Exercise 2B

1 a $s = 4t^4 - \dfrac{1}{t}$

i $v = \dfrac{ds}{dt} = 16t^3 + \dfrac{1}{t^2}$

ii $a = \dfrac{dv}{dt} = 48t^2 - \dfrac{2}{t^3}$

b $x = \dfrac{2}{3}t^3 + \dfrac{1}{t^2}$

i $v = \dfrac{dx}{dt} = 2t^2 - \dfrac{2}{t^3}$

ii $a = \dfrac{dv}{dt} = 4t + \dfrac{6}{t^4}$

c $s = (3t^2 - 1)(2t + 5)$
$= 6t^3 + 15t^2 - 2t - 5$

i $v = \dfrac{ds}{dt} = 18t^2 + 30t - 2$

ii $a = \dfrac{dv}{dt} = 36t + 30$

d $x = \dfrac{3t^4 - 2t^3 + 5}{2t} = \dfrac{3t^3}{2} - t^2 + \dfrac{5}{2t}$

i $v = \dfrac{dx}{dt} = \dfrac{9t^2}{2} - 2t - \dfrac{5}{2t^2}$

ii $a = \dfrac{dv}{dt} = 9t - 2 + \dfrac{5}{t^3}$

2 a $x = 2t^3 - 8t$
$v = \dfrac{dx}{dt} = 6t^2 - 8$
When $t = 3$, $v = 6 \times 9 - 8 = 46$
The velocity of the particle when $t = 3$ is 46 m s^{-1}.

2 b $a = \dfrac{dv}{dt} = 12t$

When $t = 2$, $a = 12 \times 2 = 24$
The acceleration of the particle when $t = 2$ is 24 m s^{-2}.

3 P is at rest when $v = 0$.
$12 - t - t^2 = 0$
$(4 + t)(3 - t) = 0$
$t = -4$ or $t = 3$
$t \geq 0$, so $t = 3$

$a = \dfrac{dv}{dt} = -1 - 2t$

When $t = 3$, $a = -1 - 2 \times 3 = -7$

The acceleration of P when P is instantaneously at rest is -7 m s^{-2}, or 7 m s^{-2} in the direction of x decreasing.

4 $x = 4t^3 - 39t^2 + 120t$
$v = \dfrac{dx}{dt} = 12t^2 - 78t + 120$

P is at rest when $v = 0$.
$12t^2 - 78t + 120 = 0$
$2t^2 - 13t + 20 = 0$
$(2t - 5)(t - 4) = 0$

P is at rest when $t = 2.5$ and $t = 4$.

When $t = 2.5$, $x = 4(2.5)^3 - 39(2.5)^2 + 120(2.5) = 118.75$

When $t = 4$, $x = 4(4)^3 - 39(4)^2 + 120(4) = 112$

The distance between the two points where P is instantaneously at rest is $118.75 - 112 = 6.75$ m.

5 $v = kt - 3t^2$

a $a = \dfrac{dv}{dt} = k - 6t$

When $t = 0$, $a = 4$
$k - 6 \times 0 = 4$
$k = 4$

5 b P is at rest when $v = 0$.
$4t - 3t^2 = 0$
$t(4 - 3t) = 0$
P is at rest when $t = 0$ and $t = \frac{4}{3}$.

When $t = \frac{4}{3}$, $a = 4 - 6 \times \frac{4}{3} = 4 - 8 = -4$
When P is next at rest, the acceleration is -4 m s^{-2}.

6 $s = \frac{1}{4}(4t^3 - 15t^2 + 12t + 30)$
$v = \frac{ds}{dt} = \frac{1}{4}(12t^2 - 30t + 12)$
The print head is at rest when $v = 0$.
$\frac{1}{4}(12t^2 - 30t + 12) = 0$
$12t^2 - 30t + 12 = 0$
$2t^2 - 5t + 2 = 0$
$(2t - 1)(t - 2) = 0$
The print head is at rest when $t = 0.5$ and $t = 2$.

