Chapter review 3

1 $\mathbf{r}_{0A} = -3\mathbf{i} + 10\mathbf{j}$, $\mathbf{v}_A = 2\mathbf{i} + 2\mathbf{j}$
$\mathbf{r}_{0B} = 6\mathbf{i} + \mathbf{j}$, $\mathbf{v}_B = -\mathbf{i} + 5\mathbf{j}$

a At t hours after noon
$\mathbf{r}_A = -3\mathbf{i} + 10\mathbf{j} + t(2\mathbf{i} + 2\mathbf{j})$
$= (2t - 3)\mathbf{i} + (2t + 10)\mathbf{j}$
$\mathbf{r}_B = 6\mathbf{i} + \mathbf{j} + t(-\mathbf{i} + 5\mathbf{j})$
$= (6 - t)\mathbf{i} + (5t + 1)\mathbf{j}$
If the ships collide then
$(2t - 3) = (6 - t)$ and $(2t + 10) = (5t + 1)$ will give the same value of t.
Solving these equations gives
$3t = 9$ and $3t = 9$
$t = 3$ for both equations, so the ships will collide.

b $\overrightarrow{AB} = (6-t)\mathbf{i} + (5t+1)\mathbf{j} - ((t-3)\mathbf{i} + (t+10)\mathbf{j})$
$= (6-t)\mathbf{i} - (t-3)\mathbf{i} + (5t+1)\mathbf{j} - (t+10)\mathbf{j}$
$= (9-2t)\mathbf{i} + (4t-9)\mathbf{j}$

c At 15:00, $t = 3$, therefore
$\overrightarrow{AB} = (9 - 2(3))\mathbf{i} + (4(3) - 9)\mathbf{j}$
$= 3\mathbf{i} + 3\mathbf{j}$
$|\overrightarrow{AB}| = \sqrt{3^2 + 3^2}$
$= 3\sqrt{2}$
$= 4.24$ km (3 s.f.)

d When B is due north of A, the **i**-component of $\overrightarrow{AB}$ will be 0, so
$9 - 2t = 0$
$t = 4.5$
Thus, B will be due north of A at 16:30.

2 $\mathbf{r}_P = 0$ and $\mathbf{v}_P = 6\mathbf{i}$
$\mathbf{r}_Q = 12\mathbf{i} + 6\mathbf{j}$ and $\mathbf{v}_Q = -3\mathbf{i} + 6\mathbf{j}$

a $\mathbf{p} = 6t\mathbf{i}$
$\mathbf{q} = 12\mathbf{i} + 6\mathbf{j} + t(-3\mathbf{i} + 6\mathbf{j})$
$= (12 - 3t)\mathbf{i} + (6 + 6t)\mathbf{j}$

2 b $\overrightarrow{PQ} = (12-3t)\mathbf{i} + (6+6t)\mathbf{j} - 6t\mathbf{i}$
$= (12-9t)\mathbf{i} + (6+6t)\mathbf{j}$
when $t = 4$
$\overrightarrow{PQ} = (12-9(4))\mathbf{i} + (6+6(4))\mathbf{j}$
$= -24\mathbf{i} + 30\mathbf{j}$
$|\overrightarrow{PQ}| = \sqrt{(-24)^2 + 30^2}$
$= 38.4$ km (3 s.f.)

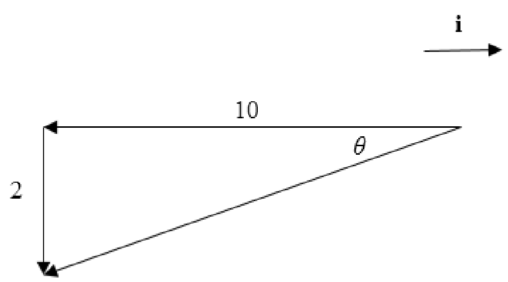
c When Q is due North of P
$12 - 9t = 0$
$9t = 12$
$t = \frac{4}{3}$

3 At $t = 0$, $\mathbf{u} = 5\mathbf{i} - 3\mathbf{j}$ and $\mathbf{a} = -3\mathbf{i} + \mathbf{j}$

a $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$= 5\mathbf{i} - 3\mathbf{j} + t(-3\mathbf{i} + \mathbf{j})$
$= (5 - 3t)\mathbf{i} + (t - 3)\mathbf{j}$
When P is moving parallel to the **i** vector
$t - 3 = 0$
$t = 3$

b When $t = 5$
$\mathbf{v} = (5 - 3t)\mathbf{i} + (t - 3)\mathbf{j}$
$= (5 - 3(5))\mathbf{i} + (5 - 3)\mathbf{j}$
$= -10\mathbf{i} - 2\mathbf{j}$
$|\mathbf{v}| = \sqrt{(-10)^2 + (-2)^2}$
$= \sqrt{104}$
$= 10.2$ m s^{-1} (3 s.f.)

c



$\tan\theta = \frac{2}{10}$
$\theta = 11.3...°$
The angle between the positive **i** vector and the motion of P is given by
$180 - 11.3… = 169°$ (3 s.f.)

