006\mathbf{j})$ N.

3 $\mathbf{r} = 5e^{-3t}\mathbf{i} + 2\mathbf{j}$ m

a When P is directly north-east of O, coefficients of **i** and **j** are identical.

$5e^{-3t} = 2$

$e^{-3t} = 0.4$

$-3t = \ln 0.4$

$t = \frac{\ln 0.4}{-3} = 0.30543...$

P is directly north-east of O at $t = 0.305$ s (3 s.f.).

b $\mathbf{v} = \dot{\mathbf{r}}$

$\mathbf{v} = -15e^{-3t}\mathbf{i}$

However, when particle is north-east of O, by part **a** we see that $e^{-3t} = 0.4$

Hence

$\mathbf{v} = -(15 \times 0.4)\mathbf{i} = 6\mathbf{i}$

The speed at this time is 6 m s^{-1}

c The velocity vector has a single component in the direction of **i** and the coefficient is always negative (since e^{-3t} is always positive) so P is always moving west.

4 **a** $\mathbf{v} = \dot{\mathbf{r}} = 8t\,\mathbf{i} + (24 - 6t)\,\mathbf{j}$

When $t = 2$,

$\mathbf{v} = (16\mathbf{i} + 12\mathbf{j})$

$|\mathbf{v}|^2 = 16^2 + 12^2 = 400$

$\Rightarrow |\mathbf{v}| = \sqrt{400} = 20$

The speed of P when $t = 2$ is $20\,\text{m s}^{-1}$

b $\mathbf{a} = \dot{\mathbf{v}} = 8\mathbf{i} - 6\mathbf{j}$

Neither component is dependent on t, hence the acceleration is a constant.

$|\mathbf{a}|^2 = 8^2 + (-6)^2 = 100$

$\Rightarrow |\mathbf{a}| = \sqrt{100} = 10$

The magnitude of the acceleration is $10\,\text{m s}^{-1}$

5 **a** $\mathbf{v} = \dot{\mathbf{r}} = (3t^2 - 12)\mathbf{i} + (8t - 6)\,\mathbf{j}$

When $t = 0$,

$\mathbf{v} = -12\mathbf{i} - 6\mathbf{j}$

$|\mathbf{v}|^2 = (-12)^2 + (-6)^2 = 180$

$\Rightarrow |\mathbf{v}| = \sqrt{180} = 6\sqrt{5}$

The speed of projection is $6\sqrt{5}\,\text{m s}^{-1}$

b When P is moving parallel to **j** the velocity has no **i**-component.

$3t^2 - 12 = 0$

$\Rightarrow t^2 = 4$

$\Rightarrow t = 2$ (since $t \geq 0$)

c When $t = 2$

$\mathbf{r} = (2^3 - 12 \times 2)\mathbf{i} + (4 \times 2^2 - 6 \times 2)\,\mathbf{j} = -16\mathbf{i} + 4\mathbf{j}$

The position vector of P at the instant when P is moving parallel to **j** is $(-16\mathbf{i} + 4\mathbf{j})$m.

d $\mathbf{r} = \left(t^3 - 12t\right)\mathbf{i} + \left(4t^2 - 6t\right)\mathbf{j}$ m, $t = 5$ s, $m = 0.5$ kg, $\mathbf{F} = ?$

$\mathbf{v} = \dot{\mathbf{r}} = \left(3t^2 - 12\right)\mathbf{i} + \left(8t - 6\right)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6t\mathbf{i} + 8\mathbf{j}$

When $t = 5$ s, $\mathbf{a} = 30\mathbf{i} + 8\mathbf{j}$

Hence, $\mathbf{F} = m\mathbf{a}$

$= 0.5\left(30\mathbf{i} + 8\mathbf{j}\right)$

$\mathbf{F} = 15\mathbf{i} + 4\mathbf{j}$

$|\mathbf{F}| = \sqrt{15^2 + 4^2}$

$= 15.524...$

The magnitude of the force acting on P at $t = 5$ s is 15.5 N (3 s.f.).

6 a $\mathbf{v} = \dot{\mathbf{r}} = (6t - 6)\mathbf{i} + (3t^2 + 2kt)\mathbf{j}$

When $t = 3$,

$\mathbf{v} = 12\mathbf{i} + (27 + 6k)\mathbf{j}$

$(12\sqrt{5})^2 = |v|^2$

$720 = 12^2 + (27 + k)^2$

$720 = 144 + 729 + 324k + 36k^2$

$0 = 36k^2 + 324k + 153$

$0 = (2k + 1)(2k + 17)$

$k = -0.5, \ -8.5$

b If $k = -0.5$

$\mathbf{v} = (6t - 6)\mathbf{i} + (3t^2 - t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t - 1)\mathbf{j}$

When $t = 1.5$,

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + 8^2 = 100$

$\Rightarrow |\mathbf{a}| = 10$

If $k = -8.5$

$\mathbf{v} = (6t - 6)\mathbf{i} + (3t^2 - 17t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t - 17)\mathbf{j}$

