Exercise 2C

1 **a** $s = \int v\,\mathrm{d}t$

$= \int \left(3t^2 - 1\right)\mathrm{d}t$

$= t^3 - t + c$, where c is a constant of integration.

When $t = 0$, $s = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$s = t^3 - t$

b $s = \int v\,\mathrm{d}t$

$= \int \left(2t^3 - \frac{3t^2}{2}\right)\mathrm{d}t$

$= \frac{t^4}{2} - \frac{t^3}{2} + c$, where c is a constant of integration.

When $t = 0$, $s = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$s = \frac{t^4}{2} - \frac{t^3}{2}$

c $s = \int v\,\mathrm{d}t$

$= \int \left(2\sqrt{t} + 4t^2\right)\mathrm{d}t$

$= \frac{4}{3}t^{\frac{3}{2}} + \frac{4t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $s = 0$:
$0 = 0 + 0 + c \Rightarrow c = 0$
$s = \frac{4}{3}t^{\frac{3}{2}} + \frac{4t^3}{3}$

2 **a** $v = \int a\,\mathrm{d}t$

$= \int \left(8t - 2t^2\right)\mathrm{d}t$

$= 4t^2 - \frac{2t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $v = 0$:
$0 = 0 - 0 + c \Rightarrow c = 0$
$v = 4t^2 - \frac{2t^3}{3}$

2 b $v = \int a\,\mathrm{d}t$

$= \int\left(6 + \frac{t^2}{3}\right)\mathrm{d}t$

$v = 6t + \frac{t^3}{9} + c$, where c is a constant of integration.

When $t = 0$, $v = 0$:
$0 = 0 + 0 + c \Rightarrow c = 0$

$v = 6t + \frac{t^3}{9}$

3 $x = \int v\,\mathrm{d}t$

$= \int\left(8 + 2t - 3t^2\right)\mathrm{d}t$

$= 8t + t^2 - t^3 + c$, where c is a constant of integration.

When $t = 0$, $x = 4$:
$4 = 0 + 0 - 0 + c \Rightarrow c = 4$
$x = 8t + t^2 - t^3 + 4$

When $t = 1$,
$x = 8 + 1 - 1 + 4 = 12$

The distance of P from O when $t = 1$ is 12 m.

4 a $v = \int a\,\mathrm{d}t$

$= \int\left(16 - 2t\right)\mathrm{d}t$

$= 16t - t^2 + c$, where c is a constant of integration.

When $t = 0$, $v = 6$:
$6 = 0 - 0 + c \Rightarrow c = 6$
$v = 16t - t^2 + 6$

b $x = \int v\,\mathrm{d}t$

$= \int\left(16t - t^2 + 6\right)\mathrm{d}t$

$= 8t^2 - \frac{t^3}{3} + 6t + k$, where k is a constant of integration.

When $t = 3$, $x = 75$:

$75 = 8 \times 3^2 - \frac{3^3}{3} + 6 \times 3 + k$

$\Rightarrow k = 75 - 72 + 9 - 18 = -6$

$x = 8t^2 - \frac{t^3}{3} + 6t - 6$

When $t = 0$,
$x = 0 - 0 + 0 - 6 = -6$

5 $v = 6t^2 - 51t + 90$
P is at rest when $v = 0$.
$6t^2 - 51t + 90 = 0$
$2t^2 - 17t + 30 = 0$
$(2t - 5)(t - 6) = 0$

P is at rest when $t = 2.5$ and when $t = 6$.

$$s = \int_{2.5}^{6} (6t^2 - 51t + 90)\,dt$$
$$= \left[2t^3 - \frac{51t^2}{2} + 90t\right]_{2.5}^{6}$$
$$= \left(2\times 6^3 - \frac{51\times 6^2}{2} + 90\times 6\right) - \left(2\times 2.5^3 - \frac{51\times 2.5^2}{2} + 90\times 2.5\right)$$
$= (432 - 918 + 540) - (31.25 - 159.375 + 225)$
$= -42.875\ldots$
$= -42.9$ (3 s.f.)

The negative sign indicates that the displacement is negative, but this can be ignored as distance is required.