When $t = 0.5$,
$s = \frac{1}{4}(4(0.5)^3 - 15(0.5)^2 + 12(0.5) + 30)$
$= \frac{1}{4}(0.5 - 3.75 + 6 + 30)$
$= 8.1875$

When $t = 2$,
$s = \frac{1}{4}(4(2)^3 - 15(2)^2 + 12(2) + 30)$
$= \frac{1}{4}(32 - 60 + 24 + 30)$
$= 6.5$

Distance between these two points $= 8.1875 - 6.5$
$= 1.6875$ cm
$= 1.7$ cm (1 d.p.)

The distance between the points when the print head is instantaneously at rest is 1.7 cm.

7 a $s = 0.4t^3 - 0.3t^2 - 1.8t + 5$
$v = \frac{ds}{dt} = 1.2t^2 - 0.6t - 1.8$
$\frac{dv}{dt} = 2.4t - 0.6$
$\frac{dv}{dt} = 0$ when $2.4t = 0.6$
$t = 0.25$

P is moving with minimum velocity at $t = 0.25$ s.

7 **b** When $t = 0.25$
$s = 0.4(0.25)^3 - 0.3(0.25)^2 - 1.8(0.25) + 5$
$= 4.54$ (3 s.f.)

When P is moving with minimum velocity, the displacement is 4.54 m.

c $v = \frac{ds}{dt} = 1.2t^2 - 0.6t - 1.8$
When $t = 0.25$, $v = 1.2 \times 0.25^2 - 0.6 \times 0.25 - 1.8$
$= -1.88$ (3 s.f.)

8 **a** $s = 4t^3 - t^4$
When $t = 4$,
$s = 4(4)^3 - 4^4 = 0$
The body returns to its starting position 4 s after leaving it.

b $s = 4t^3 - t^4 = s = t^3(4 - t)$
Since $t \geq 0$, t^3 is always positive.
Since $t \leq 4$, $(4 - t)$ is always positive.
So for $0 \leq t \leq 4$, s is always non-negative.

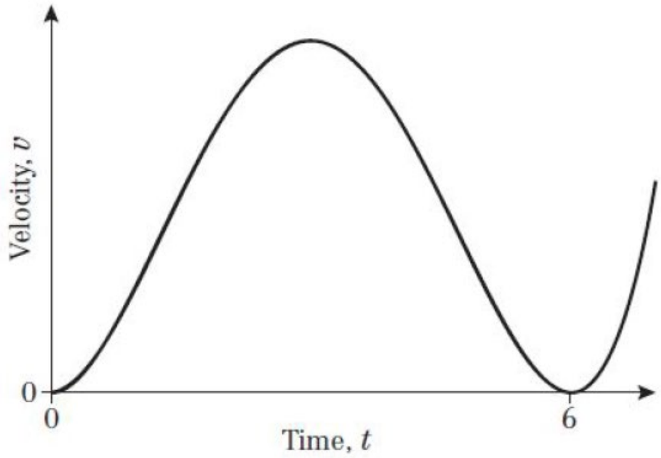
c $\frac{ds}{dt} = 12t^2 - 4t^3$
$\frac{ds}{dt} = 0$ when
$12t^2 - 4t^3 = 0$
$4t^2(3 - t) = 0$
$t = 0$ or 3

At $t = 0$, the body is at $s = 0$, so maximum displacement occurs when $t = 3$.

When $t = 3$, using factorised form of the equation of motion:
$s = 3^3(4 - 3) = 27$
The maximum displacement of the body from its starting point is 27 m.

9 a $v = t^2(6 - t)^2$

Velocity is zero when $t = 0$ and $t = 6$.
The graph touches the time axis at $t = 0$ and $t = 6$.
Graph only shown for $t \geq 0$, as this is the range over which equation is valid.



b $v = t^2(6 - t)^2$
$= t^2(36 - 12t + t^2)$
$= 36t^2 - 12t^3 + t^4$

$$\frac{dv}{dt} = 72t - 36t^2 + 4t^3$$

$\frac{dv}{dt} = 0$ when
$72t - 36t^2 + 4t^3 = 0$
$4t(18 - 9t + t^2) = 0$
$4t(3 - t)(6 - t) = 0$

The turning points are at $t = 0$, $t = 3$ and $t = 6$.
$v = 0$ when $t = 0$ and $t = 6$, therefore the maximum velocity occurs when $t = 3$.