4 At $t = 0$, $\mathbf{u} = 5\mathbf{i} - 3\mathbf{j}$ and at $t = 4$, $\mathbf{v} = -11\mathbf{i} + 5\mathbf{j}$

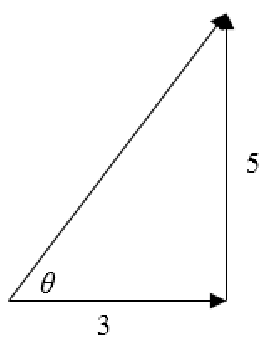
a $\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$-11\mathbf{i} + 5\mathbf{j} = 5\mathbf{i} - 3\mathbf{j} + 4\mathbf{a}$
$4\mathbf{a} = -11\mathbf{i} + 5\mathbf{j} - 5\mathbf{i} + 3\mathbf{j}$
$= -16\mathbf{i} + 8\mathbf{j}$
$\mathbf{a} = (-4\mathbf{i} + 2\mathbf{j})\text{ m s}^{-2}$

b By Newton's 2nd law $\mathbf{F} = m\mathbf{a}$, so
$\mathbf{F} = 5(-4\mathbf{i} + 2\mathbf{j})$
$= -20\mathbf{i} + 10\mathbf{j}$
$|\mathbf{F}| = \sqrt{20^2 + 10^2}$
$= 10\sqrt{5}$
$= 22.4\text{ N (3 s.f.)}$

c At $t = 6$, P has velocity
$\mathbf{v} = \mathbf{u} + \mathbf{a}t$
$= 5\mathbf{i} - 3\mathbf{j} + 6(-4\mathbf{i} + 2\mathbf{j})$
$= -19\mathbf{i} + 9\mathbf{j}$
Since the force is removed at this point, the particle will have zero acceleration from this point so will travel with constant velocity.
At $t = 6$, $\overrightarrow{OA} = 28\mathbf{i} + 6\mathbf{j}$
Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
$\mathbf{r} = 28\mathbf{i} + 6\mathbf{j} + t(-19\mathbf{i} + 9\mathbf{j})$
After 2 seconds
$\mathbf{r}_B = 28\mathbf{i} + 6\mathbf{j} + 2(-19\mathbf{i} + 9\mathbf{j})$
$= -10\mathbf{i} + 24\mathbf{j}$
$|\mathbf{r}_B| = \sqrt{(-10)^2 + 24^2}$
$= 26\text{ m}$

5 $\mathbf{v}_A = 6\mathbf{i}$
$\mathbf{v}_B = 3\mathbf{i} + 5\mathbf{j}$

a



$\tan\theta = \frac{5}{3}$
$\theta = 59.0...2°$
Since bearings are measured from North, B is moving on a bearing of
$90 - 59.0\ldots = 031°$ (to the nearest degree)

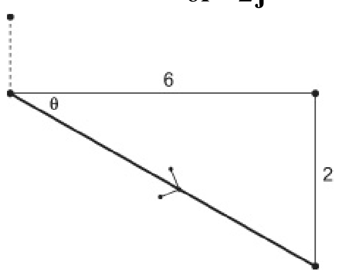
5 **b** At noon, $\mathbf{r}_A = 0$ and $\mathbf{r}_B = -10\mathbf{j}$
Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
$\mathbf{a} = 6t\mathbf{i}$
$\mathbf{b} = -10\mathbf{j} + t(3\mathbf{i} + 5\mathbf{j})$
$= 3t\mathbf{i} + (5t - 10)\mathbf{j}$

c When A is due East of B
$5t - 10 = 0$
$t = 2$
So at 14:00

d $\overrightarrow{AB} = 3t\mathbf{i} + (5t - 10)\mathbf{j} - 6t\mathbf{i}$
$= -3t\mathbf{i} + (5t - 10)\mathbf{j}$
At time t after noon the ships are d km apart, so
$\left|\overrightarrow{AB}\right| = d = \sqrt{(-3t)^2 + (5t - 10)^2}$
$d^2 = 9t^2 + 25t^2 - 100t + 100$
$= 34t^2 - 100t + 100$ as required

e At noon the ships are 10 km apart, so
$10^2 = 34t^2 - 100t + 100$
$34t^2 - 100t = 0$
$2t(17t - 50) = 0$
$t = \frac{50}{17}$
$= 2.94...$ hours
$= 2$ hours and $56.47\ldots$ minutes
So 14:56 (to the nearest minute)