The distance between the two points where P is at rest is 42.9 m (3 s.f.).

6 $s = \int v\,dt$
$= \int (12 + t - 6t^2)\,dt$
$= 12t + \frac{t^2}{2} - 2t^3 + c$, where c is a constant of integration.

When $t = 0$, $s = 0$:
$0 = 0 + 0 - 0 + c \Rightarrow c = 0$
$s = 12t + \frac{t^2}{2} - 2t^3$

$v = 0$ when
$12 + t - 6t^2 = 0$
$(3 - 2t)(4 + 3t) = 0$
$t > 0$, so $t = 1.5$

When $t = 1.5$, $s = 12 \times 1.5 + \frac{1.5^2}{2} - 2 \times 1.5^3$
$= 12.375\ldots$
$= 12.4$ (3 s.f.)

The distance of P from O when $v = 0$ is 12.4 m.

7 **a** $v = 4t - t^2$
P is at rest when $v = 0$.
$4t - t^2 = 0$
$t(4 - t) = 0$
$t > 0$, so $t = 4$

$x = \int v\,\mathrm{d}t$
$= \int (4t - t^2)\,\mathrm{d}t$
$= 2t^2 - \dfrac{t^3}{3} + c$, where c is a constant of integration.

When $t = 0$, $x = 0$
$0 = 0 + 0 + c \Rightarrow c = 0$
$x = 2t^2 - \dfrac{t^3}{3}$

When $t = 4$, $x = 2 \times 4^2 - \dfrac{4^3}{3}$
$= 10\frac{2}{3}$

b When $t = 5$, $x = 2 \times 5^2 - \dfrac{5^3}{3}$
$= 8\frac{1}{3}$

In the interval $0 \leq t \leq 5$, P moves to a point $10\frac{2}{3}$ m from O and then returns to a point $8\frac{1}{3}$ m from O.

The total distance moved is $10\frac{2}{3} + (10\frac{2}{3} - 8\frac{1}{3}) = 13$ m.

8 $x = \int v\,\mathrm{d}t$
$= \int (6t^2 - 26t + 15)\,\mathrm{d}t$
$= 2t^3 - 13t^2 + 15t + c$, where c is a constant of integration.

When $t = 0$, $x = 0$
$0 = 0 - 0 + 0 + c \Rightarrow c = 0$
$x = 2t^3 - 13t^2 + 15t$
$= t(2t^2 - 13t + 15)$
$= t(2t - 3)(t - 5)$

When $x = 0$ and t is non-zero, $t = 1.5$ or $t = 5$

P is again at O when $t = 1.5$ and $t = 5$.

9 a $x = \int v\,\mathrm{d}t$
$= \int (3t^2 - 12t + 5)\,\mathrm{d}t$
$= t^3 - 6t^2 + 5t + c$, where c is a constant of integration.

When $t = 0$, $x = 0$
$0 = 0 - 0 + 0 + c \Rightarrow c = 0$
$x = t^3 - 6t^2 + 5t$

P returns to O when $x = 0$.
$t^3 - 6t^2 + 5t = 0$
$t(t^2 - 6t + 5) = 0$
$t(t - 1)(t - 5) = 0$
P returns to O when $t = 1$ and $t = 5$.

b $v = 0$ when
$3t^2 - 12t + 5 = 0$
$$t = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(5)}}{6}$$
$= 0.473, 3.52$

So P does not turn round in the interval $2 \leq t \leq 3$.

When $t = 2$,
$x = 2^3 - 6 \times 2^2 + 5 \times 2$
$= 8 - 24 + 10$
$= -6$

When $t = 3$,
$x = 3^3 - 6 \times 3^2 + 5 \times 3$
$= 27 - 54 + 15$
$= -12$

The distance travelled by P in the interval $2 \leq t \leq 3$ is 6 m.

10 $v = \int a\,\mathrm{d}t$
$= \int (4t - 3)\,\mathrm{d}t$
$= 2t^2 - 3t + c$, where c is a constant of integration.