When $t = 3$,
$v = 3^2(6 - 3)^2 = 9 \times 9 = 81$

The maximum velocity is 81 m s^{-1} and the body reaches this 3 s after leaving O.

10 a $v = 2t^2 - 3t + 5$

For this particle to come to rest, v must be 0 for some positive value of t.

$2t^2 - 3t + 5 = 0$ must have real, positive roots.
$b^2 - 4ac = (-3)^2 - 4(2)(5)$
$= 9 - 40$
$= -31 < 0$

The equation therefore has no real roots, so v is never zero.

10 b $v = 2t^2 - 3t + 5$

$\frac{dv}{dt} = 4t - 3$

$\frac{dv}{dt} = 0$ when $4t = 3$

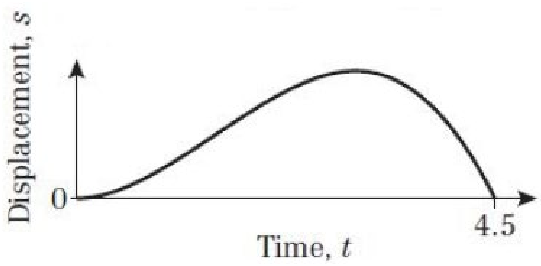
$t = 0.75$

Minimum velocity is when $t = 0.75$.
When $t = 0.75$, $v = 2(0.75)^2 - 3(0.75) + 5$
$= 1.125 - 2.25 + 5$
$= 3.875$
$= 3.88$ (3 s.f.)
The minimum velocity of the particle is 3.88 m s^{-1}.

11 a $s = \frac{9t^2}{2} - t^3$
$= t^2(4.5 - t)$

Displacement is zero when $t = 0$ and $t = 4.5$.
The graph touches the time axis at $t = 0$ and crosses it at $t = 4.5$.

Graph only shown for $0 \le t \le 4.5$, as this is range over which equation is valid.
The curve is cubic, so not symmetrical.



b For values of $t > 4.5$, s is negative. However s is a distance and can only be positive. Therefore, we must have the restriction $0 \le t \le 4.5$ for the model to be valid.

11 c $s = \dfrac{9t^2}{2} - t^3$

$\dfrac{ds}{dt} = 9t - 3t^2$

$\dfrac{ds}{dt} = 0$ when

$9t - 3t^2 = 0$

$3t(3 - t) = 0$

The turning points are at $t = 0$ and $t = 3$.
$s = 0$ when $t = 0$, so maximum distance occurs when $t = 3$.

When t =3, using factorised form of the equation of motion:
$s = 3^2(4.5 - 3) = 9 \times 1.5 = 13.5$

The maximum distance of P from O is 13.5 m.

d $v = \dfrac{ds}{dt} = 9t - 3t^2$

$a = \dfrac{dv}{dt} = 9 - 6t$

When $t = 3$,
$a = 9 - 6 \times 3 = -9$
The magnitude of the acceleration of P at the maximum distance is 9 m s^{-2}.

12 $s = 3.6t + 1.76t^2 - 0.02t^3$

$\frac{ds}{dt} = 3.6 + 3.52t - 0.06t^2$

Maximum distance occurs when $\frac{ds}{dt} = 0$.

$\frac{ds}{dt} = 0$ when

$3.6 + 3.52t - 0.06t^2 = 0$

$3t^2 - 176t - 180 = 0$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{176 \pm \sqrt{(-176)^2 + (4)(3)(180)}}{2 \times 3}$$

$$= \frac{176 \pm \sqrt{33136}}{6}$$

$$= -1.005 \text{ or } 59.67$$

$t > 0$, so maximum distance occurs when $t = 59.67$.

When $t = 59.67$, $s = 3.6(59.67) + 1.76(59.67)^2 - 0.02(59.67)^3$

$= 2230$ (3 s.f.)

The maximum distance from the start of the track is 2230 m or 2.23 km.
Since this is less than 4 km, the train never reaches the end of the track.