6 **a** $(\text{Path of } S) = (4\mathbf{i} - 6\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$



$\tan\theta = \frac{1}{3} \Rightarrow \theta = 18.43\ldots°$
Bearing $= 90° + \theta = 108°$

6 b 40 minutes is $\frac{2}{3}$ hours, so

$$\mathbf{v} = \frac{6\mathbf{i} - 2\mathbf{j}}{\frac{2}{3}}$$
$$= 9\mathbf{i} - 3\mathbf{j}$$

Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$\mathbf{s} = -2\mathbf{i} - 4\mathbf{j} + t(9\mathbf{i} - 3\mathbf{j})$

$= (9t - 2)\mathbf{i} - (4 + 3t)\mathbf{j}$

c At 11:30, *S* has position

$\mathbf{s} = (9(2.5) - 2)\mathbf{i} - (4 + 3(2.5))\mathbf{j}$

$= 20.5\mathbf{i} - 11.5\mathbf{j}$

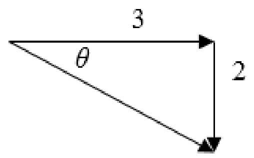
M has velocity $(p\mathbf{i} + q\mathbf{j})$ km h^{-1}

Since it intercepts *S* after half an hour

$p = 41$ and $q = -23$

7 At $t = 0$, $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$

a



$$\tan\theta = \frac{2}{3}$$
$$\theta = 33.6...°$$

So the angle between the positive **j** vector and the direction of motion of *P* is $90 + 33.6\ldots = 124°$ (to the nearest degree).

b Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$= 3\mathbf{i} - 2\mathbf{j} + t(-2\mathbf{i} + 3\mathbf{j})$

$= (3 - 2t)\mathbf{i} + (3t - 2)\mathbf{j}$

c When $t = 4$

$\mathbf{v} = (3 - 2(4))\mathbf{i} + (3(4) - 2)\mathbf{j}$

$= -5\mathbf{i} + 10\mathbf{j}$

$$|\mathbf{v}| = \sqrt{(-5)^2 + 10^2}$$
$$= 5\sqrt{5}$$
$$= 11.2 \text{ m s}^{-1} \text{ (3 s.f.)}$$

d Using $\mathbf{v} = (3 - 2t)\mathbf{i} + (3t - 2)\mathbf{j}$

When *P* is moving parallel to $\mathbf{i} + \mathbf{j}$

$3 - 2t = 3t - 2$

$5t = 5$

$t = 1$

8 $\mathbf{r}_0 = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} + 9\mathbf{j}$

a $|\mathbf{v}| = \sqrt{4^2 + 9^2}$
$= \sqrt{97}$
$= 9.85 \text{ m s}^{-1}$ (3 s.f.)

b Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + t(4\mathbf{i} + 9\mathbf{j})$
$= (3 + 4t)\mathbf{i} + (2 + 9t)\mathbf{j}$

c When the ball is due North of $29\mathbf{i} + 12\mathbf{j}$, the **i** components are equal, so
$3 + 4t = 29$
$4t = 26$
$t = 6.5$ s

d At $t = 6.5$ s
$\mathbf{r} = (3 + 4(6.5))\mathbf{i} + (2 + 9(6.5))\mathbf{j}$
$= 29\mathbf{i} + 60.5\mathbf{j}$
The footballer takes 6.5 s to get to **r** from B
This is a distance of $60.5\mathbf{j} - 12\mathbf{j} = 48.5\mathbf{j}$
$48.5 \div 6.5 = 7.46 \text{ m s}^{-1}$

9 At $t = 0$, $\mathbf{r}_P = 10\mathbf{i} + 15\mathbf{j}$ and $\mathbf{r}_Q = -16\mathbf{i} + 26\mathbf{j}$
At $t = 3$, $\mathbf{r}_P = 25\mathbf{i} + 24\mathbf{j}$
$\mathbf{v}_Q = 12\mathbf{i}$

a $\mathbf{v}_P = \dfrac{25\mathbf{i} + 24\mathbf{j} - (10\mathbf{i} + 15\mathbf{j})}{3}$
$= \dfrac{15\mathbf{i} + 9\mathbf{j}}{3}$
$= (5\mathbf{i} + 3\mathbf{j}) \text{ km h}^{-1}$

b Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$
For P
$\mathbf{p} = 10\mathbf{i} + 15\mathbf{j} + t(5\mathbf{i} + 3\mathbf{j})$
$= (10 + 5t)\mathbf{i} + (15 + 3t)\mathbf{j}$
For Q
$\mathbf{q} = -16\mathbf{i} + 26\mathbf{j} + 12t\mathbf{i}$
$= (12t - 16)\mathbf{i} + 26\mathbf{j}$