When $t = 0$, $v = 4$
$4 = 0 - 0 + c \Rightarrow c = 4$
$v = 2t^2 - 3t + 4$,

When $t = T$, $v = 4$ again
$4 = 2T^2 - 3T + 4$
$2T^2 - 3T = 0$
$T(2T - 3) = 0$
$T \neq 0$, so $T = 1.5$

11 a $v = \int a\,\mathrm{d}t$

$= \int (t-3)\,\mathrm{d}t$

$= \dfrac{t^2}{2} - 3t + c,$

When $t = 0$, $v = 4$

$4 = 0 - 0 + c \Rightarrow c = 4$

$v = \dfrac{t^2}{2} - 3t + 4$

b P is at rest when $v = 0$.

$\dfrac{t^2}{2} - 3t + 4 = 0$

$t^2 - 6t + 8 = 0$

$(t-2)(t-4) = 0$

$t = 2$ or $t = 4$

P is at rest when $t = 2$ and $t = 4$.

c $s = \displaystyle\int_2^4 \left(\frac{t^2}{2} - 3t + 4\right)\mathrm{d}t$

$= \left[\dfrac{t^3}{6} - \dfrac{3t^2}{2} + 4t\right]_2^4$

$= \left(\dfrac{4^3}{6} - \dfrac{3\times 4^2}{2} + 4\times 4\right) - \left(\dfrac{2^3}{6} - \dfrac{3\times 2^2}{2} + 4\times 2\right)$

$= \left(\dfrac{32}{3} - 24 + 16\right) - \left(\dfrac{4}{3} - 6 + 8\right)$

$= \dfrac{8}{3} - \dfrac{10}{3}$

$= -\tfrac{2}{3}$

The negative sign indicates that the displacement is negative, but this can be ignored as distance is required. The distance between the two points where P is at rest is $\tfrac{2}{3}$ m.

12 $v = \int a\,\mathrm{d}t$

$= \int (6t + 2)\,\mathrm{d}t$

$= 3t^2 + 2t + c$, where c is a constant of integration.

$s = \int v\,\mathrm{d}t$

$= \int (3t^2 + 2t + c)\,\mathrm{d}t$

$= t^3 + t^2 + ct + k$, where k is a constant of integration.

When $t = 2$, $s = 10$
$10 = 2^3 + 2^2 + 2c + k$
$2c + k = -2$ **(1)**

When $t = 3$, $s = 38$
$38 = 3^3 + 3^2 + 3c + k$
$3c + k = 2$ **(2)**
(2) – **(1)**:
$c = 4$

Substituting $c = 4$ into (**1**):
$2 \times 4 + k = -2$
$k = -10$

So the equations are:
$v = 3t^2 + 2t + 4$
$s = t^3 + t^2 + 4t - 10$

a When $t = 4$
$s = 4^3 + 4^2 + 4 \times 4 - 10$
$= 64 + 16 + 16 - 10$
$= 86$

When $t = 4$ s the displacement is 86 m.

b When $t = 4$
$v = 3 \times 4^2 + 2 \times 4 + 4$
$= 48 + 8 + 4$
$= 60$

When $t = 4$ s the velocity is 60 m s^{-1}.

13 a $a = 1 - \sin \pi t,\ t \geq 0$

$$v = \int a \, dt$$
$$= \int (1 - \sin \pi t) \, dt$$
$$= t + \frac{1}{\pi}\cos \pi t + c$$

When $t = 0$, $v = 0$, therefore:

$$(0) = (0) + \frac{1}{\pi}\cos \pi (0) + c$$
$$c = -\frac{1}{\pi}$$

Hence:
$$v = t + \frac{1}{\pi}\cos \pi t - \frac{1}{\pi} \text{ m s}^{-1}$$

b $s = \int v \, dt$

$$= \int \left(t + \frac{1}{\pi}\cos \pi t - \frac{1}{\pi} \right) dt$$
$$= \frac{1}{2}t^2 + \frac{1}{\pi^2}\sin \pi t - \frac{t}{\pi} + c$$

When $t = 0$, $s = 0$, therefore:

$$(0) = \frac{1}{2}(0)^2 + \frac{1}{\pi^2}\sin \pi (0) - \frac{(0)}{\pi} + c$$
$$c = 0$$