When $t = 1.5$,

$\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + (-8)^2 = 100$

$\Rightarrow |\mathbf{a}| = 10$

For both of the values of k the magnitude of the acceleration of P when $t = 1.5$ is $10\,\mathrm{m\,s^{-2}}$

7 a $\mathbf{v} = \dot{\mathbf{r}} = 12t\mathbf{i} + \frac{5}{2}t^{\frac{3}{2}}\mathbf{j}$

When $t = 4$,

$\mathbf{v} = 48\mathbf{i} + \frac{5}{2} \times 4^{\frac{3}{2}}\mathbf{j}$

$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$

$= 48\mathbf{i} + 20\mathbf{j}$

$|\mathbf{v}|^2 = 48^2 + 20^2 = 2704^2$

$\Rightarrow |\mathbf{v}| = \sqrt{2704} = 52$

The speed of P when $t = 4$ is $52\,\mathrm{m\,s^{-1}}$

b $\mathbf{a} = \dot{\mathbf{v}} = 12\mathbf{i} + \frac{5}{2} \times \frac{3}{2}t^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{4}t^{\frac{1}{2}}\mathbf{j}$

You need to know that $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$

When $t = 4$

$\mathbf{a} = 12\mathbf{i} + \frac{15}{4} \times 4^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \frac{15}{2}\mathbf{j}$

The acceleration of P when $t = 4$ is $\left(12\mathbf{i} + \frac{15}{2}\mathbf{j}\right)\mathrm{m\,s^{-2}}$

8 a $\mathbf{v} = \dot{\mathbf{r}} = \left(18 - 12t^2\right)\mathbf{i} + 2ct\,\mathbf{j}$

When $t = 1.5$,

$\mathbf{v} = \left(18 - 12 \times 1.5^2\right)\mathbf{i} + 3c\,\mathbf{j}$

$= -9\,\mathbf{i} + 3c\,\mathbf{j}$

$15^2 = |v|^2$

$15^2 = (-9)^2 + (3c)^2$

$9c^2 = 15^2 - 9^2$

$9c^2 = 144$

$\Rightarrow c^2 = \dfrac{144}{9} = 16$

As c is positive, $c = 4$

b $\mathbf{a} = \dot{\mathbf{v}} = -24t\,\mathbf{i} + 2c\,\mathbf{j}$

Using $c = 4$ and $t = 1.5$

$\mathbf{a} = -36\,\mathbf{i} + 8\,\mathbf{j}$

The acceleration of P when $t = 1.5$ is $\left(-36\,\mathbf{i} + 8\,\mathbf{j}\right)\text{m s}^{-2}$

Acceleration is a vector and the answer should be given in vector form.

9 $\mathbf{r} = \left(2t^2 - 3t\right)\mathbf{i} + \left(5t + t^2\right)\mathbf{j}$ m

$\mathbf{v} = \dot{\mathbf{r}} = (4t - 3)\mathbf{i} + (5 + 2t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 4\mathbf{i} + 2\mathbf{j}$

$|\mathbf{a}| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

The acceleration is constant because the expression for it does not contain t, and it has a magnitude of $2\sqrt{5}$ m s^{-2}

10 a $\mathbf{r} = \left(20t - 2t^3\right)\mathbf{i} + kt^2\mathbf{j}$ m, $t = 2$ s, $|\mathbf{v}| = 16$ m s^{-1}

$\mathbf{v} = \dot{\mathbf{r}} = \left(20 - 6t^2\right)\mathbf{i} + 2kt\mathbf{j}$

$\mathbf{v}(2) = (20 - 24)\mathbf{i} + 4k\mathbf{j}$

$= -4\mathbf{i} + 4k\mathbf{j}$

$16^2 = |\mathbf{v}(2)|^2 = (-4)^2 + (4k)^2$

$256 = 16 + 16k^2$

$k^2 = \dfrac{256 - 16}{16} = 15$

$k = \sqrt{15}$

The value of k is $\sqrt{15}$.

10 b When P is moving parallel to $\mathbf{j}$, the coefficient of the $\mathbf{i}$ component of velocity is zero.
From part **a**, since $\mathbf{v} = \left(20 - 6t^2\right)\mathbf{i} + 2kt\mathbf{j}$, P is moving parallel to $\mathbf{j}$ when:

$$20 - 6t^2 = 0$$

$$t^2 = \frac{20}{6}$$

$$t = \sqrt{\frac{10}{3}}$$

Now $\mathbf{a} = \dot{\mathbf{v}} = -12t\mathbf{i} + 2\sqrt{15}\mathbf{j}$

At $t = \sqrt{\frac{10}{3}}$ s, the acceleration is given by:

$$\mathbf{a} = -12\sqrt{\frac{10}{3}}\mathbf{i} + 2\sqrt{15}\mathbf{j}$$

$$\mathbf{a} = -4\sqrt{30}\mathbf{i} + 2\sqrt{15}\mathbf{j}$$

When P is moving parallel to $\mathbf{j}$ its acceleration is $\left(-4\sqrt{30}\mathbf{i} + 2\sqrt{15}\mathbf{j}\right)$ m s^{-2}