13 a $s = 3t^{\frac{2}{3}} + 2e^{-3t}, \ t \geq 0$

$$v = \frac{ds}{dt} = 2t^{-\frac{1}{3}} - 6e^{-3t}$$

When $t = 0.5$

$$v = 2(0.5)^{-\frac{1}{3}} - 6e^{-3(0.5)}$$

$$= 1.8110...$$

$$= 1.81 \text{ m s}^{-1} \text{ (3 s.f.)}$$

b $v = 2t^{-\frac{1}{3}} - 6e^{-3t}$

$$a = \frac{dv}{dt} = \frac{2}{3}t^{-\frac{4}{3}} + 18e^{-3t}$$

When $t = 3$

$$a = -\frac{2}{3}(3)^{-\frac{4}{3}} + 18e^{-3(3)}$$

$$= -0.1518...$$

$$= -0.152 \text{ m s}^{-2} \text{ (3 s.f.)}$$

c $F = ma$

$$= 5(-0.1518...)$$

$$= -0.7592...$$

$$= -0.759 \text{ N (3 s.f.)}$$

14 a When $t = 4$, $s = \frac{1}{2}t$

$\frac{ds}{dt} = \frac{1}{2}$

Therefore:

$v = 0.5 \text{ m s}^{-1}$

b When $t = 22$, $s = \sqrt{t+3}$

$s = (t+3)^{\frac{1}{2}}$

$\frac{ds}{dt} = \frac{1}{2}(t+3)^{-\frac{1}{2}}$

$v = \frac{1}{2}((22)+3)^{-\frac{1}{2}}$

$= 0.1 \text{ m s}^{-1}$

15 a When $t = 2$, $s = 3^t + 3t$

$\frac{ds}{dt} = 3^t \ln 3 + 3$

$v = 3^2 \ln 3 + 3$

$= 12.887...$

$= 12.9 \text{ m s}^{-1}$ (3 s.f.)

b When $t = 10$, $s = -252 + 96t - 6t^2$

$\frac{ds}{dt} = 96 - 12t$

$v = 96 - 12(10)$

$= -24 \text{ m s}^{-1}$ (3 s.f.)

c For $0 \le t \le 3$:

$s = 3^t + 3t$

The maximum displacement occurs at $t = 3$

$s = 3^3 + 3(3)$

$= 36 \text{ m}$

For $3 \le t \le 6$:

$s = 24t - 36$

The maximum displacement occurs at $t = 6$

$s = 24(6) - 36$

$= 108 \text{ m}$

For $t > 6$:

$s = -252 + 96t - 6t^2$

The maximum displacement occurs when:

$\frac{ds}{dt} = 96 - 12t = 0$

$t = 8$

Substituting $t = 8$ into $s = -252 + 96t - 6t^2$ gives:

$s = -252 + 96(8) - 6(8)^2$

$= 132 \text{ m}$

Therefore 132 m from O.

15 d For $0 \le t \le 3$:

$s = 3^t + 3t$

$v = \frac{ds}{dt} = 3^t \ln 3 + 3$

When $v = 18\text{ m s}^{-1}$

$3^t \ln 3 + 3 = 18$

$3^t = \frac{15}{\ln 3}$

$t \ln 3 = \ln\left(\frac{15}{\ln 3}\right)$

$t = \frac{\ln\left(\frac{15}{\ln 3}\right)}{\ln 3}$

$= 2.379...$

Substituting $t = 2.379\ldots$ into $s = 3^t + 3t$ gives:

$s = 20.791\ldots$

$= 20.8\text{ m (3 s.f.)}$

For $3 \le t \le 6$:

$s = 24t - 36$

$v = \frac{ds}{dt} = 24$

So the particle is moving with a constant velocity of 24 m s^{-1} in this interval.