Hence:

$$s = \frac{1}{2}t^2 + \frac{1}{\pi^2}\sin \pi t - \frac{t}{\pi} \text{ m}$$

14 a $a = \sin 3\pi t,\ t \geq 0$

$$v = \int a \, \mathrm{d}t$$
$$= \int \sin 3\pi t \, \mathrm{d}t$$
$$= -\frac{1}{3\pi}\cos 3\pi t + c$$

When $t = 0,\ v = \frac{1}{3\pi}$, therefore:

$$\frac{1}{3\pi} = -\frac{1}{3\pi}\cos 3\pi(0) + c$$
$$c = \frac{2}{3\pi}$$

Hence:

$$v = -\frac{1}{3\pi}\cos 3\pi t + \frac{2}{3\pi} \text{ m s}^{-1}$$

b The maximum speed of the particle occurs when $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$, i.e when $a = 0$

$$\sin 3\pi t = 0$$
$$3\pi t = k\pi$$
$$t = \frac{k}{3}$$

Substituting $t = 0$ into $v = -\frac{1}{3\pi}\cos 3\pi t + \frac{2}{3\pi}$ gives:

$$v = -\frac{1}{3\pi}\cos 3\pi(0) + \frac{2}{3\pi}$$
$$= \frac{1}{3\pi}$$

Substituting $t = \frac{1}{3}$ gives:

$$v = -\frac{1}{3\pi}\cos 3\pi\left(\frac{1}{3}\right) + \frac{2}{3\pi}$$
$$= \frac{1}{\pi}$$

Maximum speed is $\frac{1}{\pi}$ m s^{-1}

14 c $s = \int v \, dt$

$= \int \left(-\frac{1}{3\pi} \cos 3\pi t + \frac{2}{3\pi} \right) dt$

$= -\frac{1}{9\pi^2} \sin 3\pi t + \frac{2t}{3\pi} + c$

When $t = 0$, $s = 1$, therefore:

$(1) = -\frac{1}{9\pi^2} \sin 3\pi(0) + \frac{2(0)}{3\pi} + c$

$c = 1$

Hence:

$s = -\frac{1}{9\pi^2} \sin 3\pi t + \frac{2t}{3\pi} + 1 \text{ m}$

15 a $a = -\cos 4\pi t, \ 0 \le t \le 4$

$v = \int a \, dt$

$= -\int \cos 4\pi t \, dt$

$= -\frac{1}{4\pi} \sin 4\pi t + c$

When $t = 0$, $v = 0$, therefore:

$(0) = -\frac{1}{4\pi} \sin 4\pi(0) + c$

$c = 0$

$v = -\frac{1}{4\pi} \sin 4\pi t \text{ m s}^{-1}$

15 b The maximum speed of the particle occurs when $\frac{dv}{dt} = 0$, i.e when $a = 0$

$$-\cos 4\pi t = 0$$

$$4\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$t = \frac{1}{8}, \frac{3}{8}, \ldots.$$

Substituting $t = \frac{1}{8}$ into $v = -\frac{1}{4\pi}\sin 4\pi t$ gives:

$$v = -\frac{1}{4\pi}\sin 4\left(\frac{1}{8}\right)\pi$$

$$= -\frac{1}{4\pi}$$

Substituting $t = \frac{3}{8}$ gives:

$$v = -\frac{1}{4\pi}\sin 4\left(\frac{3}{8}\right)\pi$$

$$= \frac{1}{4\pi}$$

The maximum speed is $\frac{1}{4\pi}$ m s^{-1}

c $s = \int v \, dt$

$$= -\int \frac{1}{4\pi}\sin 4\pi t \, dt$$

$$= \frac{1}{16\pi^2}\cos 4\pi t + c$$

When $t = 0$, $s = 0$, therefore:

$$(0) = \frac{1}{16\pi^2}\cos 4\pi(0) + c$$

$$c = -\frac{1}{16\pi^2}$$

Hence:

$$s = \frac{1}{16\pi^2}\cos 4\pi t - \frac{1}{16\pi^2} \text{ m}$$

15 d Since $s = \frac{1}{16\pi^2}\cos 4\pi t - \frac{1}{16\pi^2}$

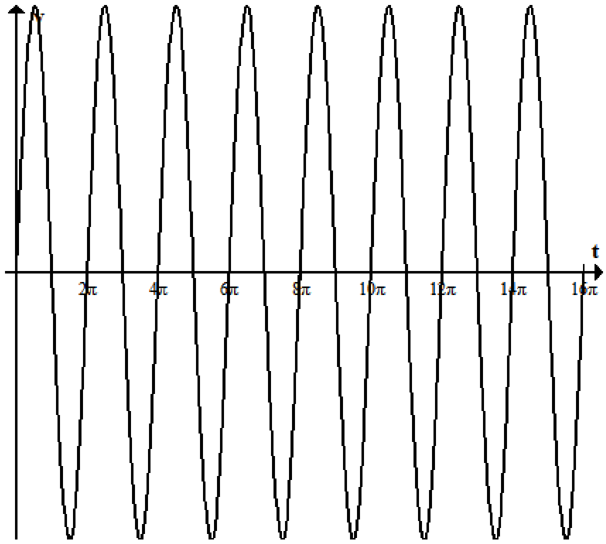
The maximum distance of the particle from O occurs when:
$\cos 4\pi t = -1$

Therefore:
$$s = -\frac{1}{16\pi^2} - \frac{1}{16\pi^2}$$
$$= -\frac{1}{8\pi^2}$$

Hence the greatest distance from O is $\frac{1}{8\pi^2}$ m

e $v = -\frac{1}{4\pi}\sin 4\pi t$

The curve of $\sin 4\pi t$ in the interval $0 \le t \le 4$ is shown below:



So the particle changes direction 16 times.

16 $a = 3\sqrt{t},\ t > 0$

$$v = \int a\,\mathrm{d}t$$
$$= 3\int t^{\frac{1}{2}}\,\mathrm{d}t$$
$$= 2t^{\frac{3}{2}} + c$$

When $t = 1$, $v = 2$, therefore:

$$(2) = 2(1)^{\frac{3}{2}} + c$$
$$c = 0$$

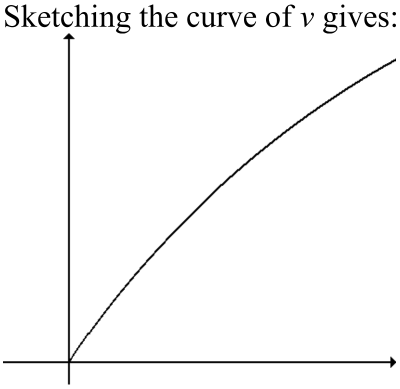
Hence:

$$v = 2t^{\frac{3}{2}}$$
$$s = \int v\,\mathrm{d}t$$
$$2\int_0^t t^{\frac{3}{2}}\,\mathrm{d}t = 16$$
$$\frac{2}{5}\left[t^{\frac{5}{2}}\right]_0^t = 8$$
$$t^{\frac{5}{2}} = 20$$
$$t = 3.314...$$
$$= 3.31\text{ s (3 s.f.)}$$

17 a In the interval $0 \le t \le 4$

$$v = 10t - 2t^{\frac{3}{2}}$$

Sketching the curve of v gives:



Therefore, $v_{\max}$ occurs when $t = 4$

Substituting $t = 4$ into $v = 10t - 2t^{\frac{3}{2}}$ gives:

$$v = 10(4) - 2(4)^{\frac{3}{2}}$$
$$= 40 - 16$$
$$= 24\text{ m s}^{-1}\text{ (3 s.f.)}$$

17 b $v = 10t - 2t^{\frac{3}{2}}$

$s = \int v \, dt$

$= \int \left(10t - 2t^{\frac{3}{2}}\right) dt$

$= 5t^2 - \frac{4}{5}t^{\frac{5}{2}} + c$

When $t = 0$, $s = 0$, therefore:

$(0) = 5(0)^2 - \frac{4}{5}(0)^{\frac{5}{2}} + c$

$c = 0$

$s = 5t^2 - \frac{4}{5}t^{\frac{5}{2}}$

When $t = 4$:

$s = 5(4)^2 - \frac{4}{5}(4)^{\frac{5}{2}}$

$= \frac{272}{5}$

$= 54.4$ m

c In the interval $t > 4$

$v = 24 - \left(\frac{t-4}{2}\right)^4$

When at rest:

$24 - \left(\frac{t-4}{2}\right)^4 = 0$

$\left(\frac{t-4}{2}\right)^4 = 24$

$(t-4)^4 = 384$

$t - 4 = \pm 4.426...$

$t > 4 \Rightarrow t = 8.426...$

$= 8.43$ s (3 s.f.)