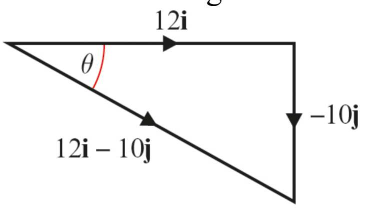
c $\overrightarrow{PQ} = (12t - 16)\mathbf{i} + 26\mathbf{j} - ((10 + 5t)\mathbf{i} + (15 + 3t)\mathbf{j})$
$= (12t - 16)\mathbf{i} - (10 + 5t)\mathbf{i} + 26\mathbf{j} - (15 + 3t)\mathbf{j}$
$= (7t - 26)\mathbf{i} + (11 - 3t)\mathbf{j}$
$|\overrightarrow{PQ}| = d = \sqrt{(7t - 26)^2 + (11 - 3t)^2}$
$d^2 = 49t^2 - 364t + 676 + 121 - 66t + 9t^2$
$= 58t^2 - 430t + 797$ as required

9 d $d^2 = 58t^2 - 430t + 797$
when $d = 13$
$169 = 58t^2 - 430t + 797$
$58t^2 - 430t + 628 = 0$
$$t = \frac{430 \pm \sqrt{(-430)^2 - 4(58)(628)}}{2(58)}$$
$$= \frac{430 \pm 198}{116}$$
$t = 5.41$ or $t = 2$
5.41 hours is 5 hours and 25 minutes (to the nearest minute)
So the lights move out of sight of the observer at 05:25.

10 a speed $|\mathbf{v}| = \sqrt{12^2 + 10^2} = \sqrt{244}$

The speed of the car is 15.6 m s^{-1} (to 3 s.f.)

b Let the acute angle made with **i** be θ, then

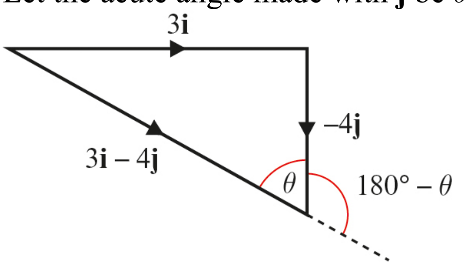


$\tan\theta = \frac{10}{12} = 0.8333$ so $\theta = 39.8°$ (to 3 s.f.)

The direction of motion of the car is 39.8° from the **i** vector.

11 a $|\mathbf{a}| = \sqrt{3^2 + 4^2} = \sqrt{25}$
The magnitude of the acceleration is 5 m s^{-2}.

b Let the acute angle made with **j** be θ



$\tan\theta = \frac{3}{4} = 0.75$ so $\theta = 36.9°$ (to 3 s.f.)

Angle required = $180° - \theta = 180° - 36.9° = 143.1°$
The direction of the acceleration is 143° from the **j** vector.

Challenge

$y = 5 - \frac{5}{3}x$

Using Pythagoras' theorem:

$x^2 + y^2 = \frac{17}{2}$

Solve the equations simultaneously.

Substitute $y = 5 - \frac{5}{3}x$ into $x^2 + y^2 = \frac{17}{2}$.

$$x^2 + \left(5 - \tfrac{5}{3}x\right)^2 = \tfrac{17}{2}$$

$$x^2 + 25 - \tfrac{50}{3}x + \tfrac{25}{9}x^2 - \tfrac{17}{2} = 0$$

$$18x^2 + 450 - 300x + 50x^2 - 153 = 0$$

$$68x^2 - 300x + 297 = 0$$

Using the quadratic formula:

$$x = \frac{300 \pm \sqrt{9216}}{136}$$

$$x = \frac{300 \pm 96}{136}$$

$$x = \frac{99}{34} \text{ or } x = \frac{51}{34}$$

When $x = \frac{99}{34}$, $y = \frac{5}{34}$

When $x = \frac{51}{34}$, $y = \frac{5}{2}$

$$\overrightarrow{OB} = \frac{99}{34}\mathbf{i} + \frac{5}{34}\mathbf{j} \text{ or } \overrightarrow{OB} = \frac{51}{34}\mathbf{i} + \frac{5}{2}\mathbf{j}$$