For $t > 6$:

$s = -252 + 96t - 6t^2$

$v = \frac{ds}{dt} = 96 - 12t$

When $v = 18\text{ m s}^{-1}$

$96 - 12t = 18$

$t = 6.5$

Substituting $t = 6.5$ into $s = -252 + 96t - 6t^2$ gives:

$s = -252 + 96(6.5) - 6(6.5)^2$

$= 118.5\text{ m}$

Therefore when the particle is moving at 18 m s^{-1}, $s = 20.8$ m or $s = 118.5$ m

16 a Since the runner completes the race in 25 s, $T = 25$ s

$s = k\sqrt{t},\ 0 \le t \le 25$

Substituting $s = 200$ and $t = 25$ into $s = k\sqrt{t}$ gives:

$200 = k\sqrt{25}$

$k = 40$

16 b $s = 40\sqrt{t} \Rightarrow s = 40t^{\frac{1}{2}}$

$v = \dfrac{ds}{dt} = 20t^{-\frac{1}{2}}$

Substituting $t = 25$ into $v = 20t^{-\frac{1}{2}}$ gives:

$v = 20(25)^{-\frac{1}{2}}$

$= 4 \text{ m s}^{-1}$

c $v = 20t^{-\frac{1}{2}} \Rightarrow v = \dfrac{20}{\sqrt{t}}$

Therefore for small values of t, v is much too large.
e.g. when $t = 0.01$ s, $v = 200 \text{ m s}^{-1}$

17 a $v = 2 + 8\sin kt,\ t \geq 0$

$a = \dfrac{dv}{dt} = 8k\cos kt$

Substituting $a = 4$ and $t = 0$ into $a = 8k\cos kt$ gives:

$8k\cos k(0) = 4$

$8k = 4$

$k = 0.5$

b $a = 0$ when:

$a = 4\cos\left(\dfrac{1}{2}t\right) = 0$

$\cos\left(\dfrac{1}{2}t\right) = 0$

$\dfrac{1}{2}t = \dfrac{\pi}{2} + k\pi$

$t = \pi + 2k\pi$

In the interval $0 \leq t \leq 4\pi$

$t = \pi$ or $t = 3\pi$

c $v = 2 + 8\sin\dfrac{t}{2} \Rightarrow \sin\dfrac{t}{2} = \dfrac{v-2}{8} \Rightarrow \sin^2\dfrac{t}{2} = \left(\dfrac{v-2}{8}\right)^2$ **(1)**

$a = 4\cos\left(\dfrac{1}{2}t\right) \Rightarrow a^2 = 16\cos^2\left(\dfrac{1}{2}t\right) \Rightarrow \cos^2\left(\dfrac{1}{2}t\right) = \dfrac{a^2}{16}$ **(2)**

Substituting **(1)** and **(2)** into $\sin^2\theta + \cos^2\theta = 1$ gives:

$\left(\dfrac{v-2}{8}\right)^2 + \dfrac{a^2}{16} = 1$

$(v-2)^2 + 4a^2 = 64$

$4a^2 = 64 - (v-2)^2$ as required

17 d $v = 2 + 8\sin\left(\frac{1}{2}t\right)$

$\frac{dv}{dt} = 4\cos\left(\frac{1}{2}t\right)$

Maximum velocity occurs when $\frac{dv}{dt} = 0$, therefore:

From part **b** this occurs when $t = \pi$ or $t = 3\pi$

Substituting $t = \pi$ into $v = 2 + 8\sin\left(\frac{1}{2}t\right)$ gives:

$$v = 2 + 8\sin\left(\frac{1}{2}(\pi)\right)$$
$$= 10 \text{ m s}^{-1}$$

$$a = 4\cos\left(\frac{1}{2}t\right)$$

Maximum acceleration occurs when $\frac{da}{dt} = 0$, therefore:

$$\frac{da}{dt} = -2\sin\left(\frac{1}{2}t\right)$$
$$-2\sin\left(\frac{1}{2}t\right) = 0$$
$$\frac{1}{2}t = 0 + k\pi$$
$$t = 2k\pi$$

So in the interval $0 \le t \le 4\pi$, maximum acceleration occurs at:
$t = 0$, $t = 2\pi$ and $t = 4\pi$

Substituting $t = 0$ into $a = 4\cos\left(\frac{1}{2}t\right)$ gives:

$$a = 4\cos\left(\frac{1}{2}(0)\right)$$
$$= 4 \text{ m s}^{-2}$$