17 d From part **b** distance travelled in the interval $0 \le t \le 4$ is:
$s = 54.4$ m
Distance travelled in the interval $4 \le t \le 8.4267....$ is:

$$\int_4^{8.4267...} 24 - \left(\frac{t-4}{2}\right)^4 \mathrm{d}t$$

$$\int_4^{10} 24 - \left(\frac{t-4}{2}\right)^4 \mathrm{d}t = \left|\left[24t - \frac{(t-4)^5}{80}\right]_4^{8.4267...}\right| + \left|\left[24t - \frac{(t-4)^5}{80}\right]_{8.4267...}^{10}\right|$$

$$= \left|\left(24(8.4267...) - \frac{((8.4267...)-4)^5}{80}\right) - \left(24(4) - \frac{((4)-4)^5}{80}\right)\right|$$

$$+ \left|\left(24(10) - \frac{((10)-4)^5}{80}\right) - \left(24(8.4267...) - \frac{((8.4267..)-4)^5}{80}\right)\right|$$

$$= |84.9931....| + |-38.1931...|$$

$$= 123.1863....$$

Total distance travelled is 54.4 + 123.1863…. = 178 m (3 s.f)

Challenge

$$v = \frac{1}{2}t^2 + 2,\ 0 \le t \le k \quad \text{and} \quad v = 10 + \frac{1}{3}t - \frac{1}{12}t^2,\ k \le t \le 10$$

For $0 \le t \le k$:

$$s = \int v \, \mathrm{d}t$$

$$s = \int_0^k \left(\frac{1}{2}t^2 + 2\right) \mathrm{d}t$$

$$= \left[\frac{1}{6}t^3 + 2t\right]_0^k$$

$$= \frac{1}{6}k^3 + 2k$$

For $k \le t \le 10$:

$$s = \int v \, \mathrm{d}t$$

$$s = \int_k^{10} \left(10 + \frac{1}{3}t - \frac{1}{12}t^2\right) \mathrm{d}t$$

$$= \left[10t + \frac{1}{6}t^2 - \frac{1}{36}t^3\right]_k^{10}$$

$$= \left(10(10) + \frac{1}{6}(10)^2 - \frac{1}{36}(10)^3\right) - \left(10k + \frac{1}{6}k^2 - \frac{1}{36}k^3\right)$$

$$= \frac{800}{9} - \left(10k + \frac{1}{6}k^2 - \frac{1}{36}k^3\right)$$

Total distance travelled:

$$s_{\text{total}} = \left|\frac{1}{6}k^3 + 2k\right| + \left|\frac{800}{9} - \left(10k + \frac{1}{6}k^2 - \frac{1}{36}k^3\right)\right| \quad \textbf{(1)}$$

At $t = k$:

$$\frac{1}{2}t^2 + 2 = 10 + \frac{1}{3}t - \frac{1}{12}t^2$$

$$\frac{7}{12}t^2 - \frac{1}{3}t - 8 = 0$$

Substituting $t = k$ gives:

$$\frac{7}{12}k^2 - \frac{1}{3}k - 8 = 0$$

$$7k^2 - 4k - 96 = 0$$

$$(7k + 24)(k - 4)$$

$$k = -\frac{24}{7} \text{ or } k = 4$$

Since k lies between 0 and 10, $k = 4$

Substituting $k = 4$ into (1) gives:

$$s_{\text{total}} = \left|\frac{1}{6}(4)^3 + 2(4)\right| + \left|\frac{800}{9} - \left(10(4) + \frac{1}{6}(4)^2 - \frac{1}{36}(4)^3\right)\right|$$

$$= \frac{56}{3} + 48$$

$$= \frac{200}{3}$$